Vol 17 No 3, March 2008
1674-1056/2008/17(03)/0967-04
Chinese Physics B
© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Scheme for the implementation of $1 \to 3$ optimal phase-covariant quantum cloning in ion-trap systems*

Yang Rong-Can(杨榕灿), Li Hong-Cai(李洪才)†,
Lin Xiu(林 秀), Huang Zhi-Ping(黄志平), and Xie Hong(谢 鸿)

School of Physics and Optoelectronics Technology, Fujian Normal University, Fuzhou 350007, *China*

(Received 3 January 2007; revised manuscript received 22 June 2007)

This paper proposes a scheme for the implementation of $1 \to 3$ optimal phase-covariant quantum cloning with trapped ions. In the present protocol, the required time for the whole procedure is short due to the resonant interaction, which is important in view of decoherence. Furthermore, the scheme is feasible based on current technologies.

Keywords: resonant interaction, $1 \to 3$ optimal phase-covariant quantum cloning, trapped ions
PACC: 4250

The existence of the no-cloning theorem shows that quantum information is fundamentally different from classical information.[1] Classical information can be copied perfectly, whereas quantum information cannot. This is a direct consequence of the linearity of quantum mechanics. Though an exact copy of an arbitrary input state cannot be achieved, one can construct an approximate cloning machine. In Ref.[2], Bužek and Hillery proposed the first theoretical model of the $1 \to 2$ universal quantum cloning machine with the fidelity 5/6. Since then, a number of modifications and extensions to their idea have been suggested.[3−10] One of the most significant examples is that the $1 \to 2$ optimal phase covariant cloning,[9] firstly proposed by Bruβ *et al*, can create two copies of an arbitrary equatorial state with the fidelity $F = 1/2 + 1/2\sqrt{2}$ (slightly higher than 5/6). This discovery motivated an intensive research into the optimal phase covariant cloning.[11−15] In Ref.[11], Fan *et al* suggested a theoretical scheme for constructing the $1 \to M$ optimal phase-covariant cloning machines. For the physical realization, there are several proposals for the implementation of such cloning.[3,15,16] The $1 \to 2$ and $1 \to 3$ optimal phase covariant cloning machines have been experimentally realized in linearly optical system.[12,17,18] However, up to now, there have been no experimental schemes reported in other systems.

The ion-trap quantum information processor, first proposed by Cirac and Zoller[19] and demonstrated experimentally shortly afterwards,[20] was considered to be a promising candidate for the implementation of quantum computation and quantum information processing.[21−26] Illuminated by these ideas, we propose a scheme for the realization of the $1 \to 3$ optimal phase covariant quantum cloning with trapped ions in the present paper. As far as we know, it is the first scheme for the demonstration of the optimal phase covariant quantum cloning with trapped ions. Our scheme might be experimentally realizable with presently available techniques.

Now, let us begin our proposal. We consider that there are three three-level ions confined in the one-dimensional potential trap. Each ion holds an excited state $|e\rangle$ and two ground states $|g\rangle$ and $|i\rangle$. The first qubit is encoded in the two ground states of the ion 1, while the second and third ones are in the bases $\{|e\rangle, |g\rangle\}$ of the ions 2 and 3, respectively. We focus three different laser pulses to drive the three ions tuned to the first lower sideband. The Rabi frequency and the phase of each laser pulse are denoted by Ω_j and ϕ_j $(j = 1, 2, 3)$, respectively. Assume that the first ionic state, which is to be cloned, is initially prepared in

$$\frac{1}{\sqrt{2}}\left(|+\rangle_1 + \mathrm{e}^{\mathrm{i}\theta}\,|-\rangle_1\right), \tag{1}$$

where $|\pm\rangle_1 = \frac{1}{\sqrt{2}}\left(|g\rangle \pm |i\rangle\right)_1$. The vibrational centre-of-mass (c.m.) mode is cooled to the

*Project supported by the National Natural Science Foundation of China (Grant Nos 10574022 and 10575022), and the Funds of the Natural Science of Fujian Province, China (Grant Nos Z0512006 and A0210014).
†Corresponding author. E-mail: hcli45@fjnu.edu.cn

vacuum state $|0\rangle_{\rm vac}$ and the other two ions to $\frac{1}{\sqrt{3}}\left(\sqrt{2}\,|e\rangle+|g\rangle\right)_2|g\rangle_3$. Thus, we have the whole system state in

$$\frac{1}{\sqrt{6}}\left(|+\rangle_1+{\rm e}^{{\rm i}\theta}\,|-\rangle_1\right)\otimes\left(\sqrt{2}\,|e\rangle+|g\rangle\right)_2|g\rangle_3\,|0\rangle_{\rm vac}\,. \quad (2)$$

We now simultaneously turn the second and third laser pulses on to drive the ions 2 and 3, respectively. In the resolved sideband limit and in the Lamb–Dicke regime, the interaction Hamiltonian, describing this situation, is given by[25]

$$\begin{aligned} H_1 \;=\; & {\rm i}\eta\Omega_2{\rm e}^{-{\rm i}\phi_2}a^+\,|g\rangle_2\,\langle e| \\ & +{\rm i}\eta\Omega_3{\rm e}^{-{\rm i}\phi_3}a^+\,|g\rangle_3\,\langle e|+{\rm H.c.}, \end{aligned} \quad (3)$$

where a and a^+ represent the annihilation and creation operators of the c.m. mode, respectively, η stands for the Lamb–Dicke parameter. After the interaction time τ_1, the state of the whole system described by Eq.(2) evolves to

$$\begin{aligned} & \frac{1}{\sqrt{6}}\left(|+\rangle_1+{\rm e}^{{\rm i}\theta}\,|-\rangle_1\right)\otimes\left\{\sqrt{2}\left[\frac{\Omega_3^2+\Omega_2^2\cos E_{23}\tau_1}{\Omega_2^2+\Omega_3^2}\,|e\rangle_2\,|g\rangle_3\,|0\rangle_{\rm vac}+\frac{\Omega_2\Omega_3{\rm e}^{{\rm i}(\phi_3-\phi_2)}\left(\cos E_{23}\tau_1-1\right)}{\Omega_2^2+\Omega_3^2}\right.\right. \\ & \left.\left.\times\,|g\rangle_2\,|e\rangle_3\,|0\rangle_{\rm vac}+\frac{\Omega_2{\rm e}^{-{\rm i}\phi_2}\sin E_{23}\tau_1}{\sqrt{\Omega_2^2+\Omega_3^2}}\,|g\rangle_2\,|g\rangle_3\,|1\rangle_{\rm vac}\right]+|g\rangle_2\,|g\rangle_3\,|0\rangle_{\rm vac}\right\}, \end{aligned} \quad (4)$$

where $E_{23}=\eta\sqrt{\Omega_2^2+\Omega_3^2}$.

We choose the two Rabi frequencies Ω_2 and Ω_3, the interaction time τ_1 and their laser phases ϕ_2 and ϕ_3 to satisfy

$$\begin{aligned} & E_{23}\tau_1=\eta\tau_1\sqrt{\Omega_2^2+\Omega_3^2}=\pi, \\ & \Omega_3=\left(\sqrt{2}+1\right)\Omega_2, \\ & \phi_3=\phi_2+\pi. \end{aligned} \quad (5)$$

The state of the whole system Eq.(4) becomes

$$\begin{aligned} & \frac{1}{\sqrt{6}}\left(|+\rangle_1+{\rm e}^{{\rm i}\theta}\,|-\rangle_1\right)\otimes\Big(\,|e\rangle_2\,|g\rangle_3+|g\rangle_2\,|e\rangle_3 \\ & +|g\rangle_2\,|g\rangle_3\Big)\,|0\rangle_{\rm vac}\,. \end{aligned} \quad (6)$$

Then, we turn off the second and third laser pulses. After that, we simultaneously turn on the first and second ones. In this case, the interaction Hamiltonian reads

$$\begin{aligned} H_2 \;=\; & {\rm i}\eta\Omega_1{\rm e}^{-{\rm i}\phi_1}a^+\,|g\rangle_1\,\langle e| \\ & +{\rm i}\eta\Omega_2{\rm e}^{-{\rm i}\phi_2}a^+\,|g\rangle_2\,\langle e|+{\rm H.c.} \end{aligned} \quad (7)$$

After the interaction time τ_2, the different initial states, composed of the state of ions 1 and 2 in the bases $\{|g\rangle_1|e\rangle_2\,,\,|i\rangle_1|e\rangle_2\,,\,|g\rangle_1|g\rangle_2\,,\,|i\rangle_1|g\rangle_2\}$ and the vacuum state $|0\rangle$, evolve as

$$\begin{aligned} |g\rangle_1\,|e\rangle_2\,|0\rangle_{\rm vac} \to\; & \frac{\Omega_1^2+\Omega_2^2\cos E_{12}\tau_2}{\Omega_1^2+\Omega_2^2}\,|g\rangle_1\,|e\rangle_2\,|0\rangle_{\rm vac}+\frac{\Omega_1\Omega_2{\rm e}^{{\rm i}(\phi_1-\phi_2)}\left(\cos E_{12}\tau_2-1\right)}{\Omega_1^2+\Omega_2^2} \\ & \times\,|e\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}+\frac{\Omega_2{\rm e}^{-{\rm i}\phi_2}\sin E_{12}\tau_2}{\sqrt{\Omega_1^2+\Omega_2^2}}\,|g\rangle_1\,|g\rangle_2\,|1\rangle_{\rm vac}\,; \end{aligned} \quad (8a)$$

$$|i\rangle_1\,|e\rangle_2\,|0\rangle_{\rm vac}\to\cos\left(\eta\Omega_2 t\right)|i\rangle_1\,|e\rangle_2\,|0\rangle_{\rm vac}+\sin\left(\eta\Omega_2 t\right){\rm e}^{{\rm i}\phi_2}\,|i\rangle_1\,|g\rangle_2\,|1\rangle_{\rm vac}\,; \quad (8b)$$

$$|g\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}\to|g\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}\,; \quad (8c)$$

$$|i\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}\to|i\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}\,, \quad (8d)$$

where $E_{12}=\eta\sqrt{\Omega_1^2+\Omega_2^2}$.

With the choice of $\Omega_1=\sqrt{3}\Omega_2$ and $\eta\Omega_2\tau_2=\pi$, $E_{12}\tau_2=2\pi$, we can obtain the controlled phase flip (CPF) operation between ions 1 and 2, i.e.

$$\begin{aligned} & |g\rangle_1\,|e\rangle_2\,|0\rangle_{\rm vac}\to|g\rangle_1\,|e\rangle_2\,|0\rangle_{\rm vac}\,; \quad |i\rangle_1\,|e\rangle_2\,|0\rangle_{\rm vac}\to-\,|i\rangle_1\,|e\rangle_2\,|0\rangle_{\rm vac}\,; \\ & |g\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}\to|g\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}\,; \quad |i\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}\to|i\rangle_1\,|g\rangle_2\,|0\rangle_{\rm vac}\,. \end{aligned} \quad (9)$$

Due to Eq.(9), we can obtain two controlled-not (CNOT) transformations in the rotating basis, namely

$$\begin{aligned} &|+\rangle_1 |e\rangle_2 |0\rangle_{\rm vac} \to |-\rangle_1 |e\rangle_2 |0\rangle_{\rm vac}\,; \quad |-\rangle_1 |e\rangle_2 |0\rangle_{\rm vac} \to |+\rangle_1 |e\rangle_2 |0\rangle_{\rm vac}\,; \\ &|+\rangle_1 |g\rangle_2 |0\rangle_{\rm vac} \to |+\rangle_1 |g\rangle_2 |0\rangle_{\rm vac}\,; \quad |-\rangle_1 |g\rangle_2 |0\rangle_{\rm vac} \to |-\rangle_1 |g\rangle_2 |0\rangle_{\rm vac}\,, \end{aligned} \tag{10a}$$

and

$$\begin{aligned} &|g\rangle_1 |\xi\rangle_2 |0\rangle_{\rm vac} \to |g\rangle_1 |\xi\rangle_2 |0\rangle_{\rm vac}\,; \quad |i\rangle_1 |\xi\rangle_2 |0\rangle_{\rm vac} \to -|i\rangle_1 |\varsigma\rangle_2 |0\rangle_{\rm vac}\,; \\ &|g\rangle_1 |\varsigma\rangle_2 |0\rangle_{\rm vac} \to |g\rangle_1 |\varsigma\rangle_2 |0\rangle_{\rm vac}\,; \quad |i\rangle_1 |\varsigma\rangle_2 |0\rangle_{\rm vac} \to -|i\rangle_1 |\xi\rangle_2 |0\rangle_{\rm vac}\,, \end{aligned} \tag{10b}$$

with the abbreviation $|\xi\rangle = \frac{1}{\sqrt{2}}(|e\rangle + |g\rangle)$ and $|\varsigma\rangle = \frac{1}{\sqrt{2}}(|e\rangle - |g\rangle)$.

Under the action of the CNOT operator Eq.(10a), the state of the whole system described by Eq.(6) becomes

$$\frac{1}{\sqrt{6}}\Big[\left(|-\rangle_1 + {\rm e}^{{\rm i}\theta}|+\rangle_1\right)|e\rangle_2|g\rangle_3 + \Big(|+\rangle_1 + {\rm e}^{{\rm i}\theta}|-\rangle_1\Big) \left(|g\rangle_2|e\rangle_3 + |g\rangle_2|g\rangle_3\right)\Big], \tag{11}$$

where we have eliminated the vacuum state $|0\rangle_{\rm vac}$.

Next, we turn off the first and second laser pulses and then simultaneously turn on the first and third ones to perform a similar CNOT operator Eq.(10a) on the ions 1 and 3, leading the state of the whole system to

$$\frac{1}{\sqrt{6}}\left[\left(|-\rangle_1 + {\rm e}^{{\rm i}\theta}|+\rangle_1\right)\left(|e\rangle_2|g\rangle_3 + |g\rangle_2|e\rangle_3\right) + \left(|+\rangle_1 + {\rm e}^{{\rm i}\theta}|-\rangle_1\right)|g\rangle_2|g\rangle_3\right]. \tag{12}$$

Now, we implement a Hadamard transformation on the ion 1, i.e.

$$|g\rangle_1 \to |+\rangle_1\,, \quad |i\rangle_1 \to |-\rangle_1\,, \tag{13a}$$

and the Hadamard transformation on the ions 2 and 3 respectively, i.e.

$$|g\rangle_j \to |\xi\rangle_j\,, |e\rangle_j \to |\varsigma\rangle_j\,. \quad j = 2,\, 3. \tag{13b}$$

The state of the whole system described by Eq.(12) will be

$$\frac{1}{\sqrt{6}}\left[\left(|i\rangle_1 + {\rm e}^{{\rm i}\theta}|g\rangle_1\right)\left(|\varsigma\rangle_2|\xi\rangle_3 + |\xi\rangle_2|\varsigma\rangle_3\right) + \left(|g\rangle_1 + {\rm e}^{{\rm i}\theta}|i\rangle_1\right)|\xi\rangle_2|\xi\rangle_3\right]. \tag{14}$$

We then perform the CNOT transformation Eq.(10b) on ions 1 and 2, and then a similar CNOT operation is implemented between the ion 1 and the ion 3, where the ion 1 is the controlled qubit. In this case, the state of the whole system evolves to

$$\begin{aligned} &\frac{1}{\sqrt{6}}\{|i\rangle_1(|\varsigma\rangle_2|\xi\rangle_3 + |\xi\rangle_2|\varsigma\rangle_3) + |g\rangle_1|\xi\rangle_2|\xi\rangle_3 \\ &+{\rm e}^{{\rm i}\theta}[|g\rangle_1(|\varsigma\rangle_2|\xi\rangle_3 + |\xi\rangle_2|\varsigma\rangle_3) \\ &+|i\rangle_1|\varsigma\rangle_2|\varsigma\rangle_3]\}\,. \end{aligned} \tag{15}$$

At last, we implement the Hadamard gate Eq.(13a) and then the rotating transformation $|g\rangle\langle i| + |i\rangle\langle g|$ on the ion 1. The whole system state will finally be

$$\begin{aligned} &\frac{1}{\sqrt{6}}[(|+\rangle_1|\varsigma\rangle_2|\xi\rangle_3 + |+\rangle_1|\xi\rangle_2|\varsigma\rangle_3 + |-\rangle_1|\xi\rangle_2|\xi\rangle_3) \\ &+{\rm e}^{{\rm i}\theta}(|-\rangle_1|\varsigma\rangle_2|\xi\rangle_3 + |-\rangle_1|\xi\rangle_2|\varsigma\rangle_3 \\ &+|+\rangle_1|\varsigma\rangle_2|\varsigma\rangle_3)]\,. \end{aligned} \tag{16}$$

For the sake of making the representation clear, we identify $|+\rangle$ and $|\xi\rangle$ as the logic $|0\rangle$ and $|-\rangle$ and $|\varsigma\rangle$ as the logic $|1\rangle$, respectively. Then the input state described by (1) and the final state of the whole system Eq.(16) can be rewritten as

$$\frac{1}{\sqrt{2}}\left(|0\rangle + {\rm e}^{{\rm i}\theta}|1\rangle\right) \tag{17a}$$

and

$$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle) + {\rm e}^{{\rm i}\theta}\frac{1}{\sqrt{3}}(|011\rangle + |110\rangle + |101\rangle)\right], \tag{17b}$$

respectively, which shows that the 1 → 3 optimal phase-covariant quantum cloning machine with fidelity of 5/6 has been realized.[11,15]

It is necessary to give a brief discussion on the feasibility of the proposed protocol. In our scheme, the above ionic levels $|g\rangle$, $|e\rangle$ and $|i\rangle$ can be obtained from the levels $S_{1/2}$ ($m_j = -1/2$), $D_{5/2}$ ($m_j = -1/2$) and $S_{1/2}$ ($m_j = 1/2$) of $^{40}{\rm Ca}^+$, respectively.[27] Since the lifetime of the metastable $D_{5/2}$ state is about $\tau \approx 1.16\,{\rm s}$, we can neglect the spontaneous radiation. Due to Refs.[28, 29], the authors also show that the vibration c.m. mode ($\nu = 2\pi \times 1.2\,{\rm MHz}$) can be cooled

to the vacuum state with the fidelity 99% and the ions to the Lamb–Dicke regime ($\eta < 0.1$) via sideband cooling. The three ions can be individually addressed by using focused laser beams with the fidelity 97%. For the single-qubit rotating, we can easily obtain the transition $|e\rangle \leftrightarrow |g\rangle$, such as the Hadamard transformation Eq.(13b) in a very short period of time.[30] And the transition $|g\rangle \leftrightarrow |i\rangle$ can be obtained via adjusting the pulse area and using Ramsey interferometry.[31] So the errors mainly owe to the imperfection of two-qubit logical gates. If we choose the Rabi frequency $\Omega_2 = 2\pi \times 47\,\mathrm{kHz}$ in Eqs.(5) and (9), the required time for preparing the entangled states described by Eq.(9) is about $40\,\mu$s and the required time for implement the CPF described by Eq.(9) is about $100\,\mu$s, which leads that our scheme may be realized in about $450\,\mu$s. Thus, we have enough time to finish the whole procedure and the error of the off-resonant excitations is less than 4%.[29] We now suppose that the fluctuation of each Rabi frequency is about 3% and Eq.(5) are satisfied, we then obtain $\cos(E_{23}\tau_1) \approx -0.998$ and $\sin(E_{23}\tau_1) \approx -0.055$ in Eq.(4). So the fidelity of having the entangled states in Eq.(6) is about 0.99. In addition, the other main sources for the imperfection are Laser frequency noise (2% error per each laser pulse and each time), and residual thermal excitation (2% error).[28,29,32] Therefore, the fidelity of our scheme is about 70%.

In summary, we have proposed a scheme for the realization of the 1→3 optimal phase covariant quantum cloning with trapped ions. The present protocol is based on the simultaneous resonant interaction between each two ions and the vibrational mode. The scheme doesn't use the vibrational mode to act as memory and the required time for the whole procedure is short based on resonant interaction, which is important in view of decoherence. Therefore, our scheme may be feasible on the basis of the presently available techniques.

References

[1] Wootters W K and Zurek W H 1982 *Nature* (London) **299** 802

[2] Bužek V and Hillery M 1996 *Phys. Rev.* A **54** 1844

[3] Bužek V, Braunstein S L, Hillery M and Bruβ D 1997 *Phys. Rev.* A **56** 3446

[4] Gisin N and Massar S 1997 *Phys. Rev. Lett.* **79** 2153

[5] Duan L M and Guo G C 1998 *Phys. Rev. Lett.* **80** 4999

[6] He G Q and Zeng G H 2005 *Chin. Phys.* **14** 541

[7] Ricci M, Sciarrino F, Sias C and De Martini F 2004 *Phys. Rev. Lett.* **92** 047901

[8] Milman P, Ollivier H and Raimond J M 2003 *Phys. Rev.* A **67** 012314

[9] Bruβ D, Cinchetti M, D'Ariano G M and Macchiavello C 2000 *Phys. Rev.* A **62** 012302

[10] Gong S Q, Wang Z Y, Feng X L and Xu Z Z 2000 *Chin. Phys.* **9** 94

[11] Fan H, Mastumoto K, Wang X B and Wadati M 2001 *Phys. Rev.* A **65** 012304

[12] Khan I A and Howell J C 2004 *Phys. Rev.* A **70** 010303 (R)

[13] Zou X B, Dong Y L and Guo G C 2006 *Phys. Lett.* A **360** 44

[14] Zhang W H, Yu L B and Ye L 2006 *Phys. Lett.* A **356** 195

[15] Zou X B and Mathis W 2005 *Phys. Rev.* A **72** 022306

[16] Song W, Yang M and Cao Z L 2004 *Phys. Lett.* A **331** 34

[17] Sciarrino F and De Martini F 2005 *Phys. Rev.* A **72** 062313

[18] Črnoch A, Bartušová L, Soubusta J, Ježek M, Fiurášek J and Dušek M 2006 *Phys. Rev.* A **74** 042327

[19] Cirac J I and Zoller P 1995 *Phys. Rev. Lett.* **74** 4091

[20] Monroe C, Meekhof D M, King B E, Itano W M and Wineland D J 1995 *Phys. Rev. Lett.* **75** 4714

[21] Poyatos J F, Cirac J I and Zoller P 1998 *Phys. Rev. Lett.* **81** 1322

[22] Mølmer K and Sørensen A 1999 *Phys. Rev. Lett.* **82** 1835

[23] Cai L H, Hai W H and Lu G B 2006 *Chin. Phys.* **15** 2275

[24] Zheng S B 2003 *Phys. Rev. Lett.* **90** 217901

[25] Zheng S B 2005 *Phys. Rev.* A **71** 062335

[26] Zheng X J, Fang M F, Liao X P, Cai J W and Cao S 2007 *Chin. Phys.* **16** 906

[27] Stevens D, Brochard J and Steane A M 1998 *Phys. Rev.* A **58** 2750

[28] Roos C F, Riebe M, Häffner H, Hänsel W, Benhelm J, Lancaster G P T, Becher C, Schnidt-kaler F and Blatt R 2004 *Science* **304** 1478

[29] Schmidt-Kaler F, Häffner H, Riebe M, Gulde S, Lancaster G P T, Deuschle T, Becher C, Roos C F, Eschner J and Blatt R 2003 *Nature* **422** 408

[30] Steane A, Roos, C F, Stevens D, Mundt A, Leibfried D, Schmidt-Kaler F and Blatt R 2000 *Phys. Rev.* A **62** 042305

[31] Home J P, McDonnell M J, Lucas D M, Imreh G, Keitch B C, Szwer D J, Thomas N R, Webster S C, Stacey D N and Steane A M 2006 *N. J. Phys.* **8** 188

[32] Schimidt-Kaler F, Häffner H, Gulde S, Riebe M, Lancaster G P T, Deuschle T, Becher C, Hänsel W, Eschner J, Roos C F and Blatt R 2003 *Appl. Phys.* B **77** 789