Vol 17 No 3, March 2008
1674-1056/2008/17(03)/0960-07
Chinese Physics B
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Instantaneous coherent polarization and coherent polarimetric spectra

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(Received 6 March 2007; revised manuscript received 3 July 2007)

The statistics of polarization in electromagnetic waves or target vector has been studied deeply, generally using Stokes parameters. Unfortunately, all the data-processing includes the assumption that the sampling data are independent of time, without a consideration of the time coherence of the polarization data, such as instantaneous Stokes parameters. In this paper, the definitions of the instantaneous coherent polarization function and polarimetric spectrum are presented for the first time, which shows the coherences of the instantaneous Stokes parameters in both time and frequency domain, even in spatial coherence. The new formula of definition can be extended to spatial propagation coherence, both in free space or any linear medium.

Keywords: instantaneous polarization, coherence, polarization spectra
PACC: 4225J, 4225K, 9265M, 0590

1. Introduction

Hurwitz[1] was the first to investigate the statistical properties of unpolarized light for which he studied probability density functions (PDFs) of the parameters characterizing the ellipse. But all the three parameters are functions of the polarization Barakat[2] studied the probability behaviour of the parameters for arbitrary degree of polarization. Eliyahu[3] presented the vector statistical distribution functions for correlated Gaussian fields. The results may be used to describe the interesting cases of polarization of multiply scattered light from disordered media based upon explicit calculations for the covariance matrix of the scattered light. Brosseau[4] examined the statistics of the normalized Stokes parameters for a stochastic plane wave field which is Gaussian distributed and gave the lower-order moments. In the recent years it was found that the degree of polarization of the beam can change upon propagation, not only in free space, but also in random media, and in a turbulent atmosphere. Touzi and Lopes[5] and Lee *et al*[6] developed the theorem to describe the statistics of the speckle of the Synthetic Aperture Radar (SAR) imagery. Korotkova[7] generated the Stokes parameters from two-point quantities other than one-point quantities, and studied the changes in statistics of the instantaneous Stokes parameters of a quasi-monochromatic electromagnetic beam on propagation. It showed that the polarization state would not change if the distance was far enough than 1 kilometer, whereas the definition may not express the space coherence clearly. Liu and LÜ[8] have given the coherence vortices of partially coherent beams in the far field.

In early radar systems, for radar's resolution is low, the backscatter wave can be seemed to be compound with many independent scattering centres, and therefore the backscatter wave is Gaussian distributed. While with the improvement of the radar's resolution, the scattering centres are not independent in space or time, and then the above assumption is not suitable. The statistical properties have not been under the consideration of the time or spatial coherence of the electromagnetic waves or target vector. But in reality the time coherence of the electromagnetic waves affects the variant rules of the polarization state which can be expressed with the Stokes parameters. In this paper, the instantaneous coherent polarization function (ICPF) is defined to describe the time coherence of the Stokes parameters, and the polarimetric spectrum is defined too to describe the coherence in frequency domain of Stokes parameters, and this is much helpful for target detection and target recognition.

Section 2 gives the definition of ICPF. The analytical expression in detail is derived for the ICPF in Section 3. Then the instantaneous coherent polarimetric spectrum (ICPS) is defined in Section 4.

Section 5 gives an instance for the ICPF and ICPS. Finally, the conclusion is presented.

2. Definition of instantaneous coherent polarization function

The coherent vector of the Stokes parameters has been defined,[7,9] but it cannot be used to describe the coherence definitely. So this definition is renamed as the first ICPF. For the description of the direct coherence of the time-sequence or spatial-propagation Stokes parameters, the second ICPF is defined to characterize the coherent Stokes parameters.

2.1. The first ICPF

Assuming that any planar incident wave can be represented as $\boldsymbol{e}(t)$, where $t \in T$, we define the time coherence of the wave vectors as

$$\boldsymbol{c}(t_1, t_1+\tau) = \boldsymbol{e}(t_1) \otimes \boldsymbol{e}^*(t_1+\tau), \quad t_1, t_1+\tau \in T,$$

where '$\otimes$' denotes the Kronecker product. It is to say, the time coherence of the wave vectors is derived by the Kronecker product of the wave vectors at two different times. Then the time-domain is presented as

$$\begin{aligned}
g_0(\tau) &= g_0(t,\tau) = \langle E_{\mathrm{H}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau)\rangle + \langle E_{\mathrm{V}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle,\\
g_1(\tau) &= g_1(t,\tau) = \langle E_{\mathrm{H}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau)\rangle - \langle E_{\mathrm{V}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle,\\
g_2(\tau) &= g_2(t,\tau) = \langle E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle + \langle E_{\mathrm{V}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau)\rangle,\\
g_3(\tau) &= g_3(t,\tau) = -j[\langle E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle - \langle E_{\mathrm{V}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau)\rangle].
\end{aligned} \tag{1}$$

In the stationary stochastic process, the coherent function is not coherent of time. Here '$\langle\cdot\rangle$' denotes the ensemble average. If the stochastic process is ergodic, the ensemble average equals to the time average. From Eq.(1), it can be seen that the first instantaneous coherent polarization is just a linear compound of the wave coherence. We cannot see the coherence of the Stokes parameters clearly.

2.2. The new ICPF

Supposing that the Stokes parameters are real signals, we define the self-coherent functions as

$$\begin{aligned}
s_0(\tau) &= \langle S_0(t)S_0(t+\tau)\rangle = \langle [E_{\mathrm{H}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t) + E_{\mathrm{V}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t)][E_{\mathrm{H}}(t+\tau)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau) + E_{\mathrm{V}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)]\rangle,\\
s_1(\tau) &= \langle S_1(t)S_1(t+\tau)\rangle = \langle [E_{\mathrm{H}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t) - E_{\mathrm{V}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t)][E_{\mathrm{H}}(t+\tau)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau) - E_{\mathrm{V}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)]\rangle,\\
s_2(\tau) &= \langle S_2(t)S_2(t+\tau)\rangle = \langle [E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t) + E_{\mathrm{V}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t)][E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau) + E_{\mathrm{V}}(t+\tau)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau)]\rangle,\\
s_3(\tau) &= \langle S_3(t)S_3(t+\tau)\rangle = -\langle [E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t) - E_{\mathrm{V}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t)][E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau) - E_{\mathrm{V}}(t+\tau)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau)]\rangle.
\end{aligned} \tag{2}$$

As is seen, the definitions are quite different among Eq.(1) and Eq.(2).

After the analysis of the equations' structure, the basic coherent polarimetric cells (BCPC) are extracted to simplify the calculation. Those are the first-order BCPC, second-order BCPC, and fourth-order BCPC. As is the following:

$$\begin{aligned}
R_{ij}(\tau) &= \langle E_i(t)E_j^{\mathrm{H}}(t+\tau)\rangle,\\
S_{ij}(\tau) &= \langle |E_i(t)E_j^{\mathrm{H}}(t+\tau)|^2\rangle,
\end{aligned}$$

where $i, j = \mathrm{H}, \mathrm{V}$.

The fourth-order BCPC is the mean of the product of the four electromagnetic elements, such as

$$\langle E_{\mathrm{H}}(t)E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle.$$

The ICPF can be derived via the compound of the three BCPCs.

By the same way, we can define the cross-coherent functions:

$$
\begin{aligned}
s_{01}(\tau) &= \langle S_0(t)S_1(t+\tau)\rangle = \langle [E_{\rm H}(t)E_{\rm H}^{\rm H}(t) + E_{\rm V}(t)E_{\rm V}^{\rm H}(t)][E_{\rm H}(t+\tau)E_{\rm H}^{\rm H}(t+\tau) - E_{\rm V}(t+\tau)E_{\rm V}^{\rm H}(t+\tau)]\rangle,\\
s_{12}(\tau) &= \langle S_1(t)S_2(t+\tau)\rangle = \langle [E_{\rm H}(t)E_{\rm H}^{\rm H}(t) - E_{\rm V}(t)E_{\rm V}^{\rm H}(t)][E_{\rm H}(t+\tau)E_{\rm V}^{\rm H}(t+\tau) + E_{\rm V}(t+\tau)E_{\rm H}^{\rm H}(t+\tau)]\rangle,\\
s_{23}(\tau) &= \langle S_2(t)S_3(t+\tau)\rangle = -j\langle [E_{\rm H}(t)E_{\rm V}^{\rm H}(t) + E_{\rm V}(t)E_{\rm H}^{\rm H}(t)][E_{\rm H}(t+\tau)E_{\rm V}^{\rm H}(t+\tau) - E_{\rm V}(t+\tau)E_{\rm H}^{\rm H}(t+\tau)]\rangle,\\
s_{13}(\tau) &= \langle S_1(t)S_3(t+\tau)\rangle = -j\langle [E_{\rm H}(t)E_{\rm H}^{\rm H}(t) - E_{\rm V}(t)E_{\rm V}^{\rm H}(t)][E_{\rm H}(t+\tau)E_{\rm V}^{\rm H}(t+\tau) - E_{\rm V}(t+\tau)E_{\rm H}^{\rm H}(t+\tau)]\rangle. \quad (3)
\end{aligned}
$$

The expressions can also be derived via the three BCPCs.

3. The calculation of BCPC

In this section the analytical expressions of BCPC are given from the compound Gaussian distribution. Then the IPCF are expressed with the BCPCs.

3.1. The calculation of BCPC

A stochastic polarized wave is ergodic owing to Gaussian distribution with zero mean. When $(\hat{h}, \hat{v})$ is selected as the coordinate, we have

$$e_{\rm HV}(t) \sim N(0, C), \quad e_{\rm HV}(t+\tau) \sim N(0, C),$$

where C is the covariance matrix, and

$$
\begin{aligned}
C &= \langle e_{\rm HV}(t)e_{\rm HV}^{\rm H}(t)\rangle = \langle e_{\rm HV}(t+\tau)e_{\rm HV}^{\rm H}(t+\tau)\rangle\\
&= \begin{bmatrix} C_{\rm HH} & C_{\rm HV} \\ C_{\rm VH} & C_{\rm VV} \end{bmatrix}.
\end{aligned}
$$

Define a variable $E_{\rm HV} = [e_{\rm H}(t),\ e_{\rm H}(t+\tau),\ e_{\rm V}(t),\ e_{\rm V}(t+\tau)]^{\rm T}$, where $e_{\rm H}(t) = x_{\rm H} + Iy_{\rm H}$, $e_{\rm H}(t+\tau) = p_{\rm H} + Iq_{\rm H}$, $e_{\rm V}(t) = x_{\rm V} + Iy_{\rm V}$, $e_{\rm V}(t+\tau) = p_{\rm V} + Iq_{\rm V}$, I is imaginary number unit.

So the PDF of the electromagnetic wave vector $E_{\rm HV}$ is

$$f(E_{\rm HV}) = \frac{1}{\pi^4|\Sigma_{\rm HV}|}\exp\{-E_{\rm HV}^{\rm H}\Sigma_{\rm HV}^{-1}E_{\rm HV}\},$$

where $\Sigma_{\rm HV}$ is the covariance matrix:

$$
\Sigma_{\rm HV} = \langle E_{\rm HV}E_{\rm HV}^{\rm H}\rangle = \begin{bmatrix}
\sigma_{\rm HH} & R_{\rm HH}(\tau) & \sigma_{\rm HV} & R_{\rm HV}(\tau)\\
R_{\rm HH}(-\tau) & \sigma_{\rm HH} & R_{\rm VH}(\tau) & \sigma_{\rm HV}\\
\sigma_{\rm VH} & R_{\rm VH}(\tau) & \sigma_{\rm VV} & R_{\rm VV}(\tau)\\
R_{\rm HV}(\tau) & \sigma_{\rm VH} & R_{\rm VV}(-\tau) & \sigma_{\rm VV}
\end{bmatrix}.
$$

The elements of the wave vector $E_{\rm HV}$ own to Feynman path assumption. The elements in the matrix $\Sigma_{\rm HV}$ are complex.

For the BCPC calculation, we can derive them via complex Gaussian distribution of the wave vector.

Firstly we try to obtain the expressions of $\langle E_i(t)E_j^{\rm H}(t+\tau)\rangle$, $\langle |E_i(t)E_j^{\rm H}(t+\tau)|^2\rangle$, i, j= H,V.

Generally we let $E_i(t) = x_i + Iy_i$, $E_j(t+\tau) = p_j + Iq_j$, where I is imaginary number unit. Thus the covariance matrix of $E_i(t)$ and $E_j(t+\tau)$ is

$$
\Sigma_{ij} = \begin{bmatrix} \sigma_{ij\rm RR}(\tau) & \sigma_{ij\rm RI}(\tau) \\ \sigma_{ij\rm IR}(\tau) & \sigma_{ij\rm II}(\tau) \end{bmatrix},
$$

where $\sigma_{ij\rm RR}(\tau) = \sigma_{ji\rm II}(\tau)$, $\sigma_{ij\rm RI}(\tau) = -\sigma_{ij\rm IR}(\tau)$.
$\sigma_{ij\rm RR}(\tau)$ denotes the real part of the coherent function and $\sigma_{ij\rm II}(\tau)$ is the imaginary part of the coherent function. The other two elements of the covariance matrix are the coherent functions between real part and imaginary part.

From Feynman path assumption of the complex Gaussian distribution, we can get

$$
\begin{aligned}
&\langle x_ip_j\rangle = \langle y_iq_j\rangle = \sigma_{ij\rm RR}(\tau) = \sigma_{ij\rm II}(\tau),\\
&\langle y_ip_j\rangle = -\langle x_iq_j\rangle = \sigma_{ij\rm IR}(\tau) = -\sigma_{ij\rm RI}(\tau).
\end{aligned}
$$

The first-order BCPC is

$$
\begin{aligned}
R_{ij}(\tau) =& \langle E_i(t)E_j^{\rm H}(t+\tau)\rangle\\
=& 2[\sigma_{ij\rm RR}(\tau) + I\sigma_{ij\rm IR}(\tau)]. \quad (4)
\end{aligned}
$$

The second-order BCPC is

$$\begin{aligned} S_{ij}(\tau) =&\langle |E_i(t)E_j^{\mathrm{H}}(t+\tau)|^2\rangle \\ =&4[\sigma_{ii\mathrm{RR}}(\tau)\sigma_{jj\mathrm{RR}}(\tau)+\sigma_{ij\mathrm{RR}}^2(\tau) \\ &+\sigma_{ij\mathrm{RI}}^2(\tau)]. \end{aligned} \tag{5}$$

If there are four Gaussian random variables with zero mean X_1, X_2, X_3, X_4, it is easy to know

$$\begin{aligned} E[X_1X_2X_3X_4] =&E[X_1X_2]E[X_3X_4] \\ &+E[X_1X_3]E[X_2X_4] \\ &+E[X_2X_3]E[X_1X_4]. \end{aligned} \tag{6}$$

Then for a fourth-order BCPC, there is

$$\begin{aligned} &\langle E_{\mathrm{H}}(t)E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle \\ =&\langle E_{\mathrm{H}}(t)E_{\mathrm{H}}(t+\tau)\rangle\langle E_{\mathrm{V}}^{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle \\ &+\langle E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t)\rangle\langle E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle \\ &+\langle E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t)\rangle\langle E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle. \end{aligned}$$

Based on Feynman path, it is known

$$\begin{aligned} &\langle E_{\mathrm{H}}(t)E_{\mathrm{H}}(t+\tau)\rangle = \langle E_{\mathrm{V}}^{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle = 0, \\ &\langle E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t)\rangle = \langle E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle = \sigma_{\mathrm{HV}}. \end{aligned}$$

And then we can get

$$\begin{aligned} &\langle E_{\mathrm{H}}(t)E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle \\ =&\langle E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t)\rangle\langle E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle \\ &+\langle E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t)\rangle\langle E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle. \end{aligned} \tag{7}$$

Then we derive the fourth-order BCPC

$$\begin{aligned} &\langle E_{\mathrm{H}}(t)E_{\mathrm{H}}(t+\tau)E_{\mathrm{V}}^{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle \\ =&R_{\mathrm{HV}}(-\tau)R_{\mathrm{HV}}(\tau)+R_{\mathrm{HV}}(0)R_{\mathrm{HV}}(0). \end{aligned} \tag{8}$$

By the same method, the other fourth-order BCPC can be obtained as

$$\begin{aligned} &\langle E_{\mathrm{V}}(t)E_{\mathrm{H}}(t+\tau)E_{\mathrm{H}}^{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t+\tau)\rangle \\ =&R_{\mathrm{HH}}(-\tau)R_{\mathrm{VV}}(\tau)+R_{\mathrm{HH}}(0)R_{\mathrm{VV}}(0), \end{aligned} \tag{9}$$

$$\begin{aligned} &\langle E_{\mathrm{H}}(t)E_{\mathrm{V}}^{\mathrm{H}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau)E_{\mathrm{V}}(t+\tau)\rangle \\ =&R_{\mathrm{VV}}(-\tau)R_{\mathrm{HH}}(\tau)+R_{\mathrm{VV}}(0)R_{\mathrm{HH}}(0), \end{aligned} \tag{10}$$

$$\begin{aligned} &\langle E_{\mathrm{V}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t)E_{\mathrm{H}}^{\mathrm{H}}(t+\tau)E_{\mathrm{V}}(t+\tau)\rangle \\ =&R_{\mathrm{VH}}(\tau)R_{\mathrm{VH}}(-\tau)+R_{\mathrm{VH}}(0)R_{\mathrm{VH}}(0). \end{aligned} \tag{11}$$

3.2. The analytical expression of the first ICPFs

Inserting Eqs.(4,5) into Eq.(1), we can obtain the analytical expressions of the first ICPFs as

$$\begin{aligned} g_0(\tau) &= R_{\mathrm{HH}}(\tau)+R_{\mathrm{VV}}(\tau), \\ g_1(\tau) &= R_{\mathrm{HH}}(\tau)-R_{\mathrm{VV}}(\tau), \\ g_2(\tau) &= R_{\mathrm{HV}}(\tau)+R_{\mathrm{VH}}(\tau), \\ g_3(\tau) &= -j[R_{\mathrm{HV}}(\tau)-R_{\mathrm{VH}}(\tau)]. \end{aligned} \tag{12}$$

3.3. The analytical expression of the second ICPFs

Inserting Eqs.(8–11) into Eq.(2), we can derive the expressions of the second ICPFs as

$$\begin{aligned} s_0(\tau) =&S_{\mathrm{HH}}(\tau)+S_{\mathrm{HV}}(\tau)+S_{\mathrm{VH}}(\tau)+S_{\mathrm{VV}}(\tau), \\ s_1(\tau) =&S_{\mathrm{HH}}(\tau)-S_{\mathrm{HV}}(\tau)-S_{\mathrm{VH}}(\tau)+S_{\mathrm{VV}}(\tau), \\ s_2(\tau) =&2\mathrm{Re}[R_{\mathrm{HV}}(-\tau)R_{\mathrm{HV}}(\tau)+R_{\mathrm{HH}}(-\tau)R_{\mathrm{VV}}(\tau) \\ &+R_{\mathrm{HV}}(0)R_{\mathrm{HV}}(0)+R_{\mathrm{HV}}(0)R_{\mathrm{VH}}(0)], \\ s_3(\tau) =&-2\mathrm{Re}[R_{\mathrm{HV}}(-\tau)R_{\mathrm{HV}}(\tau)-R_{\mathrm{HH}}(-\tau)R_{\mathrm{VV}}(\tau) \\ &+R_{\mathrm{HV}}(0)R_{\mathrm{HV}}(0)-R_{\mathrm{HV}}(0)R_{\mathrm{VH}}(0)]. \end{aligned} \tag{13}$$

3.4. The cross-coherent polarization function

Inserting Eqs.(8–11) into Eq.(3), we can derive the expressions of the cross-coherent polarization function as

$$\begin{aligned} s_{01}(\tau) &= \langle S_0(t)S_1(t+\tau)\rangle = S_{\mathrm{HH}}(\tau)-S_{\mathrm{HV}}(\tau)+S_{\mathrm{VH}}(\tau)-S_{\mathrm{VV}}(\tau), \\ s_{12}(\tau) &= \langle S_1(t)S_2(t+\tau)\rangle = 2\mathrm{Re}[R_{\mathrm{HH}}(0)R_{\mathrm{HV}}(0)+R_{\mathrm{HV}}(\tau)R_{\mathrm{HH}}(-\tau)-R_{\mathrm{VV}}(0)R_{\mathrm{HV}}(0)-R_{\mathrm{VV}}(\tau)R_{\mathrm{HV}}(-\tau)], \\ s_{23}(\tau) &= \langle S_2(t)S_3(t+\tau)\rangle = 2\mathrm{Im}[R_{\mathrm{HV}}(0)R_{\mathrm{HV}}(0)+R_{\mathrm{HV}}(\tau)R_{\mathrm{HV}}(-\tau)+R_{\mathrm{VV}}(\tau)R_{\mathrm{HH}}(-\tau)], \\ s_{13}(\tau) &= \langle S_1(t)S_3(t+\tau)\rangle = 2\mathrm{Im}[R_{\mathrm{HH}}(0)R_{\mathrm{HV}}(0)+R_{\mathrm{HV}}(\tau)R_{\mathrm{HH}}(-\tau)+R_{\mathrm{VV}}(0)R_{\mathrm{VH}}(0)+R_{\mathrm{VH}}(\tau)R_{\mathrm{VV}}(-\tau)]. \end{aligned} \tag{14}$$

Other cross-coherent polarization functions can be obtained by replacing τ with $(-\tau)$.

4. The ICPS

If $g_i(\tau)$ satisfies the following assumption,

$$G_g(\omega) = \int_{-\infty}^{\infty} |g(\tau)| \mathrm{d}\tau < \infty,$$

we can define the ICPS as

$$G_g(\omega) = \int_{-\infty}^{\infty} g(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau. \tag{15}$$

By the same condition and definition, we can get the ICPS for $s_i(\tau)$, $s_{ij}(\tau)$ as

$$G_{ss}(\omega) = \int_{-\infty}^{\infty} s(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau, \tag{16}$$

$$G_{sc}(\omega) = \int_{-\infty}^{\infty} s_{ij}(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau. \tag{17}$$

The spectra definitions describe the coherence of Stokes parameters in frequency domain.

5. The instance for the gaussian correlated fields

For the convenience of the simulation, we suppose the wave vector independence between horizontal and vertical polarization channels. The coherent function of the horizontal channel is $R_{\mathrm{HH}}(\tau) = \langle E_{\mathrm{H}}(t) E_{\mathrm{H}}^{\mathrm{H}}(t+\tau) \rangle = 2\sigma_{\mathrm{HHRR}}(\tau)$, on the assumption of $\sigma_{\mathrm{HHIR}}(\tau) = 0$. The coherent function of the vertical channel is $R_{\mathrm{VV}}(\tau) = \langle E_{\mathrm{V}}(t) E_{\mathrm{V}}^{\mathrm{H}}(t+\tau) \rangle = 2\sigma_{\mathrm{VVRR}}(\tau)$, where $\sigma_{\mathrm{VVIR}}(\tau) = 0$. Then the first ICPFs can be simplified as

$$\begin{aligned} g_0(\tau) &= 2[\sigma_{\mathrm{HHRR}}(\tau) + \sigma_{\mathrm{VVRR}}(\tau)], \\ g_1(\tau) &= 2[\sigma_{\mathrm{HHRR}}(\tau) - \sigma_{\mathrm{VVRR}}(\tau)], \\ g_2(\tau) &= 0, \\ g_3(\tau) &= 0. \end{aligned}$$

The second ICPFs are simplified as following:

$$\begin{aligned} s_0(\tau) =& 8[\sigma_{\mathrm{HHRR}}^2(\tau) + \sigma_{\mathrm{VVRR}}^2(\tau) \\ &+ \sigma_{\mathrm{HHRR}}(\tau)\sigma_{\mathrm{VVRR}}(\tau)], \\ s_1(\tau) =& 8[\sigma_{\mathrm{HHRR}}^2(\tau) + \sigma_{\mathrm{VVRR}}^2(\tau) \\ &- \sigma_{\mathrm{HHRR}}(\tau)\sigma_{\mathrm{VVRR}}(\tau)], \\ s_2(\tau) =& 8[\sigma_{\mathrm{HHRR}}(-\tau)\sigma_{\mathrm{VVRR}}(\tau)], \\ s_3(\tau) =& 8[\sigma_{\mathrm{HHRR}}(-\tau)\sigma_{\mathrm{VVRR}}(\tau)]. \end{aligned}$$

Here we let $\sigma_{\mathrm{HHRR}}(\tau) = \frac{1}{4}\mathrm{e}^{-\alpha|\tau|}$, $\sigma_{\mathrm{VVRR}}(\tau) = \frac{1}{4}\mathrm{e}^{-\beta|\tau|}$. Under this condition we can obtain the ICPS of $g_i(\tau)$ as

$$\begin{aligned} G_{g_0}(\omega) &= \int_{-\infty}^{\infty} g_0(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau \\ &= \frac{\alpha}{\alpha^2 + \omega^2} + \frac{\beta}{\beta^2 + \omega^2}, \\ G_{g_1}(\omega) &= \int_{-\infty}^{\infty} g_1(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau \\ &= \frac{\alpha}{\alpha^2 + \omega^2} - \frac{\beta}{\beta^2 + \omega^2}, \\ G_{g_2}(\omega) &= \int_{-\infty}^{\infty} g_2(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau = 0, \\ G_{g_3}(\omega) &= \int_{-\infty}^{\infty} g_3(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau = 0. \end{aligned}$$

The ICPS of $s_i(\tau)$ are

$$\begin{aligned} G_{s_0}(\omega) &= \int_{-\infty}^{\infty} s_0(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau \\ &= \frac{2\alpha}{4\alpha^2 + \omega^2} + \frac{2\beta}{4\beta^2 + \omega^2} + \frac{(\alpha+\beta)}{(\alpha+\beta)^2 + \omega^2}, \\ G_{s_1}(\omega) &= \int_{-\infty}^{\infty} s_1(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau \\ &= \frac{2\alpha}{4\alpha^2 + \omega^2} + \frac{2\beta}{4\beta^2 + \omega^2} - \frac{(\alpha+\beta)}{(\alpha+\beta)^2 + \omega^2}, \\ G_{s_2}(\omega) &= \int_{-\infty}^{\infty} s_2(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau \\ &= \frac{(\alpha+\beta)}{(\alpha+\beta)^2 + \omega^2}, \\ G_{s_3}(\omega) &= \int_{-\infty}^{\infty} s_3(\tau) \mathrm{e}^{-\mathrm{j}\omega\tau} \mathrm{d}\tau \\ &= \frac{(\alpha+\beta)}{(\alpha+\beta)^2 + \omega^2}. \end{aligned}$$

The Figs.1–6 are given to show the coherence of the Stokes parameters for this case, when $\alpha = 1$, $\beta = 2$. The horizontal coordinate means the coherent time τ and the vertical coordinate means the values of the coherent function $g_i(\tau)$ or $s_i(\tau)$.

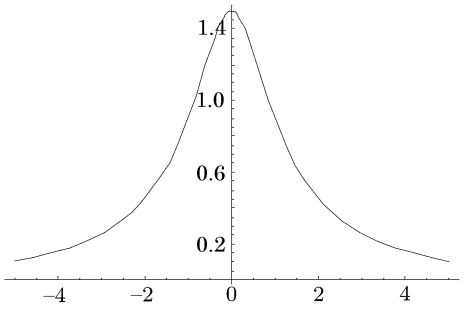


Fig.1. The first ICPS of $g_0(\tau)$.

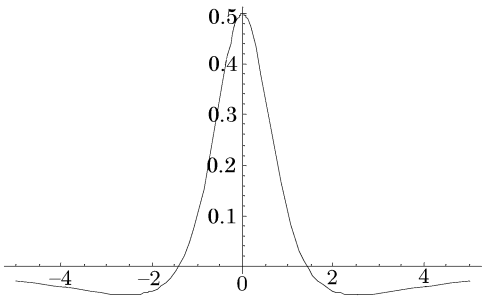


Fig.2. The first ICPS of $g_1(\tau)$.

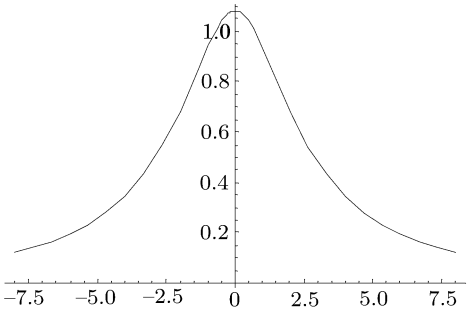


Fig.3. The second ICPS of $s_0(\tau)$.

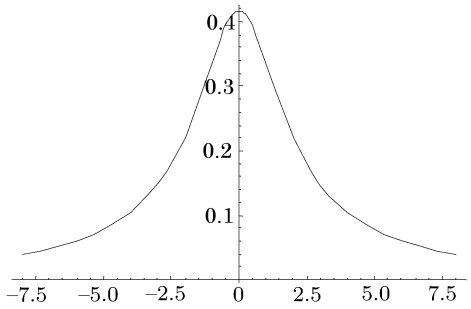


Fig.4. The second ICPS of $s_1(\tau)$.

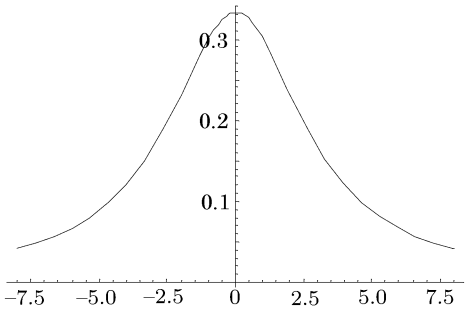


Fig.5. The second ICPS of $s_2(\tau)$.

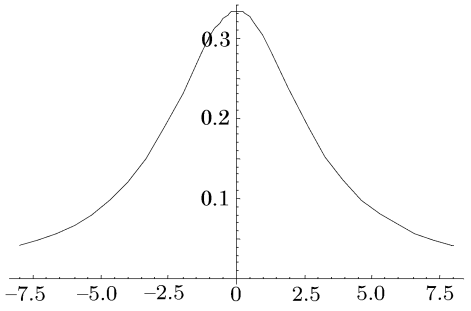


Fig.6. The second ICPS of $s_3(\tau)$.

From the above figures we can see that the band of the first ICPS is narrower than the second definition, which means that the second definition has shorter coherent time. And the first ICPS $g_1(\tau)$ has a trend of little fluctuation while its band of frequency is narrowest.

6. Conclusion

In this paper the coherence of the instantaneous Stokes parameters at different times is studied, which describes the coherence of the polarization states. The ICPF, ICPS and the cross-polarization function are defined as the variables of the polarization state shifts. In reality, the Stokes parameters may be seen as a kind of real signals just describing the polarization state in time or frequency domain. All the results derived in the paper may be extended new definition of the coherence of Stokes parameters in space propagation.[7] and would be used to target detection, recognition and other aspects.

As we know, the distributions of the electromagnetic waves are of many kinds, how to obtain their ICPF, ICPS, and the cross-polarization function will be our work in future.

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