Vol 17 No 3, March 2008
1674-1056/2008/17(03)/0857-08

Chinese Physics B

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Stochastic period-doubling bifurcation in biharmonic driven Duffing system with random parameter*

Xu Wei(徐 伟), Ma Shao-Juan(马少娟)†, and Xie Wen-Xian(谢文贤)

Department of Applied Mathematics, Northwestern Polytechnical University, Xi'an 710072, China

(Received 12 April 2007; revised manuscript received 9 September 2007)

Stochastic period-doubling bifurcation is explored in a forced Duffing system with a bounded random parameter as an additional weak harmonic perturbation added to the system. Firstly, the biharmonic driven Duffing system with a random parameter is reduced to its equivalent deterministic one, and then the responses of the stochastic system can be obtained by available effective numerical methods. Finally, numerical simulations show that the phase of the additional weak harmonic perturbation has great influence on the stochastic period-doubling bifurcation in the biharmonic driven Duffing system. It is emphasized that, different from the deterministic biharmonic driven Duffing system, the intensity of random parameter in the Duffing system can also be taken as a bifurcation parameter, which can lead to the stochastic period-doubling bifurcations.

Keywords: random parameter, stochastic Duffing system, stochastic period-doubling bifurcation, orthogonal polynomial approximation

PACC: 0547

1. Introduction

Duffing system is a deterministic mathematical model for an electro-mechanical system suggested by Duffing in nineteenth century. Since then it has become a paradigm for studying nonlinear dynamics and chaos, and thousands of papers related to Duffing systems have been presented today. In general, there are two kinds of excitations that are subjected to Duffing system: stochastic excitation and harmonic excitation. For a Duffing system subjected to harmonic excitation there are many important conclusions about period-doubling (PD for short) bifurcation, quasi-periodic motion and chaos.[1−6] Recently multiple harmonic excitations subjected to a forced Duffing system has been paid more attention. Qu *et al*[7]and Yang *et al*[8] explored the phase effect of additional weak harmonic perturbation on suppressing chaos in forced Duffing system. References [9–12] studied the chaos control and synchronization by analytical and numerical methods for a Duffing system with two harmonic excitations. Occurrence of multiple period-doubling bifurcation routes to chaos in periodically pulsed forced Duffing systems is discussed in Ref.[13]. Jing *et al*[14,15] studied the complex dynamics in a forced Duffing system and a three-well Duffing system with two harmonic excitations.

The differential equation of a forced Duffing system with an additional weak harmonic perturbation is as follows:

$$\ddot{x} + \alpha\dot{x} + \varepsilon x^3 = F\cos(\omega t) + \beta F\cos(\Omega t + \phi), \quad (1)$$

where $\alpha, \varepsilon, F, \omega, \beta, \Omega$ and ϕ are real parameters. The first periodic force $F\cos(\omega t)$ drives the system to the different dynamical states while the second one $\beta F\cos(\Omega t)$ is a weak periodic excitation sensitively modifying the system dynamics, and β is the amplitude of the perturbation, which is assumed to be very small ($\beta < 0.1$). In fact, the parameter α or ε of Duffing system (1) is a random parameter in real world due to the practical factors, such as installation, material and design.

If the parameter ε is a random parameter, the Duffing system (1) is called a stochastic Duffing system in this article. For this kind of stochastic dynamical system three numerical methods are available for analysing its dynamical behaviour: one is the Monte Carlo method,[16] which is simple and universal, but usually involves a quite amount of computation effort. The second is the stochastic finite element method,[17,18] which needs least computation, but usually is restricted to systems with random variables of

*Project supported by the National Natural Science Foundation of China (Grant Nos 10472091 and 10332030).

†Corresponding author. E-mail: mashaojuan@mail.nwpu.edu.cn, shaojuan-ma@163.com

small perturbation only. The third is the orthogonal polynomial approximation method,[19−22] which is based on the expansion theory of orthogonal polynomials without the limitation to small perturbation. References [23–28] have already discussed the PD bifurcation, chaos and chaos control in a nonlinear dynamical system with random parameters using orthogonal polynomial approximation. Next we will apply the orthogonal polynomial approximation to the study of stochastic PD bifurcation in a stochastic Duffing system subjected to two harmonic excitations.

This paper is organized as follows. In Section 2, the stochastic Duffing system is reduced to the equivalent deterministic one by orthogonal polynomial approximation. The stochastic PD bifurcation and its random feature in a stochastic Duffing system are presented in Section 3. In Section 4 we draw some conclusions about the stochastic dynamical behaviour in the stochastic Duffing system with two harmonic excitations.

2. Orthogonal polynomial approximation for stochastic system (1)

System (1) can be rewritten as follows:

$$\ddot{x} + \alpha\dot{x} + \varepsilon x^3 = F\cos(\omega t) + \beta F\cos(\Omega t + \phi), \quad (2)$$

where $\alpha, F, \omega, \beta, \Omega$ and ϕ are real parameters. We suppose that ε is a random parameter in the form

$$\varepsilon = \bar{\varepsilon} + \delta u, \quad (3)$$

where $\bar{\varepsilon}$ and $\delta/2$ are the mean value and the standard deviation of ε respectively, δ is taken as the intensity of ε and is named random intensity. Suppose that U is a random variable defined on $[-1, 1]$ with a given probability density function (PDF for short), say an arch-like PDF which can be expressed as

$$p(u) = \begin{cases} (2/\pi)\sqrt{1-u^2}, & \text{as} \quad |u| \le 1; \\ 0, & \text{as} \quad |u| > 1. \end{cases}$$

According to the theory of orthogonal polynomials,[29] the PDF $p(u)$ is a weight function of the second Chebyshev polynomial $U_i(u)$ $(i = 0, 1, \ldots, N)$. That is to say, the orthogonality of Chebyshev polynomials of the second kind can be expressed as

$$\int_{-1}^{1} \frac{2}{\pi}\sqrt{1-u^2}U_i(u)U_j(u)\,\mathrm{d}u = \begin{cases} 1, & (i=j), \\ 0, & (i \ne j). \end{cases}$$

The recurrent formula for Chebyshev polynomials of the second kind is

$$uU_i(u) = \frac{1}{2}\left(U_{i-1}(u) + U_{i+1}(u)\right). \quad (4)$$

Owing to the orthogonality of Chebyshev polynomials, any function $f(u) \subset L^2$ can be expanded into the following form:

$$f(u) = \sum_{i=0}^{\infty} c_i U_i(u),$$

where $c_i = \int_{-1}^{1} p(u) f(u) U_i(u)\,\mathrm{d}u$. So the response $x(t, u)$ of system (1) can be expressed by the following series

$$x(t, u) = \sum_{i=0}^{N} x_i(t) U_i(u), \quad (5)$$

where $U_i(u)$ represents the ith Chebyshev polynomial, N represents the largest order of the polynomials we have taken, and $x_i(t) = \int_{-1}^{1} p(u) x(t, u) U_i(u)\,\mathrm{d}u$.

Substituting Eqs.(3) and (5) into Eq.(2), we have

$$\begin{aligned} &\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\frac{\mathrm{d}}{\mathrm{d}t}\right)\sum_{i=0}^{N} x_i(t) U_i(u) \\ &+\bar{\varepsilon}\left(\sum_{i=0}^{N} x_i(t) U_i(u)\right)^3 \\ &+\delta u\left(\sum_{i=0}^{N} x_i(t) U_i(u)\right)^3 \\ &= F\cos(\omega t) + \beta F\cos(\Omega t + \phi). \end{aligned} \quad (6)$$

The nonlinear term $\left(\sum_{i=0}^{N} x_i(t) U_i(u)\right)^3$ on the left side of Eq.(6) can be expanded into

$$\begin{aligned} &\left(\sum_{i=0}^{N} x_i(t) U_i(u)\right)^3 \\ &= x_0^3(t) U_0^3(u) + \cdots + x_N^3(t) U_N^3(u) \\ &+3x_0^2(t) x_1(t) U_0^2(u) U_1(u) + \cdots \\ &+3x_{N-1}(t) x_N^2(t) U_{N-1}(u) U_N^2(u) \\ &+6x_0(t) x_1(t) x_2(t) U_0(u) U_1(u) U_2(u) + \cdots \\ &+6x_{N-2}(t) x_{N-1}(t) x_N(t) \\ &\times U_{N-2}(u) U_{N-1}(u) U_N(u). \end{aligned} \quad (7)$$

By the recurrent formula, Eq.(4), the triple product polynomial of Eq.(6) can be further reduced into

a linear combination of the related single polynomials. As for Chebyshev polynomials, $U_i(u)$, we have

$$\begin{aligned}
&U_0^3(u) = U_0(u),\\
&3U_0^2(u)U_1(u) = 3U_1(u),\\
&3U_1^2(u)U_2(u) = 3U_4(u) + 6U_2(u) + 3U_0(u),\\
&\cdots\cdots
\end{aligned} \tag{8}$$

By denoting the coefficient of $U_i(u)$ in the linear combination as $X_i(t)$, Eq.(7) can be finally expressed as

$$\begin{aligned}
&\left(\sum_{i=0}^{N} x_i(t)U_i(u)\right)^3\\
&= X_0(t)U_0(u) + \cdots + X_{3N}(t)U_{3N}(u)\\
&= \sum_{i=0}^{3N} X_i(t)U_i(u).
\end{aligned} \tag{9}$$

The expressions of $X_i(t), i = 0, 1, 2, \ldots, 3N$, can be derived through computer algebraic system, such as MAPLE.[30] By the recurrent formula of Chebyshev polynomials Eq.(4) and by Eq.(9), the third term of Eq.(6) can be reduced to

$$\begin{aligned}
&u\delta\left(\sum_{i=0}^{N} x_i(t)U_i(u)\right)^3\\
&= \delta\left(u\sum_{i=0}^{3N} X_i(t)U_i(u)\right)\\
&= \frac{1}{2}\delta\sum_{i=0}^{3N} X_i(t)\left[U_{i-1}(u) + U_{i+1}(u)\right]\\
&= \frac{1}{2}\delta\sum_{i=0}^{3N}\left[X_{i-1}(t) + X_{i+1}(t)\right]U_i(u),
\end{aligned} \tag{10}$$

where $X_{-1}(t)$ is supposed to be zero due to the approximation, Eq.(5). Substituting Eqs.(9) and (10) into Eq.(6), we have

$$\begin{aligned}
&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\frac{\mathrm{d}}{\mathrm{d}t}\right)\sum_{i=0}^{N} x_i(t)U_i(u)\\
&+\bar{\varepsilon}\left(\sum_{i=0}^{3N} X_i(t)U_i(u)\right)\\
&+\frac{1}{2}\delta\left(\sum_{i=0}^{3N}\left[X_{i-1}(t) + X_{i+1}(t)\right]U_i(u)\right)\\
&= F\cos(\omega t) + \beta F\cos(\Omega t + \phi).
\end{aligned} \tag{11}$$

Multiplying both sides of Eq.(11) by $U_i(u)$ in sequence and taking expectation, owing to the orthogonality of the Chebyshev polynomial, we finally have

$$\begin{aligned}
&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\frac{\mathrm{d}}{\mathrm{d}t}\right)x_0(t) + \bar{\varepsilon}X_0(t) + \frac{1}{2}\delta X_1(t) = F\cos(\omega t) + \beta F\cos(\Omega t + \phi),\\
&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\frac{\mathrm{d}}{\mathrm{d}t}\right)x_1(t) + \bar{\varepsilon}X_1(t) + \frac{1}{2}\delta\left[X_0(t) + X_2(t)\right] = 0,\\
&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\frac{\mathrm{d}}{\mathrm{d}t}\right)x_2(t) + \bar{\varepsilon}X_2(t) + \frac{1}{2}\delta\left[X_1(t) + X_3(t)\right] = 0,\\
&\quad\cdots\\
&\left(\frac{\mathrm{d}^2}{\mathrm{d}t^2} + a\frac{\mathrm{d}}{\mathrm{d}t}\right)x_N(t) + \bar{\varepsilon}X_N(t) + \frac{1}{2}\delta\left[X_{N-1}(t) + X_{N+1}(t)\right] = 0.
\end{aligned} \tag{12}$$

This is just the equivalent nonlinear deterministic system reduced by the Chebyshev polynomial approximation for the stochastic Duffing system with a random variable U of an arch-like PDF. When $N \to \infty$ in Eq.(5), $\sum_{i=0}^{N} x_i(t)U_i(u)$ is strictly equivalent to $x(t,u)$ of the stochastic dynamical system. Otherwise, if N is finite, Eq.(5) is just approximately valid with a minimal residual error. In the following numerical analysis, we take $N = 9$ and obtain the numerical solution $x_i(t)$, $(i = 0, 1, \ldots, 9)$ of Eq.(12) by effective numerical methods available. Then, the approximate random response of the original stochastic Duffing system can be expressed as

$$x(t,u) \approx \sum_{i=0}^{9} x_i(t)U_i(u). \tag{13}$$

If the random parameter $u = 0$, namely $\varepsilon = \bar{\varepsilon}$, we obtain the deterministic Duffing system, which is significant for the following text. We also name this system as the mean parameter system:

$$\ddot{x} + \alpha\dot{x} + \bar{\varepsilon}x^3 = F\cos(\omega t) + \beta F\cos(\Omega t + \phi). \quad (14)$$

The ensemble mean response of the stochastic Duffing system can be obtained:

$$E[x(t,u)] \approx \sum_{i=0}^{9} x_i(t) E[U_i(u)] = x_0(t). \quad (15)$$

Then, the sample response of the mean parameter system may be expressed as

$$\begin{aligned} & x(t,0) \\ \approx & \sum_{i=0}^{9} x_i(t) U_i(0) \\ = & x_0(t) - x_2(t) + x_4(t) - x_6(t) + x_8(t). \end{aligned} \quad (16)$$

3. Stochastic PD bifurcation in stochastic system (1)

In this section we will show the various types of responses corresponding to a series of specific conditions for studying stochastic PD bifurcations. We will find the ensemble mean response $E[x(t,u)]$ (EMR for short) by Eq.(15), and then further check it by the sample response $x(t,0)$ of mean parameter system (SRM for short) by Eq.(16). Let IC denote the initial condition for the deterministic equivalent system, Eq.(12),

$$\boldsymbol{x}(0) = [x_0(0), \ldots, x_9(0)]^{\mathrm{T}},$$

$$\boldsymbol{y}(0) = [\dot{x}_0(0), \ldots, \dot{x}_9(0)]^{\mathrm{T}},$$

and the ICs those for the deterministic Duffing system $x_0 = x(0)$, $y_0 = \dot{x}(0)$. In this paper, the parameter δ is taken as a small quantity, and the same IC for the deterministic equivalent system, Eq.(12), is taken as the IC for the deterministic Duffing system, namely

$$\begin{aligned} & x_0 = x_0(0), \quad y_0 = \dot{x}_0(0), \\ & x_i(0) = \dot{x}_i(0) = 0, \quad (i = 1, 2, 3, \ldots, 9). \end{aligned}$$

In this paper, the IC is chosen as follows:

$$\begin{aligned} & \boldsymbol{x}(0) = [3.0, 0, \ldots, 0, 0]^{\mathrm{T}}, \\ & \boldsymbol{y}(0) = [2.0, 0, \ldots, 0, 0]^{\mathrm{T}}, \\ & x_0 = 3.0, \quad y_0 = 2.0. \end{aligned}$$

3.1. phase effect on the dynamical behaviour in a stochastic Duffing system

The system parameters are chosen to be $\alpha = 0.3$, $\bar{\varepsilon} = 1.0$, $\omega = 1.0$ (throughout the article). Compared with other parameters, the phase of the additional weak harmonic perturbation can cause an unimaginable effect. As the parameter $\delta = 0.0$, the stochastic system is changed into a deterministic Duffing system. For the parameter $\beta = 0.0$, the bifurcation diagram and the maximal Lyapunov exponent with respect to F in the deterministic Duffing system are shown in Figs.1(a) and 1(b) respectively. We can find from the figure that chaos is the main state in a usual Duffing system, and the width of the period windows is very narrow. For the parameters $F = 8.85$, $\beta = 0.075$ and $\Omega = 3.0$, the bifurcation diagram and its maximal Lyapunov exponent with respect to ϕ are shown in Figs.2(a) and 2(b) respectively. The chaos in the biharmonic-driven Duffing system is replaced by periodic motion as the phase of the additional weak harmonic perturbation is increased. From Figs.1 and 2 it is clear that after adding the weak harmonic perturbation, $0.075F\cos(\Omega t + \phi)$, the dynamical behaviour of the Duffing system is changed drastically.

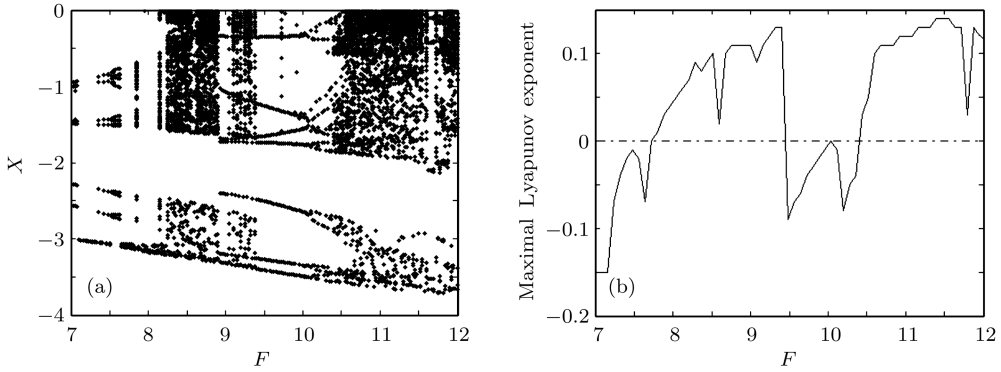


Fig.1. Diagram of bifurcation (a) and the maximal Lyapunov exponents diagram (b) against F for $\alpha = 0.3$, $\bar{\varepsilon} = 1.0$, $\omega = 1.0$, $\beta = 0.0$.

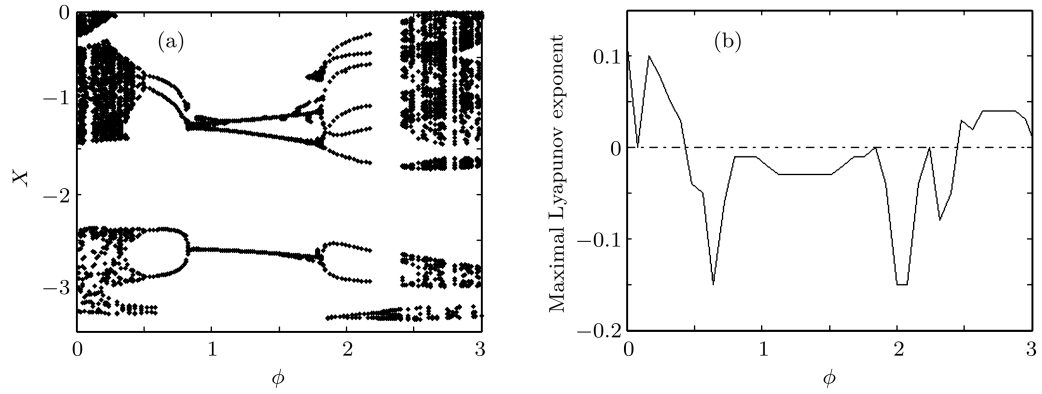


Fig.2. Bifurcation diagram (a) and the maximal Lyapunov exponents diagram (b) against ϕ for $\alpha = 0.3,\ \bar{\varepsilon} = 1.0,\ \omega = 1.0,\ F = 8.85,\ \beta = 0.075,\ \Omega = 3.0$.

The same phase effect on dynamical behaviour is also shown in the stochastic Duffing system. When the parameters are chosen to be $F = 8.85,\ \beta = 0.075,\ \Omega = 3.0,\ \delta = 0.01$, the bifurcation diagrams against ϕ of SRM and EMR are shown in Figs.3(a) and 3(b) respectively. It is clear that similar to the deterministic Duffing system (14), there are three series of PD bifurcations in the stochastic Duffing system. Next, we will discuss the stochastic PD bifurcation and its random feature in the stochastic Duffing system.

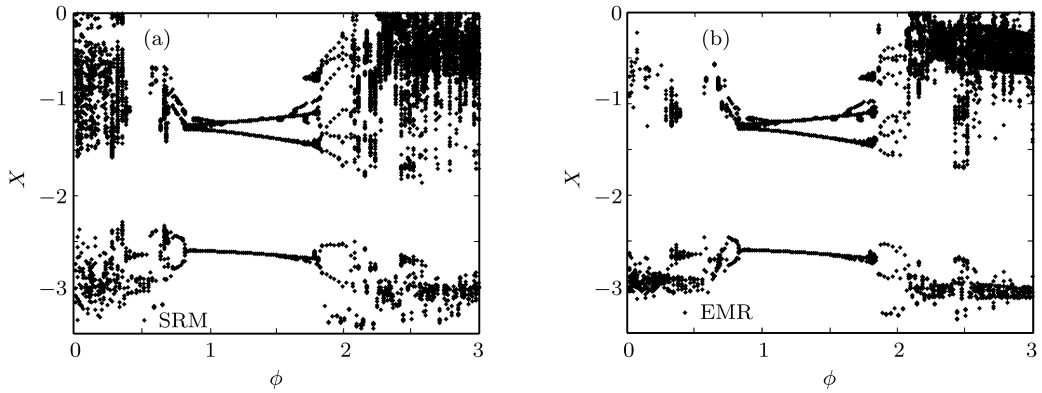


Fig.3. Diagrams of bifurcation against ϕ in (a) SRM; (b) EMR for $\alpha = 0.3,\ \bar{\varepsilon} = 1.0,\ \omega = 1.0,\ F = 8.85,\ \beta = 0.075,\ \Omega = 3.0,\ \delta = 0.01$.

3.2. Phase effect on the stochastic PD bifurcation in a stochastic Duffing system

When the system parameter ϕ is increased from 1.6 to 2.4, it is well known that there is a series of PD bifurcation period in the deterministic Duffing system (14) as shown in Fig.2, which shows the change of periodic motion from $1\,T$ to $2\,T$, $2\,T$ to $4\,T$. For a stochastic Duffing system if it has the parameters $\delta = 0.01$ and $\phi = 1.7$, the responses of SRM and EMR in stochastic Duffing system are in a stable state with period $1\,T$. Figures 4(a) and 4(b) are Poincaré section diagram and phase portrait of responses respectively. If the phase ϕ is increased to 2.1, we can find that there are two '.' and two '+' in Fig.4(c), and two stable state trajectories with period $2\,T$ in Fig.4(d). As ϕ is increased to 2.3, the signs representing responses of SRM and EMR in the stochastic Duffing system are changed from two to four in Fig.4(e). There are two stable state periodic trajectories with period $4\,T$ in Fig.4(f).

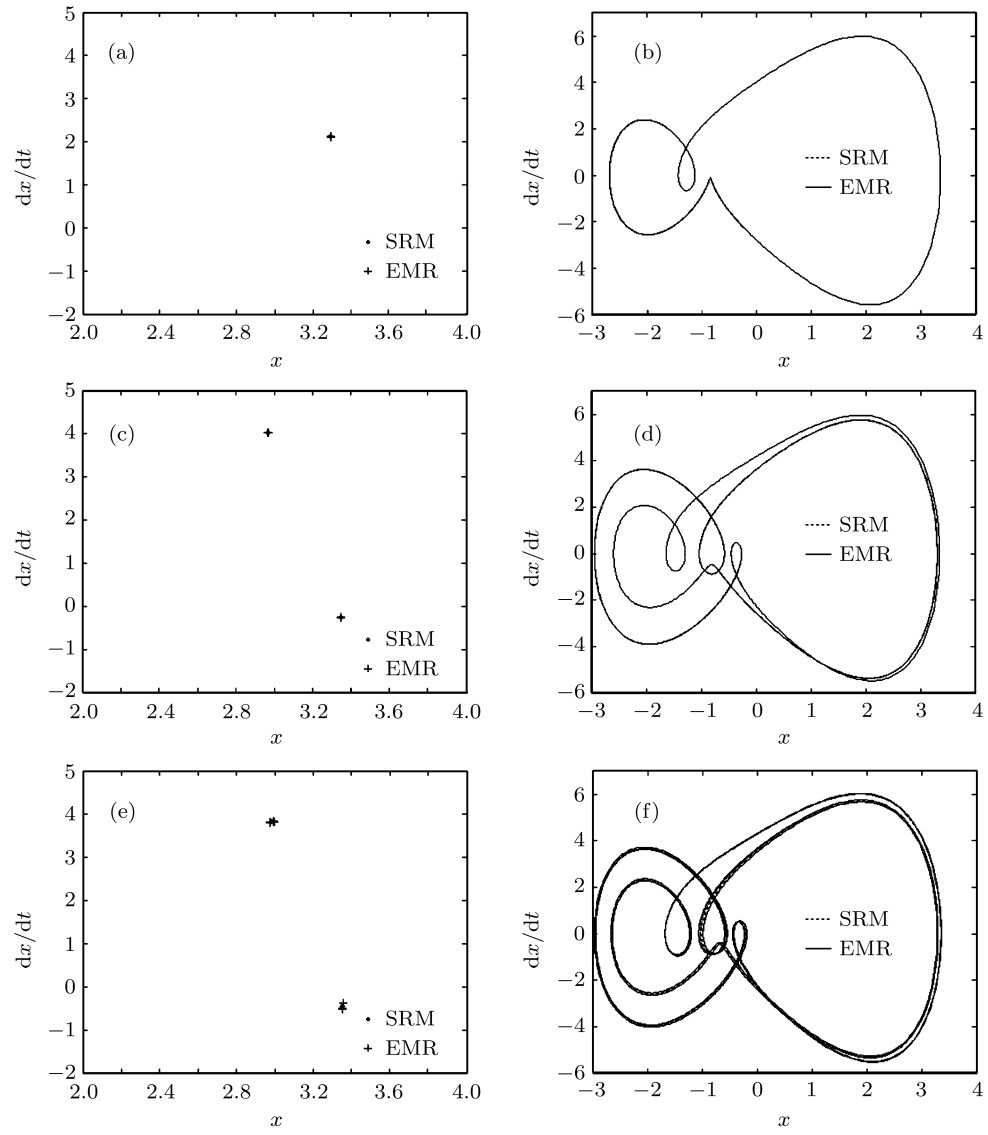


Fig.4. The Poincaré section diagrams (a), (c), (e) and phase portraits (b),(d),(f) for $\alpha = 0.3,\ \bar{\varepsilon} = 1.0,\ \omega = 1.0,\ F = 8.85,\ \beta = 0.075,\ \Omega = 3.0,\ \delta = 0.01$. (a, b)$\phi = 1.7$; (c, d)$\phi = 2.1$; (e, f)$\phi = 2.3$.

From Fig.4 we can see that similar to the deterministic Duffing system, the PD bifurcation exists in the stochastic Duffing system only for lower periodic state, and the higher periodic state is changed into irregular motion due to the effect of the random parameter.

3.3. Features of stochastic PD bifurcation in a stochastic Duffing system

After adding a weak harmonic perturbation to a deterministic forced Duffing system, the phase effect on the dynamical behaviour is prominent. Now the weak harmonic perturbation is added to the forced Duffing system with a random parameter, the phase effect on the stochastic PD bifurcation is more complicated. We will study the effect of random intensity on the stochastic PD bifurcation in the stochastic Duffing system subjected to two harmonic excitations.

Let $\phi = 1.75$ in this subsection. When the random intensity δ is equal to 0.01, there are only one '.' and one '+' in the Poincaré section, Fig.5(a); and there are two stable-state periodic trajectories with period $1\,T$ as shown in Fig.5(b). As the random intensity $\delta = 0.1$, the signs that represent responses of SRM and EMR in the stochastic Duffing system change from two to four in Fig.5(c). There are two stable-state periodic trajectories with period $2\,T$, as shown in Fig.5(d).

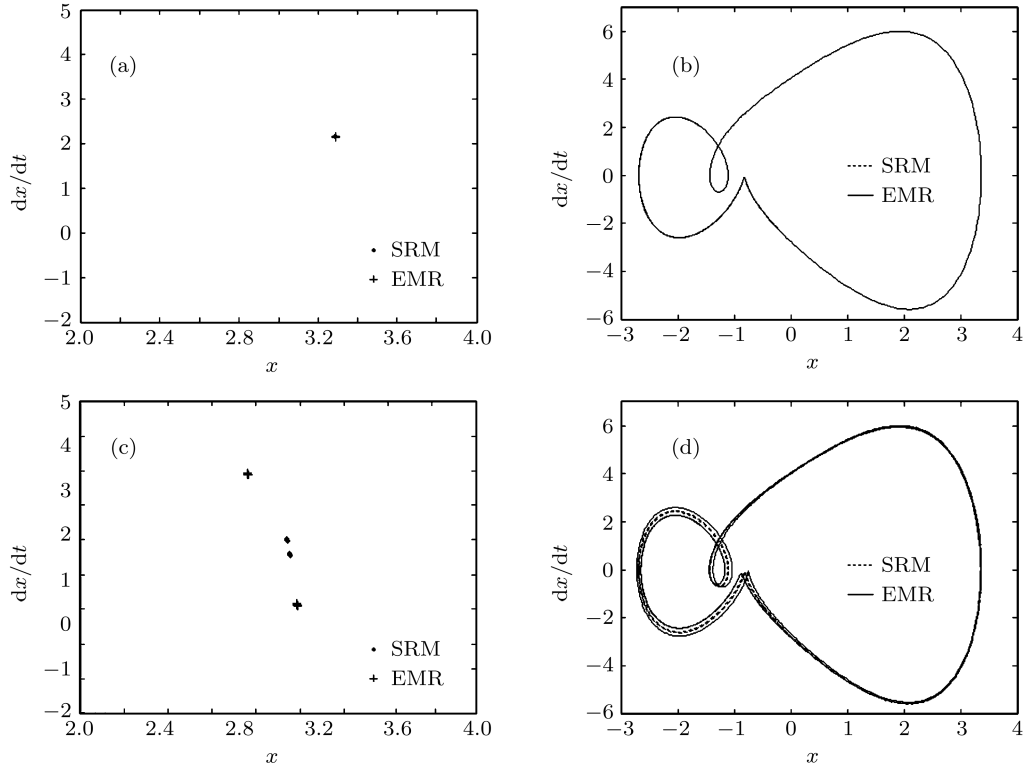


Fig.5. The Poincaré section diagrams (a), (c) and phase portraits (b), (d) for $\alpha = 0.3$, $\bar{\varepsilon} = 1.0$, $\omega = 1.0$, $F = 8.85$, $\beta = 0.075$, $\Omega = 3.0$, $\phi = 1.75$. (a, b) $\delta = 0.01$; (c, d) $\delta = 0.1$.

From Fig.5 we can draw a conclusion that the random intensity can be taken to be a PD bifurcation parameter, which can change the periodic state of the stochastic response from period $1\,T$ to period $2\,T$.

4. Conclusions

In this paper we have applied an orthogonal polynomial approximation to discuss the stochastic PD bifurcation in biharmonic driven Duffing system with random parameter. The stochastic Duffing system is firstly reduced into its equivalent deterministic one, and then the responses of stochastic system can be obtained by available effective numerical methods. Finally numerical simulations show that the phase of the additional weak harmonic perturbation plays an important role in leading to PD bifurcation. In this stochastic system we also find that the random intensity can also produce PD bifurcation and is taken to be the bifurcation parameter. So we can apply the proposed method to explore the phase effect on other dynamical behaviours of the stochastic Duffing system subjected to two harmonic excitations, such as chaos, chaos control and synchronization.

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