Vol 17 No 3, March 2008
1674-1056/2008/17(03)/0836-06

Chinese Physics B

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Adaptive coupled synchronization of non-autonomous systems in ring networks*

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(Received 19 December 2006; revised manuscript received 10 September 2007)

The adaptive coupled synchronization method for non-autonomous systems is proposed. This method can avoid estimating the value of coupling coefficient. Under the uniform Lipschitz assumption, we derive the asymptotical synchronization for a general coupling ring network with N identical non-autonomous systems, even when N is large enough. Strict theoretical proofs are given. Numerical simulations illustrate the effectiveness of the present method.

Keywords: non-autonomous system, adaptive synchronization, chaotic synchronization, ring networks

PACC: 0545

1. Introduction

Synchronization is a fundamental phenomenon that enables the coherent behaviour to occur in a coupled system. Since Pecora and Carroll[1] first introduced two identical chaotic autonomous systems under different initial conditions in 1990, chaos synchronization has been widely investigated due to its potential application in many areas such as secure communication,[2] information processing, biological systems, and chemical reaction.[3,4] Up to now, a wide variety of approaches have been used to design a synchronization controller such as impulsive control method, adaptive control,[5−9] and active control,[10−12] etc.

In fact, non-autonomous systems for modelling the behaviours of many engineering systems, such as offshore platforms, building earthquakes and so on have been widely explored.[13−15] Many complex phenomena in this kind of nonlinear dynamic systems have been demonstrated. Different chaotic motions and their attractive zones in parameter space have been studied by numerical calculation and experiments, which have been described in a great number of papers on non-autonomous systems. However, to our knowledge, compared with chaotic autonomous systems, synchronization of chaotic non-autonomous systems has seldom been considered so far.[16−21]

Recently, the adaptive-feedback control has been applied to synchronize two identical chaotic non-autonomous systems.[22] It has been shown[23−25] that synchronization of the nearest-neighbour coupling networks in a ring is hard to achieve if the number N of nodes is too large. Lu *et al*[25] tried to present the coupled adaptive control and derive a sufficient condition to synchronize a ring of N special multi-Lorenz systems.[25] In this paper, we theoretically derive an adaptive scheme to synchronize a general unidirectional ring network of any identical N non-autonomous systems globally under a certain assumption. Strict theoretical proofs are given. Numerical simulations in this paper show the effectiveness of the proposed method, even when the number N of nodes is very large.

The rest of the paper is organized as follows. In Section 2, the chaotic non-autonomous system used in this work is presented, and adaptive synchronization between two coupled systems is considered. This method is extended to a general unidirectional ring network of N non-autonomous systems and strict theoretical proofs are given to adaptive synchronization in Section 3. In Section 4, two representative examples are considered and numerical simulations are done to validate the proposed approach. Finally, conclusions are drawn in Section 5.

*Project supported by the National Natural Science Foundation of China (Grant No 10372054) and the Science Foundation of Jiangnan University, China (Grant No 000408).
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2. Adaptive synchronization between two coupled non-autonomous systems

Consider a class of chaotic non-autonomous system described by

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t), \tag{1}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}} \in \boldsymbol{R}^n$ is state vector of the system and $\boldsymbol{f} = (f_1, f_2, \ldots, f_n)^{\mathrm{T}} \in \boldsymbol{R}^n$ is a nonlinear function of the system (1). For the vector function $\boldsymbol{f}(\boldsymbol{x}, t)$, we make the following assumption.[25]

Assumption 1 For any $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \boldsymbol{R}^n$, there exists a positive constant l such that

$$\begin{aligned} &|f_i(x,t) - f_i(y,t)| \\ &\le l \max\{|x_j - y_j|, \quad j = 1, 2, \ldots, n\}. \end{aligned} \tag{2}$$

Here, it should be noted that a wide variety of chaotic systems satisfy the above condition. In fact, if $\boldsymbol{f}(\boldsymbol{x}, t)$ degenerates into $\boldsymbol{f}(\boldsymbol{x})$, i.e. $\boldsymbol{f}(\boldsymbol{x}, t)$ is independent of time t, then the above condition is called a uniform Lipschitz condition and l refers to the uniform Lipschitz constant. Hence system (1) degenerates into an autonomous one that satisfies this condition and includes almost all well-known autonomous chaotic systems, such as the Rossler system, Chua's system, Lorenz system, Chen system, etc. In this paper, we consider the non-autonomous chaotic case, in which the vector function $\boldsymbol{f}(\boldsymbol{x}, t)$ explicitly depends on time t. Also, almost all well-known non-autonomous chaotic systems satisfy this condition, for example, the excited chaotic pendulum system, Duffing equation, etc.

We consider two coupled non-autonomous systems

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{d}(\boldsymbol{x} - \boldsymbol{y}), \tag{3}$$

$$\dot{\boldsymbol{y}} = \boldsymbol{f}(\boldsymbol{y}, t) - \boldsymbol{d}(\boldsymbol{y} - \boldsymbol{x}). \tag{4}$$

Suppose

$$\dot{\boldsymbol{d}} = \boldsymbol{k}\boldsymbol{e}^2, \tag{5}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^{\mathrm{T}}$, $\boldsymbol{y} = (y_1, y_2, \ldots, y_n)^{\mathrm{T}}$, $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in \boldsymbol{R}^n$, $(k_i \ge 0,\ i = 1, 2, \ldots, n)$, $\boldsymbol{d} = (d_1, d_2, \ldots, d_n)^{\mathrm{T}}$, $\boldsymbol{e} = (e_1, e_2, \ldots, e_n)^{\mathrm{T}}$. Then we have an adaptive coupled system with two non-autonomous systems, which contains systems (3), (4) and also system (5).

Denote

$$\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t) \tag{6}$$

as the synchronizing error. Thus the dynamics for the error system is written as

$$\dot{\boldsymbol{e}} = \boldsymbol{f}(\boldsymbol{x}, t) - \boldsymbol{f}(\boldsymbol{y}, t) - 2\boldsymbol{d}\boldsymbol{e}. \tag{7}$$

If $e_i(t) \to 0$ as $t \to \infty$ for all $i = 1, 2, \ldots, n$, the coupled synchronization of systems (3) and (4) is achieved. The problem of synchronization here can be transformed into a problem of how to realize the asymptotical stabilization of the error system (6). So our aim is to design a coupled controller u for systems (3) and (4),

$$u_i = d_i(x_i - y_i), \quad i = 1, 2, \ldots, n, \tag{8}$$

where $d_i < 0$ is a variable feedback strength. Instead of the usual linear feedback, the feedback strength d_i will be adapted according to the following update law:

$$\dot{d}_i = k_i e_i^2, \quad i = 1, 2, \ldots, n, \tag{9}$$

where $k_i \ge 0$ is an arbitrary positive constant.

Then the error dynamical system (7) can be rewritten as

$$\dot{e}_i = f_i(x,t) - f_i(y,t) - 2d_i e_i, \quad i = 1, 2, \ldots, n. \tag{10}$$

Theorem 1 If the function $\boldsymbol{f}(\boldsymbol{x}, t)$ of non-autonomous systems (3) and (4) satisfies Assumption 1, systems (3) and (4) will globally asymptotically synchronize for any initial values of $\boldsymbol{x}(0)$, $\boldsymbol{y}(0)$, $\boldsymbol{d}(0)$.

Proof Construct a Lyapunov function in the following form:

$$V = \frac{1}{2}\sum_{i=1}^{n} e_i^2 + \sum_{i=1}^{n} \frac{1}{k_i}(d_i - L)^2, \tag{11}$$

where L is a sufficiently large constant, i.e. $L > \frac{1}{2}nl$. Differentiating V with respect to time t, we obtain

$$\begin{aligned} \dot{V} &= \sum_{i=1}^{n} e_i \dot{e}_i + 2\sum_{i=1}^{n} \frac{1}{k_i}(d_i - L)\dot{d}_i \\ &= \sum_{i=1}^{n} e_i(f_i(x,t) - f_i(y,t) - 2d_i e_i) \\ &\quad + 2\sum_{i=1}^{n} (d_i - L)e_i^2 \\ &= \sum_{i=1}^{n} e_i(f_i(x,t) - f_i(y,t)) \\ &\quad - 2\sum_{i=1}^{n} L e_i^2 \le (nl - 2L)\sum_{i=1}^{n} e_i^2 \le 0. \end{aligned} \tag{12}$$

It is obvious that $\dot{V} = 0$ if and only if $e_i = 0$, $i = 1, 2, \ldots, n$. Note that the construction of the Lyapunov function and the above inequality implies the boundedness of all the errors e_i, $(i = 1, 2, \ldots, n)$. According to the well-known Lyapunov–Lasalle type theorem for functional differential equations, the trajectories of the error dynamical system, starting with an arbitrary initial value, converges asymptotically to the largest invariant set $E = \{(e, d) \in R^{2n} : e = 0, d = d^0\}$ contained in $\dot{V} = 0$ as $t \to \infty$, thus implying that the two systems (3) and (4) will globally asymptotically synchronize. The proof is completed.

3. Adaptive synchronization of N coupled non-autonomous systems

A general system with $N \geq 3$ coupled with non-autonomous systems in a ring can be described by

$$\begin{aligned}
\dot{x}_1 &= f(x_1, t) - d(x_1 - x_2),\\
\dot{x}_2 &= f(x_2, t) - d(x_2 - x_3),\\
&\vdots\\
\dot{x}_{N-1} &= f(x_{N-1}, t) - d(x_{N-1} - x_N),\\
\dot{x}_N &= f(x_N, t) - d(x_N - x_1),
\end{aligned} \tag{13}$$

where $x_j = (x_{j1}, x_{j2}, \ldots, x_{jn})^{\mathrm{T}} \in \boldsymbol{R}^n$, $j = 1, 2, \ldots, N$.

We design

$$\dot{d} = k \left[\sum_{j=1}^{N} e_j^2 - \sum_{j=1}^{N-1} e_j e_{j+1} - e_N e_1 \right] \in R^n, \tag{14}$$

i.e.

$$\begin{aligned}
\dot{d}_i &= k_i \left[\sum_{j=1}^{N} e_{ji}^2 - \sum_{j=1}^{N-1} e_{ji} e_{j+1i} - e_{Ni} e_{1i} \right],\\
k_i &\geq 0, \quad i = 1, 2, \ldots, n,
\end{aligned} \tag{15}$$

where

$$\begin{aligned}
&e_j = x_j - x_{j+1}, j = 1, 2, \ldots, N-1;\\
&e_N = x_N - x_1.
\end{aligned} \tag{16}$$

It is obvious that

$$e_N = -e_1 - e_2 - \cdots - e_{N-1}. \tag{17}$$

The dynamics for the error system is

$$\begin{aligned}
\dot{e}_1 &= f(x_1, t) - f(x_2, t) - d(e_1 - e_2),\\
\dot{e}_2 &= f(x_2, t) - f(x_3, t) - d(e_2 - e_3),\\
&\vdots\\
\dot{e}_{N-1} &= f(x_{N-1}, t) - f(x_N, t) - d(e_{N-1} - e_N),\\
\dot{e}_N &= f(x_N, t) - f(x_1, t) - d(e_N - e_1).
\end{aligned} \tag{18}$$

We construct the Lyapunov function in the following form:

$$V = \frac{1}{2} \left[\sum_{i=1}^{n} \sum_{j=1}^{N} e_{ji}^2 + \sum_{i=1}^{n} \frac{1}{k_i} (d_i - L)^2 \right]. \tag{19}$$

Denote e_{ji} the ith state variable of the jth dynamical system, here L should be a sufficiently large constant. Differentiating V with respect to time t, we obtain

$$\begin{aligned}
\dot{V} &= \sum_{i=1}^{n} \left[e_{1i}\dot{e}_{1i} + e_{2i}\dot{e}_{2i} + \cdots + e_{Ni}\dot{e}_{Ni} + \frac{1}{k_i}(d_i - L)\dot{d}_i \right]\\
&= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{N-1} e_{ji}[f_i(x_j, t) - f_i(x_{j+1}, t) - d_i(e_{ji} - e_{(j+1)i})] + e_{Ni}[f_i(x_N, t) - f_i(x_1, t) - d_i(e_{Ni} - e_{1i})] \right.\\
&\quad \left. + (d_i - L) \left[\sum_{j=1}^{N} e_{ji}^2 - \sum_{j=1}^{N-1} e_{ji} e_{j+1i} - e_{Ni} e_{1i} \right] \right\}\\
&= \sum_{i=1}^{n} \left\{ \sum_{j=1}^{N-1} e_{ji}[f_i(x_j, t) - f_i(x_{j+1}, t)] + e_{Ni}[f_i(x_N, t) - f_i(x_1, t)] - L \left[\sum_{j=1}^{N} e_{ji}^2 - \sum_{j=1}^{N-1} e_{ji} e_{j+1i} - e_{Ni} e_{1i} \right] \right\}\\
&\leq \sum_{i=1}^{n} \left\{ nl \sum_{j=1}^{N} e_{ji}^2 - L \left[\sum_{j=1}^{N} e_{ji}^2 - \sum_{j=1}^{N-1} e_{ji} e_{j+1i} - e_{Ni} e_{1i} \right] \right\}
\end{aligned}$$

$$\triangleq -\sum_{i=1}^{n} \tilde{e}_{*i}^{\mathrm{T}} P \tilde{e}_{*i}, \tag{20}$$

where

$$\tilde{e}_{*i} = (e_{1i}, e_{2i}, \ldots, e_{Ni})^{\mathrm{T}}, \tag{21}$$

$$P = \begin{pmatrix} L-nl & -\frac{L}{2} & 0 & \cdots & 0 \\ -\frac{L}{2} & L-nl & -\frac{L}{2} & \cdots & 0 \\ 0 & -\frac{L}{2} & L-nl & \ddots & \vdots \\ & \ddots & \ddots & & 0 \\ \vdots & & -\frac{L}{2} & L-nl & -\frac{L}{2} \\ 0 & \cdots & 0 & -\frac{L}{2} & L-nl \end{pmatrix}_{N\times N}. \tag{22}$$

For any fixed $n, l, N(N \geq 3)$, there always exists a constant $L = 2N^2nl$ such that the matrix $\boldsymbol{P}$ is positive definite. Similar to the proof of Theorem 1, $\dot{V} \leq 0$, then a ring networks coupled with N non-autonomous systems can synchronize by adding the following adaptive feedback gain equation (13). Thus we have the following theorem.

Theorem 2 If the function $\boldsymbol{f}(\boldsymbol{x}, t)$ of non-autonomous systems (13) satisfies Assumption 1, then the system (13) will globally asymptotically synchronize for any initial values of $x_j(0)$, $j = 1, 2, \ldots, N$, $d(0)$ under the adaptive control (14).

Remark 1 Theorem 2 includes Theorem 1 as a special case when $N = 2$. At the same time, it is worth noting that the global asymptotic synchronization in the ring network will occur under the Assumption 1 theoretically, even when the number N of nodes is very large according to Theorem 2.

Remark 2 The convergence of the error system (18) can be adjusted by modifying the parameters $k_i > 0$, $i = 1, 2, \ldots, n$ in Eq.(15). It is sufficient but not necessary for some particular models to use all the variables of the system as feedback signals. For example, if $|e_i| \leq |e_j|$ $(i, j = 1, 2, \ldots, n)$, in view of the procedure of the above proof, we can set $k_i = 0$, which implies that it does not need to add the feedback control to the respective variable x_i.

Remark 3 Compared with the method proposed in Ref.[20], the above scheme is more symmetrical and general. It is also easy to design programme and to gain high-quality, which will be illustrated through several examples in the next section.

4. Numerical simulation

In this section, global asymptotic synchronization in a class of chaotic coupled non-autonomous systems presented in this paper is demonstrated by two examples of different dimensions.

Example 1

Consider a ring network model with N dynamical nodes, in which each node is a two-dimensional Duffing system described by

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -0.3y + x - x^3 + A\cos(\omega t), \end{aligned} \tag{23}$$

where $A = 0.4$, $\omega = 1.2$. This system appears chaotic behaviours in Fig.1 and satisfies Assumption 1 with $l = 16$. We set the ring network model with $N = 2$

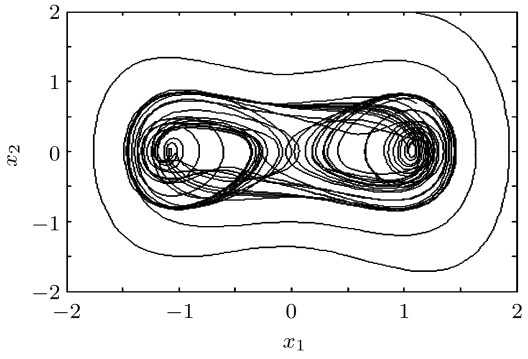


Fig.1. The chaotic behaviour of state variables $(x_1(t),\ x_2(t))^{\mathrm{T}}$ of a Duffing system.

nodes, in which each node is a Duffing system to satisfy the condition in Theorem 1 and choose the parameters $k_1 = k_2 = 0.01$. We can conclude that the two coupled non-autonomous systems are globally asymptotically synchronized for any initial values. The numerical simulation results are shown in Fig.2. In Fig.3. the two adaptive update laws $d_1 \to 1.007$ and $d_2 \to 2.2023$ vary very quickly, as $t \to \infty$, with $d_1(0) = 1$ and $d_2(0) = 2$.

If we set the number $N = 50$ nodes, $k_1 = 0.01$, $k_2 = 0$, it was found that k_2 vanishes, indicating that we do not need to add the feedback control to the respective variable y. The systems with 50 nodes will globally asymptotically synchronize for any initial values of $x_j(0)$, $(j = 1, 2, \ldots, 50)$ and $d(0)$. The simulation results are illustrated in Fig.3.

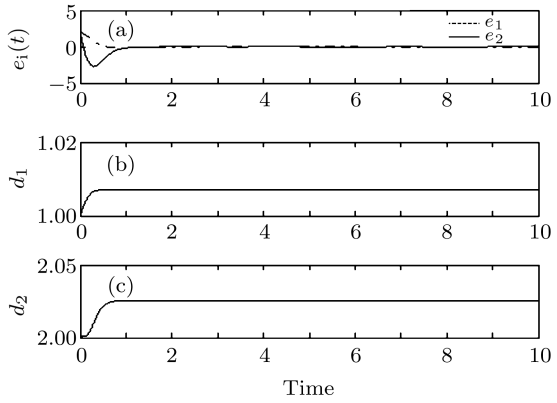


Fig.2. (a) The synchronization of error variables of the two coupled duffing system: $e_1 = x_1 - y_1$; $e_2 = x_2 - y_2$. (b) and (c) the variations of update laws $d_1 - t$ and $d_2 - t$ respectively, for $N = 2$, $k_1 = k_2 = 0.01$, $d_1(0) = 1$, $d_2(0) = 2$.

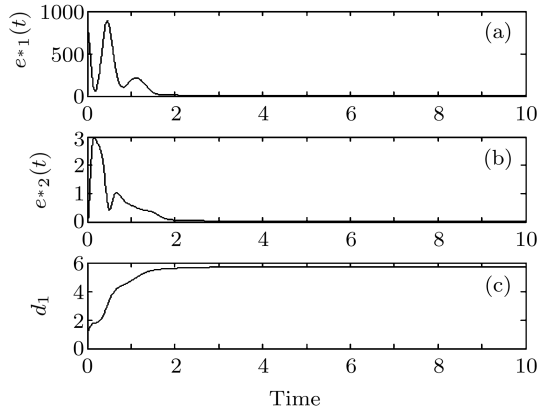


Fig.3. (a) The 1st global error states $e_{*1} = \sqrt{\sum_{j=1}^{50} e_{j1}^2}$; (b) the 2nd global error states $e_{*2} = \sqrt{\sum_{j=1}^{50} e_{j2}^2}$; (c) the update law $d_1 - t$; for $N = 50$, $k_1 = 0.01$, $k_2 = 0$, $d_1(0) = 1$.

Example 2

Consider the ring network model, in which each node is the 4D system,[15] of 4 dimensions, described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) + x_2 x_3 x_4, \\ \dot{x}_2 &= b(x_1 + x_2) - x_1 x_3 x_4, \\ \dot{x}_3 &= -[c_1 + c_2 \sin(t)] x_3 + x_1 x_2 x_4, \\ \dot{x}_4 &= -d x_4 + x_1 x_2 x_3, \end{aligned} \tag{24}$$

where x_i $(i = 1, 2, 3, 4)$ are the state variables of the system, and a, b, c_1, c_2 and d are all positive real constant parameters. The time history of the 4D system for fixed parameters $a = 30$, $b = 10$, $c_1 = 95$, $c_2 = 10$ and $d = 10$ is given in Fig.4. The chaotic behaviours of the system are clearly seen; at the same time, the system satisfies Assumption 1 with $l = 800$.

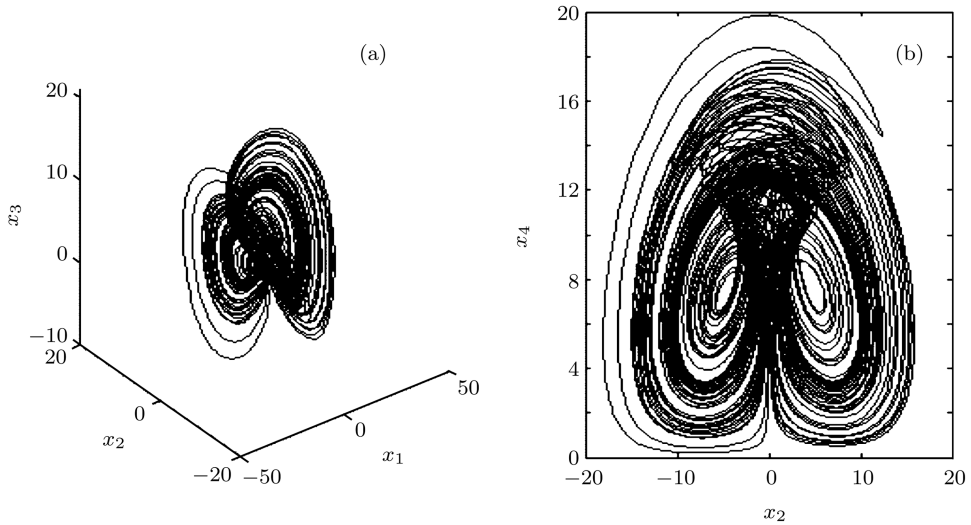


Fig.4. The phase portrait of the chaotic 4D system: (a) $x_1 - x_2 - x_3$; (b) $x_2 - x_4$.

Let $N = 50$. To globally asymptotically synchronize the nodes systems more quickly, we set $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $k_4 = 0$ and choose any initial values of $x_j(0)$, $j = 1, 2, \ldots, 50$, $d(0)$. The corresponding simulation results are illustrated in Fig.5. Figure 6 show the three adaptive update laws $d_1 \to 40.1515$, $d_2 \to 37.4635$, $d_2 \to 24.2344$ vary quite quickly, as $t \to \infty$, $d_1(0) = 1$, $d_2(0) = 2$, $d_3(0) = 3$.

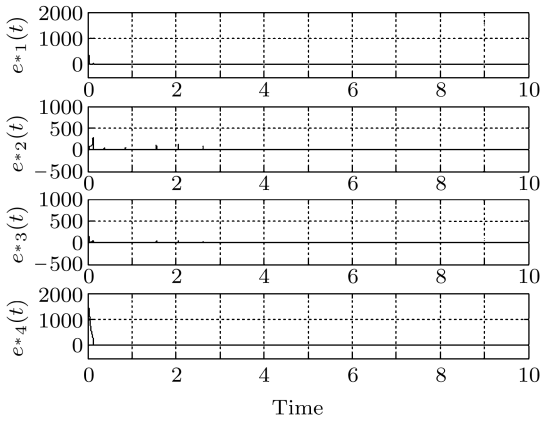


Fig.5. The 4 global error states $e_{*i} = \sqrt{\sum_{j=1}^{50} e_{ji}^2}$ $(i = 1, 2, 3, 4)$ for $N = 50$, $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $k_4 = 0$.

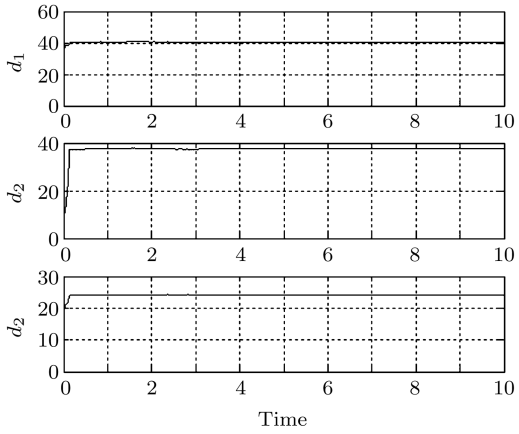


Fig.6. The update laws $(d_1 - t, d_2 - t, d_3 - t)$ for $N = 50$, $k_1 = 1$, $k_2 = 1$, $k_3 = 1$, $k_4 = 0$.

5. Conclusions

The synchronization of coupled chaotic systems is very important for application. The method of adaptive synchronization of coupled chaotic non-autonomous systems has been presented in this paper, which can avoid the difficulty of estimating the value of a coupling coefficient for the coupled systems. We have derived asymptotical synchronization of a general coupling ring network with N identical non-autonomous systems under Assumption 1. Strict theoretical proofs have been given. Numerical simulations in this paper show the effectiveness of the proposed method, even when N is quite large. In fact, the result is that the networks will be synchronized by adding only part of state variables of adaptive feedback gain equation. The method can also be applied to non-autonomous or autonomous systems in other types of networks.

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