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Noether's and Poisson's methods for solving differential equation $x_s^{(m)} = F_s\left(t, x_k^{(m-2)}, x_k^{(m-1)}\right)$*

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This paper studies integration of a higher-order differential equation which can be reduced to a second-order ordinary differential equation. The solution of the second-order equation can be obtained by the Noether method and the Poisson method. Then the solution of the higher-order equation can be obtained by integrating the solution of the second-order equation.

Keywords: Noether's method, Poisson's method, higher order ordinary differential equation, integration

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1. Introduction

The change of ordinary differential equation into mechanical system equation which can be solved by the special integration method of analytical mechanics is called mechanization of the differential equation, and the solving method is named the analytical mechanics method of the ordinary differential equation. The ordinary differential equation which can be changed into Lagrange's or Hamilton's equation can be solved by the classical Hamilton–Jacobi method,[1−3] with strict restriction,[4] and the modern symmetry method, such as Noether symmetry,[5−7] Lie symmetry[6] and form invariance[8] and so on, may be useful for the ordinary differential equation which can be partially changed into Lagrange's or Hamilton's equation. In recent years, research on the analytical mechanics solution of ordinary differential equations has been developed.[9−15] In this paper, Noether's and Poisson's methods of solving a kind of higher-order ordinary differential equation are presented. The main idea of these methods is the following: reducing higher-order equation to a second-order equation; and then giving the holonomic mechanics representation of the second-order equation; getting the integration of it by the Noether theory and the Poisson theory in the end.

2. Higher-order equation reduced to second-order equation

We study the integration of the following higher-order ordinary differential equation:

$$x_s^{(m)} = F_s\left(t, x_k^{(m-2)}, x_k^{(m-1)}\right) \times \begin{pmatrix} s,k = 1,2,\ldots,n \\ m = 2,3,\ldots \end{pmatrix}. \tag{1}$$

Let

$$x_s^{(m-2)} = q_s, \tag{2}$$

then Eq.(1) can be expressed as a second-order equation:

$$\ddot{q}_s = F_s\left(t, q_k, \dot{q}_k\right) \quad (s,k = 1,2,\ldots,n). \tag{3}$$

Regarding it as holonomic mechanics system equation, we can obtain its Lagrangian

$$L = \frac{1}{2}\dot{q}_s\dot{q}_s \tag{4}$$

and the nonpotential generalized force

$$Q_s = F_s, \tag{5}$$

then we can rewrite differential equation (3) in the following form:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s \quad (s = 1,2,\ldots,n). \tag{6}$$

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If $q_s = q_s(t)$ can be obtained from it, then $x_s = x_s(t)$ can be solved by integrating Eq.(2) after substituting $q_s = q_s(t)$ into it.

3. Noether symmetry method

For system (6), the Noether identity is

$$L\dot{\xi}_0 + X^{(1)}(L) + Q_s(\xi_s - \dot{q}_s\xi_0) + \dot{G}_{\mathrm{N}} = 0, \tag{7}$$

where

$$X^{(1)} = \xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)\frac{\partial}{\partial \dot{q}_s} \tag{8}$$

then there exists the following Noether conserved identity:

$$I_{\mathrm{N}} = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G_{\mathrm{N}} = \text{ const.} \tag{9}$$

4. Poisson method

Let

$$\begin{aligned} &p_s = \dot{q}_s, \quad H = \frac{1}{2}p_s p_s, \quad \tilde{Q}_s = F_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ &a^s = q_s \ \ (s = 1, 2, \ldots, n), \quad a^{n+s} = p_s, \end{aligned} \tag{10}$$

then Eq.(3) can be expressed as the following contravariant algebraic form

$$a^\mu = S^{\mu v}\frac{\partial H}{\partial a^v}, \tag{11}$$

where

$$(S^{\mu v}) = \begin{pmatrix} O_{n\times n} & I_{n\times n} \\ -I_{n\times n} & O_{n\times n} \end{pmatrix} + \begin{pmatrix} O_{n\times n} & 0 \\ 0 & -\Omega_{n\times n} \end{pmatrix},$$

$$\tilde{Q}_s = -\Omega_{sk}\frac{\partial H}{\partial p_k}.$$

There exists the following generalized Poisson condition for Eq.(11):

$$\frac{\partial I}{\partial t} + \frac{\partial I}{\partial a^\mu}S^{\mu v}\frac{\partial H}{\partial a^v} = 0, \tag{12}$$

where $I = I(t, \boldsymbol{a})$ which is the integration of Eq.(11), and the integration can be obtained by solving Eq.(12).

5. Example

To illustrate the application of the above methods, an example is given below.

Find the integration of the following third-order ordinary differential equation by using Noether theory and Poisson theory:

$$\dddot{x} = \left(1 + \ddot{x}^2\right)^{\frac{1}{2}}. \tag{13}$$

Firstly, by the Noether theory, let

$$\dot{x} = q, \tag{14}$$

then Eq.(13) can express a mechanical system as follows:

$$L = \frac{1}{2}\dot{q}^2, \quad Q = \left(1 + \dot{q}^2\right)^{\frac{1}{2}}, \tag{15}$$

the Noether identity from Eq.(7) is

$$\frac{1}{2}\dot{q}^2\dot{\xi}_0 + \dot{q}(\dot{\xi} - \dot{q}\dot{\xi}_0) + \left(1 + \dot{q}^2\right)^{\frac{1}{2}}(\xi - \dot{q}\xi_0) + \dot{G}_{\mathrm{N}} = 0, \tag{16}$$

whose solutions are:

$$\begin{aligned} &\xi_{0_1} = 0, \quad \xi_1 = \dot{q}\left(1 + \dot{q}^2\right)^{-\frac{1}{2}}, \\ &G_{\mathrm{N}_1} = \left(1 + \dot{q}^2\right)^{-\frac{1}{2}} - q, \\ &\xi_{0_2} = 0, \quad \xi_2 = \left(1 + \dot{q}^2\right)^{-\frac{1}{2}}, \end{aligned} \tag{17}$$

$$\begin{aligned} G_{\mathrm{N}_2} = &\ln\left[\dot{q} + \left(1 + \dot{q}^2\right)^{\frac{1}{2}}\right] \\ &-\dot{q}\left(1 + \dot{q}^2\right)^{-\frac{1}{2}} - t, \end{aligned} \tag{18}$$

their corresponding conserved quantities are respectively

$$I_{\mathrm{N}_1} = \left(1 + \dot{q}^2\right)^{\frac{1}{2}} - q = c_1, \tag{19}$$

$$I_{\mathrm{N}_2} = \operatorname{arsh}\dot{q} - t = c_2, \tag{20}$$

$\dot{q}$ can be eliminated via Eq.(19) and Eq.(20), then we have

$$q = \operatorname{ch}(t + c_2) - c_1. \tag{21}$$

Integrating Eq.(14) after substituting Eq.(21) into it, we get the general solution of Eq.(13) as follows:

$$x = \operatorname{sh}(t + c_2) - c_1 t + c_3. \tag{22}$$

Secondly, by Poisson theory, let

$$a^1 = q, \quad a^2 = p = \dot{q},$$

then system (15) can be expressed as follows:

$$\dot{a}^\mu = S^{\mu v}\frac{\partial H}{\partial a^v},$$

where

$$(S^{\mu v}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \left[1 + \left(a^2\right)^2\right]^{\frac{1}{2}}\left(a^2\right)^{-1} \end{pmatrix},$$

the generalized Poisson condition from Eq.(12) is as follows:

$$\frac{\partial I}{\partial t}+\frac{\partial I}{\partial a^1}a^2+\frac{\partial I}{\partial a^2}\left[1+\left(a^2\right)^2\right]^{\frac{1}{2}}=0, \tag{23}$$

which can be presented as

$$\frac{\mathrm{d}t}{1}=\frac{\mathrm{d}a^1}{a^2}=\frac{\mathrm{d}a^2}{\left[1+\left(a^2\right)^2\right]^{\frac{1}{2}}}.$$

From it we obtain

$$\mathrm{d}a^1=\frac{a^2\mathrm{d}a^2}{\left[1+\left(a^2\right)^2\right]^{\frac{1}{2}}},\quad \mathrm{d}t=\frac{\mathrm{d}a^2}{\left[1+\left(a^2\right)^2\right]^{\frac{1}{2}}}.$$

Integrating it, we obtain

$$\left[1+\left(a^2\right)^2\right]^{\frac{1}{2}}-a^1=c_1, \tag{24}$$

$$\operatorname{arsh}a^2=t+c_2, \tag{25}$$

a^2 can be eliminated via Eq.(24) and Eq.(25), and we have

$$a^1=\operatorname{ch}\left(t+c_2\right)-c_1. \tag{26}$$

Integrating it, we obtain

$$x=\operatorname{sh}\left(t+c_2\right)-c_1t+c_3. \tag{27}$$

6. Summary

This paper presents two sorts of methods, Noether's method and Poisson's method, using analytical mechanics for solving the higher-order differential equation which can be reduced to the second-order differential equation.

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