Vol 17 No 3, March 2008
1674-1056/2008/17(03)/0817-05
Chinese Physics B
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Tunnelling effect of the non-stationary Kerr black hole*

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(Received 12 December 2006; revised manuscript received 11 April 2007)

Extending Parikh and Wilczek's work to the non-stationary black hole, we study the Hawking radiation of the non-stationary Kerr black hole by the Hamilton–Jacobi method. The result shows that the radiation spectrum is not purely thermal and the tunnelling probability is related to the change of Bekenstein–Hawking entropy, which gives a correction to the Hawking thermal radiation of the black hole.

Keywords: Hamilton–Jacobi equation, radiation spectrum, self-gravitation interaction, non-stationary Kerr black hole

PACC: 0420, 9760L

1. Introduction

Recently, Parikh and Wilczek's research on the Hawking radiation[1,2] of the static Schwarzschild black hole and Reissner–Nordström black hole shows that when the energy conservation and the self-gravitation interaction are considered, the actual radiation spectrum deviates from the precisely thermal one and the tunnelling probability is related to the change of Bekenstein–Hawking entropy.[3−5] In their work, the essence of the methodology is dynamic treatment of the Hawking radiation. In the discussion, the derivation of the radiation particle's action is the crux. For deriving the action, they introduced Painlevé coordinates that are well regular at the horizon of the black hole and found out the motion equation of the particle, and then calculated Hamilton equation to get it. Finally, using the WKB approximation, the relationship between the tunnelling probability and imaginary part of the action was derived, which satisfies $\Gamma \sim \exp(-2\mathrm{Im}I)$. Following the method, people studied the Hawking radiation of various background space–times,[6−18] such as Hemming and Keski–Vakkuri on the anti-de Sitter space–time,[6] Medved on the de Sitter space–time[7] and Zhang and Zhao *et al* on the stationary axisymmetric space–times.[10−13] In 2005, Zhang and Zhao extended the work to the Hawking radiation of massive charge particles and further discussed the Hawking radiation of sphere symmetry and axisymmetric black holes.[19,20] All of these researches played important roles in the further cognition and research on black holes.[21−23]

However, the researches were focused on the Hawking radiation of the static and stationary black holes only. In 2006, Ren and Zhao studied the Hawking radiation of the sphere-symmetrically non-stationary black hole.[24,25] The case of the non-stationary axisymmetric black holes have not been deeply investigated yet. As we know, due to the absorption and evaporation, the mass and other parameters of black holes should be unfixed and change with time. Thus, generally speaking, black holes are non-stationary, and the stationary black holes are only idealized models. So the study of the Hawking radiation of the non-stationary black holes is meaningful.

In this paper, the Hawking radiation of the non-stationary Kerr black hole is discussed by taking account of the self-gravitation interaction and the unfixed background space–time. The result shows the actual radiation spectrum deviates from the precisely thermal one and the tunnelling probability is related to the change of Bekenstein–Hawking entropy. In our discussion, the background space–time is unfixed, thus the energy conservation and angular momentum conservation should be taken into account. However, since the black hole is non-stationary, the mass and angular momentum and other parameters change with time. Thus how to account for the unfixed background space–time is a difficult problem in this paper. Our treatment is as follows. When a particle with energy

*Project supported by the National Natural Science Foundation of China (Grant No 10773008).

ω and angular momentum j tunnels out, the mass and angular momentum of the black hole change into $M-\omega$ and $J-j$ respectively. Although the black hole is non-stationary, and the mass and angular momentum change with time, yet these are fixed at a certain time v'. Thus at this time, the energy and angular momentum are certain. Meanwhile, taking the instantaneous effect of the particle's Hawking radiation and the condition of the thermal equilibrium in the local area into account, we can discuss the Hawking radiation at time v'. And then considering the arbitrariness of the selected time, we can get the Hawking radiation of the non-stationary Kerr black hole.

The paper is organized as follows. For the convenience of the discussion of the Hawking radiation, we perform the coordinate transformations in Section 2. In Section 3, the Hawking radiation as tunnelling from the non-stationary Kerr black hole is discussed by Hamilton–Jacobi method and the actual radiation spectrum is derived. Section 4 includes some discussion and conclusions.

2. Coordinate transformations

The line element of the non-stationary Kerr black hole[26] represented by the advanced Eddington–Finkelstein coordinates is given by

$$\begin{aligned}\mathrm{d}s^2 =& -\left(1-\frac{2Mr}{\rho^2}\right)\mathrm{d}v^2+2\mathrm{d}v\mathrm{d}r\\ &-2\frac{2Mra\sin^2\theta}{\rho^2}\mathrm{d}v\mathrm{d}\phi-2a\sin^2\theta\mathrm{d}r\mathrm{d}\phi+\rho^2\mathrm{d}\theta^2\\ &+\left[\left(r^2+a^2\right)+\frac{2Mra^2\sin^2\theta}{\rho^2}\right]\sin^2\theta\mathrm{d}\phi^2, \quad (1)\end{aligned}$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $M = M(v)$ and $a = a(v)$. According to the null hypersurface equation $g^{\mu\nu}\partial_\mu f\partial_\nu f = 0$, we can get the horizon equation as

$$\begin{aligned}&a^2\dot{r}_{\mathrm{h}}^2\sin^2\theta + r_{\mathrm{h}}^2 - 2Mr_{\mathrm{h}} + a^2\\ &+ r'^2_{\mathrm{h}} - 2\left(r_{\mathrm{h}}^2+a^2\right)\dot{r}_{\mathrm{h}} = 0, \quad (2)\end{aligned}$$

where $\dot{r}_{\mathrm{h}} = \dfrac{\partial r_{\mathrm{h}}}{\partial v}$ and $r'_{\mathrm{h}} = \dfrac{\partial r_{\mathrm{h}}}{\partial\theta}$ express the location of the horizon changes with time and angle. Thus the event horizon can be easily gotten as

$$r_{\mathrm{h}} = \frac{M+\sqrt{M^2-(1-2\dot{r}_{\mathrm{h}})\left(a^2+a^2\dot{r}_{\mathrm{h}}^2\sin^2\theta+r'^2_{\mathrm{h}}-2a^2\dot{r}_{\mathrm{h}}\right)}}{1-2\dot{r}_{\mathrm{h}}}. \quad (3)$$

For the convenience of research of the Hawking radiation, the particle is treated as a shell tunnelling across the event horizon in radial direction and let $r'_{\mathrm{h}} = 0$. Since the black hole is non-stationary and the event horizon changes with time, it is not convenient to discuss the Hawking radiation of the non-stationary Kerr black hole. For eliminating the inconvenience, we should find out a moving coordinate system which changes with the event horizon and discuss the Hawking radiation in it. Therefore, let $R = r - r_{\mathrm{h}}(v)$, we have

$$\mathrm{d}R = \mathrm{d}r - \dot{r}_{\mathrm{h}}\mathrm{d}v. \quad (4)$$

Substituting Eq.(4) into (1) yields

$$\begin{aligned}\mathrm{d}s^2 =& -\left(1-\frac{2Mr}{\rho^2}-2\dot{r}_{\mathrm{h}}\right)\mathrm{d}v^2+2\mathrm{d}v\mathrm{d}R\\ &-2a\sin^2\theta\left(\dot{r}_{\mathrm{h}}+\frac{2Mr}{\rho^2}\right)\mathrm{d}v\mathrm{d}\phi\\ &-2a\sin^2\theta\mathrm{d}R\mathrm{d}\phi+\rho^2\mathrm{d}\theta^2\\ &+\left[\left(r^2+a^2\right)+\frac{2Mra^2\sin^2\theta}{\rho^2}\right]\sin^2\theta\mathrm{d}\phi^2. \quad (5)\end{aligned}$$

Considering the rotation of the black hole, there is a frame dragging effect of the coordinate system in the rotating space–time and the matter field in the ergosphere near the horizon must be dragged. Which is also not convenient for discussing the Hawking radiation and an effective approach to describe it should be adopting the dragging coordinate system. So we perform the following dragging coordinate transformation,

$$\frac{\mathrm{d}\phi}{\mathrm{d}v} = \frac{a\left(\rho^2\dot{r}_{\mathrm{h}}+2Mr\right)}{\rho^2\left(r^2+a^2\right)+2Mra^2\sin^2\theta}, \quad (6)$$

on the line element (5) and order $\Delta = r^2 - 2Mr + a^2$, we can obtain

$$\begin{aligned}\mathrm{d}s^2 =& -\rho^2\frac{\Delta+a^2\dot{r}_{\mathrm{h}}^2\sin^2\theta-2\dot{r}_{\mathrm{h}}\left(r^2+a^2\right)}{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}\mathrm{d}v^2\\ &+\frac{2\rho^2\left(r^2+a^2-a^2\dot{r}_{\mathrm{h}}^2\sin^2\theta\right)}{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}\mathrm{d}v\mathrm{d}R+\rho^2\mathrm{d}\theta^2. \quad (7)\end{aligned}$$

In the line element (7), the metric satisfies Landau's condition of the coordinate clock synchroniza-

tion, which provides the possibility to study the Hawking radiation. Meanwhile, the event horizon and the outer infinite red-shift surface are coincident with each other, which means the geometrical optics limit can be reliable here. Utilizing the WKB approximation,[27,28] the relationship between the tunnelling probability and the imaginary part of the radiation particle's action can be derived. In our discussion, the derivation of the action is the key. For the case of the stationary black holes, we can get the action through Parikh and Wilczek's method. However, since the black hole is non-stationary and the mass and other parameters change with time, so the subsequent treatment is different from Parikh and Wilczek's. Ignoring the self-gravitation interaction, we first explore the action by Hamilton–Jacobi equation. And then correct the action and get the true radiation spectrum of the non-stationary Kerr black hole. Now we turn to the treatment.

3. Hawking radiation as tunnelling from the non-stationary Kerr black hole

The action of the radiation particle satisfies the relativistic Hamilton–Jacobi equation, namely,

$$g^{\mu\nu}\partial_\mu I\partial_\nu I + u^2 = 0, \tag{8}$$

where u and I are the mass and action of the radiation particle respectively, and $g^{\mu\nu}$ is the inverse metric tensor derived from the line element (7). Substituting them into Eq.(8) we obtain

$$2\partial_v I\partial_R I + \frac{\Delta + a^2\dot{r}_{\rm h}^2\sin^2\theta - 2\dot{r}_{\rm h}\left(r^2+a^2\right)}{r^2+a^2-a^2\dot{r}_{\rm h}^2\sin^2\theta}\left(\partial_R I\right)^2 + \frac{g_{01}}{\rho^2}\left(\partial_\theta I\right)^2 + g_{01}u^2 = 0, \tag{9}$$

where

$$g_{01} = \frac{\rho^2\left(r^2+a^2-a^2\dot{r}_{\rm h}^2\sin^2\theta\right)}{\left(r^2+a^2\right)^2 - \Delta a^2\sin^2\theta}.$$

It is difficult to solve the action I from Eq.(9) directly for it is a function of v, R, θ and ϕ. Considering the property of the black hole space–time and the dragging effect, we carry out the separation of variables[29]

$$I = W\left(v,R,\theta\right) + {\rm j}\phi, \tag{10}$$

and let $\partial_v I = \partial_v W = -\omega$. Inserting Eq.(9) into (10), we obtain

$$W\left(v,R,\theta\right) = \int \frac{\left(r^2+a^2-a^2\dot{r}_{\rm h}^2\sin^2\theta\right){\rm d}R}{\Delta + a^2\dot{r}_{\rm h}^2\sin^2\theta - 2\dot{r}_{\rm h}\left(r^2+a^2\right)}\left[(\omega - {\rm j}\Omega)\right.$$
$$\left. + \sqrt{\left(\omega-{\rm j}\Omega\right)^2 - \rho^2\frac{\Delta + a^2\dot{r}_{\rm h}^2\sin^2\theta - 2\dot{r}_{\rm h}\left(r^2+a^2\right)}{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}\left(\left(\frac{\partial_\theta W\left(v,R,\theta\right)}{\rho^2}\right)^2 + u^2\right)}\right]. \tag{11}$$

In Eq.(11), $\Omega = \dfrac{{\rm d}\phi}{{\rm d}v}$ expresses the angular velocity and the imaginary part can be produced at a pole that corresponds to the event horizon. Near the event horizon, we have

$$\Delta + a^2\dot{r}_{\rm h}^2\sin^2\theta - 2\dot{r}_{\rm h}\left(r^2+a^2\right) = \left[2r_{\rm h}\left(1-2\dot{r}_{\rm h}\right) - 2M\right]\left(r-r_{\rm h}\right) + \cdots. \tag{12}$$

And then

$$W\left(v,R,\theta\right) = \frac{1}{2}\frac{r_{\rm h}^2+a^2-a^2\dot{r}_{\rm h}\sin^2\theta}{r_{\rm h}\left(1-2\dot{r}_{\rm h}\right)-M}\int\frac{{\rm d}R}{r-r_{\rm h}}$$
$$\times\left[(\omega-{\rm j}\Omega) + \sqrt{\left(\omega-{\rm j}\Omega\right)^2 - \rho^2\frac{\left[2r_{\rm h}\left(1-2\dot{r}_{\rm h}\right)-2M\right]\left(r-r_{\rm h}\right)}{\left(r^2+a^2\right)^2-\Delta a^2\sin^2\theta}\left(\left(\frac{\partial_\theta W\left(v,R,\theta\right)}{\rho^2}\right)^2+u^2\right)}\right]. \tag{13}$$

Finishing the integral and substituting the result in Eq.(10), the imaginary part of the action can be obtained as

$$\mathrm{Im}I = \pi \frac{r_{\mathrm{h}}^2 + a^2 - a^2 \dot{r}_{\mathrm{h}} \sin^2\theta}{r_{\mathrm{h}}\left(1 - 2\dot{r}_{\mathrm{h}}\right) - M} \left(\omega - \mathrm{j}\Omega_{\mathrm{h}}\right), \tag{14}$$

where

$$\Omega_{\mathrm{h}} = \frac{a\left[\left(r_{\mathrm{h}}^2 + a^2\cos^2\theta\right)\dot{r}_{\mathrm{h}} + 2Mr_{\mathrm{h}}\right]}{\left(r_{\mathrm{h}}^2 + a^2\cos^2\theta\right)\left(r_{\mathrm{h}}^2 + a^2\right) - 2Mr_{\mathrm{h}}a^2\sin^2\theta}$$

denotes the angular velocity of the radiation particle at the event horizon and $\dot{r}_{\mathrm{h}}$ is treated as a constant. Using the WKB approximation, we can obtain the tunnelling probability and the radiation spectrum, but the radiation spectrum is precisely thermal, which is not in accordance with the result of recent researches. The reason is that the self-gravitation interaction and the unfixed background space–time have not been taken into account in the discussion. Incorporating them, we go on discussing the Hawking radiation of the non-stationary Kerr black hole. When a particle with energy ω and angular momentum j tunnels out, the mass and angular momentum of the black hole will become $M-\omega$ and $J-j$, respectively. Considering the instantaneous effect that the particle tunnelling across the event horizon and the condition of the thermal equilibrium in the local area, we can study the Hawking radiation at an arbitrarily fixed time v'. Now let us firstly review the condition. In a short time Δv (Δv expresses the shortest time after measuring the mass M of the black hole), the change of the mass of the black hole is ΔM. When $\dfrac{\Delta M}{M} \ll 1$, the condition is satisfied. So, at this time, the imaginary part of the actual action should be

$$\begin{aligned}\mathrm{Im}I =& \pi \int_{(0,0)}^{(\omega,j)} \frac{\tilde{r}_{\mathrm{h}}^2 + a^2 - a^2\dot{r}_{\mathrm{h}}\sin^2\theta}{\tilde{r}_{\mathrm{h}}\left(1-2\dot{r}_{\mathrm{h}}\right) - \left(M-\omega'\right)}\left(\mathrm{d}\omega' - \tilde{\Omega}_{\mathrm{h}}\mathrm{d}j'\right)\\ =& -\pi \int_{M}^{M-\omega} \frac{\tilde{r}_{\mathrm{h}}^2 + a^2 - a^2\dot{r}_{\mathrm{h}}\sin^2\theta}{\tilde{r}_{\mathrm{h}}\left(1-2\dot{r}_{\mathrm{h}}\right) - \left(M-\omega'\right)}\left[\mathrm{d}\left(M-\omega'\right) - a\tilde{\Omega}_{\mathrm{h}}\mathrm{d}\left(M-\omega'\right)\right],\end{aligned} \tag{15}$$

where

$$\begin{aligned}\tilde{\Omega}_{\mathrm{h}} =& \frac{a\left[\left(\tilde{r}_{\mathrm{h}}^2 + a^2\cos^2\theta\right)\dot{r}_{\mathrm{h}} + 2\left(M-\omega'\right)\tilde{r}_{\mathrm{h}}\right]}{\left(\tilde{r}_{\mathrm{h}}^2 + a^2\cos^2\theta\right)\left(\tilde{r}_{\mathrm{h}}^2 + a^2\right) - 2\left(M-\omega'\right)\tilde{r}_{\mathrm{h}}a^2\sin^2\theta},\\ \tilde{r}_{\mathrm{h}} =& \frac{\left(M-\omega'\right) + \sqrt{\left(M-\omega'\right)^2 - \left(1-2\dot{r}_{\mathrm{h}}\right)\left(a^2 + a^2\dot{r}_{\mathrm{h}}^2\sin^2\theta - 2a^2\dot{r}_{\mathrm{h}}\right)}}{1-2\dot{r}_{\mathrm{h}}}.\end{aligned}$$

The surface gravitation with the particle emission can be obtained and is just

$$\tilde{\kappa} = \frac{\tilde{r}_{\mathrm{h}}\left(1-2\dot{r}_{\mathrm{h}}\right) - \left(M-\omega'\right)}{\tilde{r}_{\mathrm{h}}^2 + a^2\left(1 - \dot{r}_{\mathrm{h}}\sin^2\theta\right)}. \tag{16}$$

Clearly, therefore, the imaginary part can be gotten as

$$\begin{aligned}\mathrm{Im}I =& -\pi \int_{M}^{M-\omega} \frac{1}{\tilde{\kappa}}\left[\mathrm{d}\left(M-\omega'\right) - a\tilde{\Omega}_{\mathrm{h}}\mathrm{d}\left(M-\omega'\right)\right]\\ =& -\frac{1}{2}\int_{S_{\mathrm{BH}}(M)}^{S_{\mathrm{BH}}(M-\omega)} \mathrm{d}\tilde{S} = -\frac{1}{2}\Delta S_{\mathrm{BH}},\end{aligned} \tag{17}$$

where $\Delta S_{\mathrm{BH}} = S_{\mathrm{BH}}\left(M-\omega\right) - S_{\mathrm{BH}}\left(M\right)$ is the change of Bekenstein–Hawking entropy. So the tunnelling probability is

$$\Gamma \sim \exp\left(-2\mathrm{Im}I\right) = \exp\left(\Delta S_{\mathrm{BH}}\right). \tag{18}$$

Obviously, the tunnelling probability is related to the change of Bekenstein–Hawking entropy and the radiation spectrum deviates from the precisely thermal one. Although the discussion is done at the fixed time v', the selected time v' is at random. Therefore we can obtain the Hawking radiation of the non-stationary Kerr black hole at any other time.

4. Discussion and conclusion

When M and a do not change with time and $\dot{r}_{\rm h}=0$, from Eq.(15), we can obtain

$$\begin{aligned}\mathrm{Im}I =& -\pi\int_{M}^{M-\omega}\frac{2\left(M-\omega'\right)^2+2\left(M-\omega'\right)\sqrt{\left(M-\omega'\right)^2-a^2}}{\sqrt{\left(M-\omega'\right)^2-a^2}}\mathrm{d}\left(M-\omega'\right)\\ =& -\pi\left[\left(M-\omega\right)^2-M^2+\left(M-\omega\right)\sqrt{\left(M-\omega\right)^2-a^2}-M\sqrt{M^2-a^2}\right]\\ =& -\frac{1}{2}\Delta S_{\rm BH},\end{aligned}\tag{19}$$

which is in full accordance with the result obtained from the stationary Kerr case.[12]

In this paper, we have discussed the Hawking radiation of the non-stationary Kerr black hole by considering the self-gravitation interaction and the unfixed background space–time. The result is in accordance with that obtained from the stationary black hole and gives a correction to the Hawking radiation of the non-stationary black hole. In the discussion, there are some key points. Firstly, the condition of thermal equilibrium in local area must be satisfied, under which we can introduce the thermal equilibrium in the locale area to discuss the Hawking radiation and to define the local area temperature. Moreover, since the black hole is non-stationary and the mass parameter *et al* change with time, it is difficult to discuss the Hawking radiation by considering the changing background space–time. But at an arbitrarily fixed time v', the space–time is fixed and the energy and angular momentum of the black hole are certain. Meanwhile, considering the instantaneous effect that radiation particle tunnels across the event horizon and the condition of the thermal equilibrium in the local area, we can study the Hawking radiation of the non-stationary black hole by considering the changing background space–time at time v'. Considering the arbitrariness of the selected time, we can obtain the Hawking radiation of the black hole at any other time.

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