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# Formation of three-body entanglement via a vacuum optical cavity induction in Tavis–Cummings model

Li Yu-Liang(李玉良)† and Li Xue-Mei(李雪梅)

*Department of Physics and Electronic Information Engineering, Minjiang University, Fuzhou* 350108, *China*

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After briefly introducing Akhtarshenas, concurrence vector and subvector for describing many-body entanglement, we investigate the entanglement formation for a system which contains three bodies, i.e. two identical atoms and a single-model cavity field, in the Tavis–Cummings model by calculating the concurrences. The results show that the coupling strength between two atoms, the decay cavity and the atomic spontaneous emission can change the entanglement of formation according to different modes: these factors destroy periodicity and symmetry of all concurrences, and that the coupling strength of two atoms does not change the peak value of concurrence ($C$), but the strength of decay cavity and the atomic spontaneous emission decline in the peak value of concurrence ($C$) and the latter is more serious than the former under the same strengths. The concurrence vector and subvector are a useful measure of entanglement for a pure state of the many-body system, in that it can give novel pictures about the entanglements for the entire system and between its inner bodies.

**Keywords:** quantum three-body entanglement, concurrence vector, cavity QED
**PACC:** 0367, 4250

## 1. Introduction

Entanglement is a striking feature of quantum systems and also a key physical resource to realize quantum information tasks such as quantum teleportation,[1] quantum cryptography[2] and quantum computation,[3] which cannot be accounted for by classical physics and so has been studied intensively in various ways. Nevertheless, there are still a number of open problems about the entanglement, for example, what is the best way to quantify the amount of entanglement for higher-dimensional systems and systems of more than two qubits.

For a two-qubit pure or mixed state, the Wootters' concurrence $C$[4−6] is known as a good measure of entanglement, from it we can directly derive the explicit formula for the entanglement of formation. For a pure two-qubit state $|\Psi\rangle$, which is a vector in the Hilbert space $C^2 \otimes C^2$, the concurrence $C(|\Psi\rangle)$ measures the degree of bipartite entanglement shared between the qubits. In terms of the coefficients $\{a_{11}, a_{12}, a_{21}, a_{22}\}$ of $|\Psi\rangle$ with an orthonormal basis, the concurrence takes the form $C = 2|a_{11}a_{22} - a_{12}a_{21}|$. On the other hand, because of the relation between concurrence and entanglement of formation, it is interesting to ask whether the concurrence can be generalized to larger quantum systems, such as a many-body or even three-qubit system in a pure (mixed) state. Up to now, many investigators have made a great effort to quantitatively measure entanglement.[7−14] In Ref.[4], Wootters provided a basis for the quantification of three-qubit entanglement by introducing the so-called residual tangle (or the 3-tangle) which represents a collective property of all the three qubits that are unchanged by permutations, and led to an understanding of important subtleties in multipartite quantum correlations.[15−18] In Ref.[12], Akhtarshenas generalized a concept of the concurrence vector, as a measure of entanglement for a given pure state of many-body systems, by defining the representation of generators of the corresponding rotation groups.

We adopt the concurrence vector proposed by Akhtarshenas as a measure of three-body entanglement in the present paper. For a pure three-body state

$$|\Psi\rangle = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{3} a_{ijk}|i,j,k\rangle \in C^2 \otimes C^2 \otimes C^3, \quad (1)$$

†Corresponding author. E-mail: lee9002@163.com

the concurrence vector $C$ is defined in such a way that it involves the whole entanglement shared between all pairs of bodies. Moreover, we can consider vector $C$ as a direct sum of elementary subvectors $C^{\{ij\}}$, i.e.

$$C = \sum_{\oplus ij} C^{\{ij\}} = C^{\{12\}} \oplus C^{\{13\}} \oplus C^{\{23\}} \tag{2}$$

such that each subvector $C^{\{ij\}}$ corresponds to the pair, $i$ and $j$, of bodies and is defined as $C^{\{ij\}} = \langle \Psi | L_i \otimes L_j | \Psi^* \rangle$, where $|\Psi^*\rangle$ is the complex conjugate of $|\Psi\rangle$, $L_\alpha (\alpha = 1, 2)$ is the only generator of two-dimensional rotation group SO(2) and $L_3$ corresponds to the group SO(3). For a pure state $|\Psi\rangle = \sum_{i=1}^{2} \sum_{j=1}^{N} a_{ij} |i, j\rangle \in C^2 \otimes C^N$, we can obtain $C^{\{12\}}$ expressed in the following form:

$$C^{\{12\}} = 2 \sqrt{\sum_{j<k}^{N} |a_{1j} a_{2k} - a_{2j} a_{1k}|^2}. \tag{3}$$

Accordingly the entanglement contribution of pair, $i$ and $j$, in the state $|\Psi\rangle = a_{212}|2,1,2\rangle + a_{122}|1,2,2\rangle + a_{221}|2,2,1\rangle + a_{113}|1,1,3\rangle \in C^2 \otimes C^2 \otimes C^3$ can be defined as the norm of the concurrence subvector $C^{\{ij\}}$, that is

$$C^{\{12\}} = (|2a_{122}a_{212}|^2 + |a_{113}a_{221}|)^{1/2}, \tag{4a}$$

$$C^{\{13\}} = (|2a_{122}a_{221}|^2 + 2|a_{221}a_{113}|^2 + |2a_{113}a_{212}|^2)^{1/2}, \tag{4b}$$

$$C^{\{23\}} = (|2a_{212}a_{221}|^2 + 2|a_{221}a_{113}|^2 + |2a_{113}a_{122}|^2)^{1/2}. \tag{4c}$$

Finally, the total concurrence of $|\Psi\rangle$ is defined as the norm of the concurrence vector $C$, i.e.

$$C = |C| = \sqrt{|C^{\{12\}}|^2 + |C^{\{13\}}|^2 + |C^{\{23\}}|^2}. \tag{5}$$

This feature of $C^{\{ij\}}$ shows that it can be considered as the pairwise entanglement between the subsystems $i$ and $j$. It should be stressed that the $C^{\{ij\}}$ is not obtained through tracing out the remaining subsystems, but its definition comes directly from a generalization of the bipartite case. It is clear that if $|\Psi\rangle$ is completely separable, then all $C^{\{ij\}}$ are zero and the total concurrence $C$ becomes zero. The total concurrence $C$ is an entanglement monotone and invariant under local unitary transformation, and it cannot increase under local operation and classical communication. For a state $|\Psi\rangle = (|00\rangle + |11\rangle)/\sqrt{2} \otimes (\alpha|0\rangle + \beta|1\rangle)$ with $|\alpha|^2 + |\beta|^2 = 1$, we obtain $C^{\{12\}} = 1$, $C^{\{23\}} = C^{\{13\}} = 0$ and $C = 1$. For a three-qubit system with a Greenberger–Horne–Zeilinger (GHZ) state, $|\mathrm{GHZ}_3\rangle = (|000\rangle + |111\rangle)/\sqrt{2}$, it gives $C^{\{ij\}} = 1/\sqrt{2}$ (for all $i, j$), and $C = \sqrt{3/2} > 1$ respectively.

The essential idea of cavity quantum electrodynamics (QED) is to trap several atoms in a small high-finesse optical cavity, and the entanglement character between the trapped atoms and cavity field is utilized to process quantum information.[3] Therefore, it is important to investigate the entanglement of formation and quantification of the system including two atoms and a cavity field in the Tavis–Cummings (T–C) model by Akhtarshenas's concurrence vector. In this paper, we study the formation of three-body entanglement in the two different cases: one is the case in which two identical two-level atoms interact with a singe-model field and the two atomic coupling effect is not neglected, and the other is the case in which two atoms interact with a decay cavity and their spontaneous emission takes place. To make it clear how those factors influence the entanglement of formation, all concurrences need calculating carefully.

## 2. Two coupling atoms interacting with a resonant cavity

We consider two identical two-level atoms simultaneously interacting with a single-mode cavity field. Assuming that the exact resonance between the cavity mode and an atomic transition takes place and the atomic interaction is so strong that it cannot be neglected, thus, the interaction Hamiltonian in the interaction picture can be written as[3,19]

$$H_i = g \sum_{j=1,2} (a^+ S_j^- + a S_j^+) + \varepsilon (S_1^+ S_2^- + S_2^+ S_1^-), \tag{6}$$

where $S_j^+ = |e_j\rangle\langle g_j|$ and $S_j^- = |g_j\rangle\langle e_j|$, with $|e_j\rangle$ and $|g_j\rangle (j = 1, 2)$ being the excited and the ground states of the $j$th atom; $a^+$ and $a$ are the creation and annihilation operators for the cavity mode respectively; $g$ and $\varepsilon$ are the atom–cavity and two atomic coupling coefficients respectively. Since an operator of the total system excitation $N = a^+ a + \sum_{j=1,2} S_j^+ S_j^-$ commutes with the Hamiltonian $[N, H_i] = 0$, $N$ is an exchange constant. For example, when the system is initially in the separable state $|e_1, g_2, m\rangle$, the system state vector is always in the same excitation $N = m + 1$ and change in the Hilbert subspace $C^2 \otimes C^2 \otimes C^3$ with an orthonormal basis $\{|e_1, g_2, m\rangle, |g_1, e_2, m\rangle, |g_1, g_2, m+1\rangle, |e_1, e_2, m-1\rangle\}$. The state evolution of the system can be obtained from the Schrödinger equation as[20]

$$
\begin{aligned}
|\Psi_1(0)\rangle =& |e_1, g_2, 0\rangle \xrightarrow{=|2,1,2\rangle} |\Psi_1(t)\rangle \\
=& \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{3} a_{ijk}|i,j,k\rangle = \frac{\mathrm{i}g}{\Omega}\exp(-\mathrm{i}\varepsilon t/2)\sin(\Omega t)|g_1, g_1, 1\rangle \\
&+ \frac{1}{2}\left[\exp(\mathrm{i}\varepsilon t) + \exp(\mathrm{i}\varepsilon t/2)\left(\cos(\Omega t) - \frac{\mathrm{i}\varepsilon}{4\Omega}\sin(\Omega t)\right)\right]|e_1, g_2, 0\rangle \\
&+ \frac{1}{2}\left[-\exp(\mathrm{i}\varepsilon t) + \exp(\mathrm{i}\varepsilon t/2)\left(\cos(\Omega t) - \frac{\mathrm{i}\varepsilon}{4\Omega}\sin(\Omega t)\right)\right]|g_1, e_1, 0\rangle, \qquad (7)
\end{aligned}
$$

where $\Omega = \sqrt{2g^2 + \frac{1}{4}\varepsilon^2}$, and $|i,j,k\rangle = |i\rangle \otimes |j\rangle \otimes |k\rangle \in C^2 \otimes C^2 \otimes C^3$ represents the separable state of the system in which atom '1', atom '2' and cavity field are in the states $|i\rangle$,$|j\rangle$ and $|k\rangle$ respectively, the state $|2,1,2\rangle$, for example, means $|e_1, g_2, 0\rangle$ in which atom '1' is in the excitation state $|e\rangle$, atom '2' is in the ground state $|g\rangle$ and the cavity field is in a vacuum state $|0\rangle$. Now, from expressions (4a)–(4c) and (5), we can calculate $C^{\{ij\}}$ and $C$ by using the state $|\Psi(t)\rangle$ referring to expression (7), and depict them in Fig.1 with different values of coupling coefficient $\varepsilon$ between two atoms. It is obvious that all concurrences of the system, $C$ and $C^{\{ij\}}$, change periodically and show themselves symmetrically when the coupling strength $\varepsilon$ between two atoms is negligible compared with the atom–cavity coupling strength $g$. These characters are destroyed, but a maximum three-body entanglement ($C_{\max}$) can be retained as the value of $\varepsilon$ increases. It is easy to understand since the coupling strength $\varepsilon$ is in a phase factor $\exp(\mathrm{i}\varepsilon t)$ and $\Omega$ refers to that in expression (7). At the initial time $t = 0$, the state is separable, i.e. it has no genuine three-body entanglement. A maximum three-body entanglement can be always established at some other fixed times with the evolution of system state.

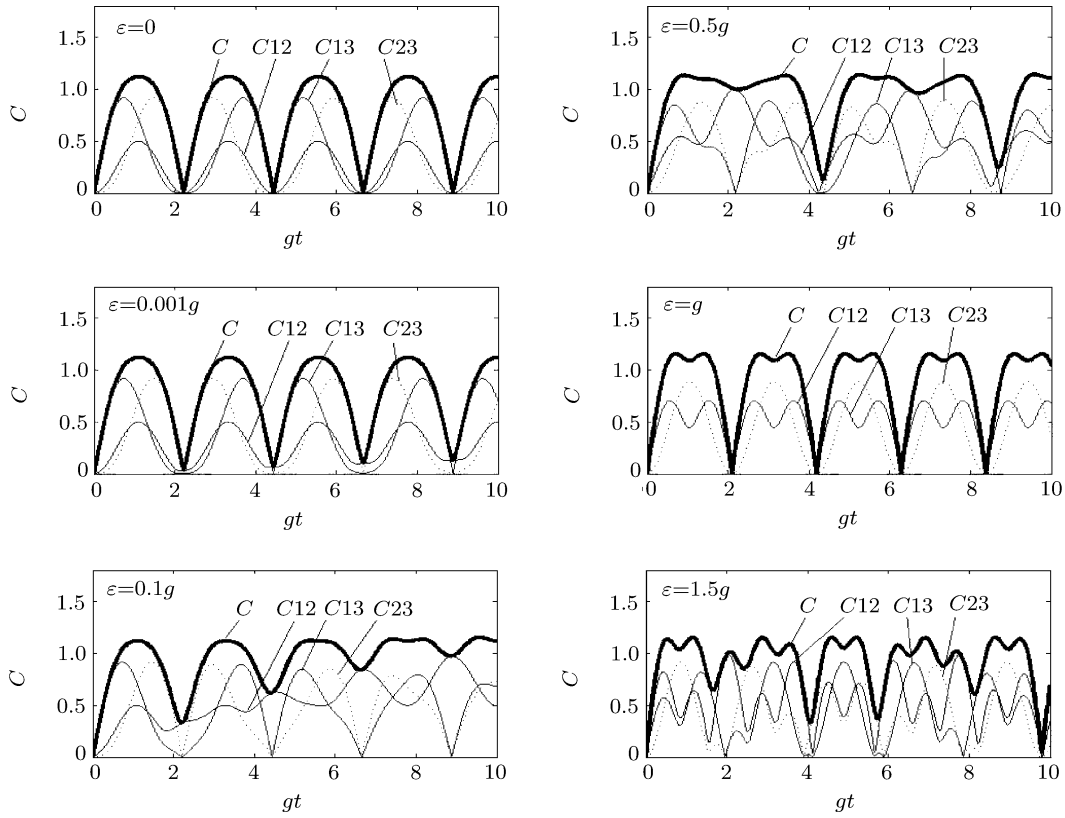



**Fig.1.** The entanglement formation of the system that contains two atoms and a single-mode cavity field in T–C model at the different values of atomic coupling strength $\varepsilon$. The $C12$, $C13$ and $C23$ denote the concurrence subvectors between atom–atom, 'atom 1'–'cavity field' and 'atom 2'–'cavity field', and $C$ is the total concurrence of the system. The system is in a state $|e_1, g_2, 0\rangle$ initially.

## 3. Two atoms interacting with a resonant decay cavity and their spontaneous emission

Now, we discuss another case in which two atoms interact with a resonant decay cavity and their spontaneous emission takes place. The interaction Hamiltonian in the interaction picture can be written as[3,17,18]

$$H_i = g\sum_{j=1,2}(a^+S_j^- + aS_j^+) - \mathrm{i}\frac{\kappa}{2}a^+a - \mathrm{i}\frac{\Gamma}{2}\sum_{j=1,2}|e_j\rangle\langle e_j|, \tag{8}$$

where $\kappa$ and $\Gamma$ are respectively the decay strength of cavity and the spontaneous emission strength of an atom, the other symbols have the same meanings as in expression (6). The state evolution of the system can be obtained from the Schrödinger equation as

$$\begin{aligned}|\Psi_1(0)\rangle =& |e_1,g_2,0\rangle \xrightarrow{=|2,1,2\rangle} |\Psi_1(t)\rangle\\ =& \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{3}a_{ijk}|i,j,k\rangle = \frac{\mathrm{i}g}{\Omega}\exp[-\mathrm{i}(\kappa+\Gamma)t/4]\sin(\Omega t)|g_1,g_1,1\rangle\\ &+\frac{1}{2}\left\{\exp[-(\kappa+\Gamma)t/4]\left(\cos(\Omega t)+\frac{\kappa-\Gamma}{4\Omega}\sin(\Omega t)\right)+\exp(-\Gamma t/2)\right\}|e_1,g_2,0\rangle\\ &+\frac{1}{2}\left\{\exp[-(\kappa+\Gamma)t/4]\left(\cos(\Omega t)+\frac{\kappa-\Gamma}{4\Omega}\sin(\Omega t)\right)-\exp(-\Gamma t/2)\right\}|g_1,e_1,0\rangle. \end{aligned}\tag{9}$$

From expressions (4a)-(4c) and (5), we can calculate $C^{\{ij\}}$ and $C$ by using the state $|\Psi(t)\rangle$ referring to expression (9), and depict them in Fig.2 again with different values of coefficients $\kappa$ and $\Gamma$. There appear

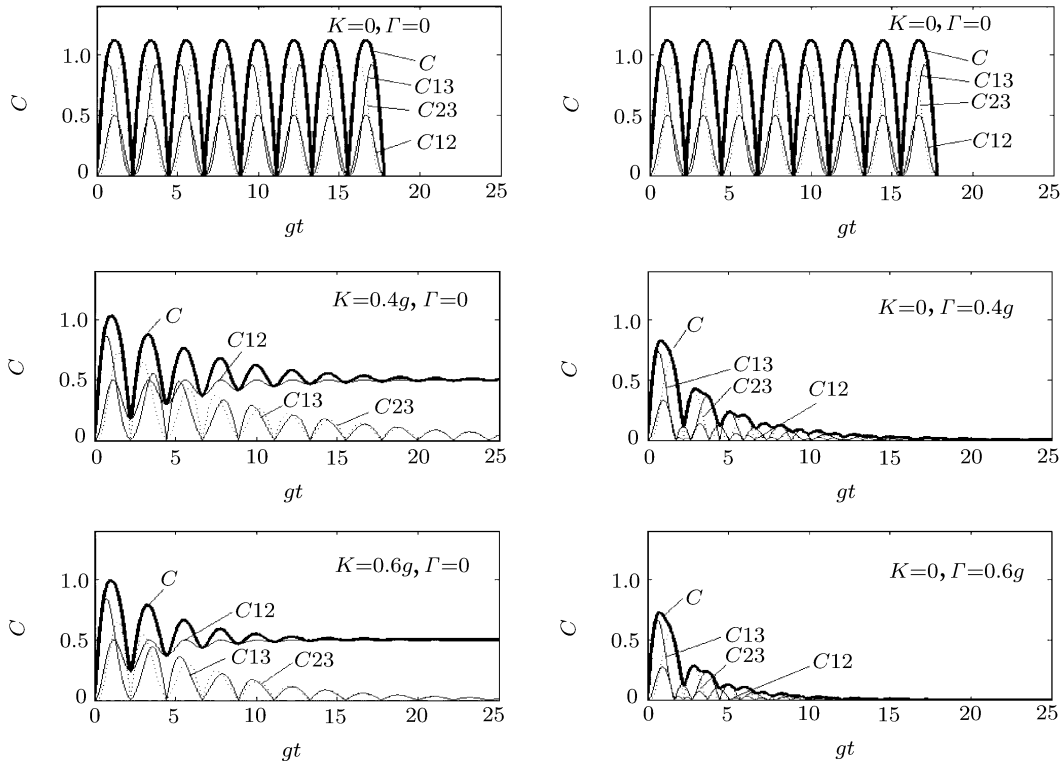



**Fig.2.** The entanglement formation of the system that contains two atoms and a single-mode cavity field in T–C model at the different values of coefficients $\kappa$ and $\Gamma$. The $C12$, $C13$ and $C23$ denote the concurrence subvectors between atom–atom, 'atom 1'–'cavity field' and 'atom 2'–'cavity field', and $C$ is the total concurrence of the system. The system is in an state $|e_1,g_2,0\rangle$ initially.

some behaviours of asymptote for all concurrences in Fig.2. When $\Gamma = 0$ and $\kappa$ increase from zero, $C$ and $C12$ approach to 0.5, but $C13$ and $C23$ tend to zero (see the left panels in Fig.2). When $\kappa = 0$ and $\Gamma$ increases from zero, all concurrences tend to zero (see the right panels in Fig.2). It shows that cavity decay and atomic spontaneous emission seriously influence the entanglement of formation in the system and inner parties, and the latter is more serious than the former under the same strengths. On the other hand, $\kappa$ and $\Gamma$ are not in the same states referring to expression (9), $\Gamma$ is in an exponential function $\exp(-\Gamma t)$ and presses all coefficients tending to zero in Eq.(9) as time $t \to \infty$, while $\kappa$ determines partly the value of coefficients though it is also in an exponential function $\exp(-kt)$. Further, we can obtain from expression (9) the following conclusions: (i) when $\Gamma = 0$, $\kappa \neq 0$ and $t \to \infty$, the system is in a state of $|\Psi(\infty)\rangle = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}(|e_1, g_2\rangle - |g_1, e_2\rangle)\right]|0\rangle$, therefore $C = C12 = 1/2$ and $C13 = C23 = 0$; (ii) when $\kappa = 0$, $\Gamma \neq 0$ and $t \to \infty$, the system is in a state of $|\Psi(\infty)\rangle \geq 0$, therefore all concurrences are zero.

# 4. Conclusions

We have investigated the entanglement formation for a system which contains three bodies, i.e. two identical atoms and a single-model cavity field, in T–C model by calculating Akhtarshenas's concurrences. The results show that the coupling strength between atoms, the decay cavity and the atomic spontaneous emission can change the entanglement of formation according to different modes: these factors destroy periodicity and symmetry of all concurrences, and that the coupling strength of two atoms does not change the peak value of concurrence ($C$), but the strength of decay cavity and the atomic spontaneous emission decline in the peak value of concurrence ($C$) and the latter is more serious than the former under the same strengths. The concurrence vector and subvector are a useful measure of entanglement for a pure state of many-body system in that it can give novel pictures about the entanglement for the entire system and between its inner bodies.

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