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Exact self-similar solitary waves and collisions in nonlinear optical media*

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This paper analyses bright and dark spatial self-similar waves propagation and collision in graded-index nonlinear waveguide amplifiers with self-focusing and self-defocusing Kerr nonlinearities. It finds an appropriate transformation for the first time such that the nonlinear Schrödinger equation (NLSE) with varying coefficients transform into standard NLSE. It obtains one-solitonlike, two-solitonlike and multi-solitonlike self-similar wave solutions by using the transformation. Furthermore, it analyses the features of the self-similar waves and their collisions.

Keywords: exact self-similar solitary wave, graded-index nonlinear waveguide amplifier, collision
PACC: 0340K, 4265S, 4281F, 0290

1. Introduction

Studies of self-similar solutions of the nonlinear differential equations have been of great value in understanding widely different nonlinear physical phenomena.[1] The self-similar evolution of nonlinear wave implies that the wave profile remains unchanged and its amplitude and width simply scale with time or propagation distance. Self-similar solutions have been extensively studied in many fields such as hydrodynamics,[1,2] quantum field theory,[1] plasma physics,[3] nuclear physics[1] and optics.[4−8] Recently, exact self-similar solitary waves have been found in optical fibres whose dispersion, nonlinearity and gain profile are allowed to change with the propagation distance, but the functional forms of these parameters cannot be chosen independently.[6,7] Such self-similar solitary waves were shown to have many features similar to ideal solitons[6] and have important applications such as the amplification and compression of pulses in optical fibre amplifiers.[9] More recently, Ponomarenko and Agrawal presented exact analytical solutions describing spatial bright and dark self-similar waves inside planar, graded-index nonlinear waveguide amplifiers with self-focusing and self-defocusing Kerr nonlinearities, respectively, by using the symmetry group analysis method.[10] The new self-similar waves are supported by such media if the magnitudes of the gain and the linear inhomogeneity of the waveguide are adjusted to maintain a static quadratic phase chirp of each wave.[10] The main difference between the new self-similar solitary waves and those obtained in Refs.[6–8] is that the former can be realized in practice under much less restrictive conditions.

However, the interaction between optical solitons influences directly the capacity and quality of communication.[11,12] In this paper, We focus our attention on multi-solitonlike self-similar waves and their collisions in inhomogeneous nonlinear media. The rest of this paper is organized as follows. In Section 2, we find an appropriate transformation for the first time such that the nonlinear Schrödinger equation (NLSE) with varying coefficients[13−20] transform into standard NLSE. We can obtain one-solitonlike, two-solitonlike and multi-solitonlike self-similar wave solutions of NLSE with varying coefficients by using the transformation. Then we further investigate the main features of various analytical solutions obtained by using direct computer simulations. The results in this paper include some results in the Ref.[10]. The last section is a short summary and discussion.

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2. Transformation and exact self-similar solitary waves in graded-index media

2.1. Transformation

We start by considering the propagation of a continuous-wave optical beam inside a planar, graded-index nonlinear waveguide amplifier with the refractive index

$$n = n_0 + n_1 x^2 + n_2 I, \tag{1}$$

where n_0 is the linear refractive index, n_1 governs the change in refractive index in the transverse dimension x, n_2 is the nonlinear parameter responsible for self-focusing or self-defocusing, and I is the light intensity. We assume $n_1 > 0$ so that the graded-index nonlinear waveguide acts as a linear defocusing lens.

The nonlinear wave equation governing beam propagation in such a waveguide can be written as

$$\begin{aligned} &\mathrm{i}\frac{\partial u}{\partial z} + \frac{1}{2k_0}\frac{\partial^2 u}{\partial x^2} - \frac{\mathrm{i}g}{2}u \\ &+ \frac{1}{2}k_0 n_1 x^2 u + k_0 n_2 |u|^2 u = 0, \end{aligned} \tag{2}$$

where $u = u(x,t)$ is the complex envelope of the electric field in an optical waveguide, g is the gain coefficient and $k_0 = 2\pi n_0/\lambda$, λ being the wavelength of the optical source generating the beam. If we introduce the normalized variables $X = x/\omega_0$, $Z = z/L_{\mathrm{D}}$, $G = gL_{\mathrm{D}}$, $U = (k_0|n_2|L_{\mathrm{D}})^{1/2}u$, where $L_{\mathrm{D}} = k_0\omega_0^2$ is the diffraction length associated with the characteristic transverse scale $\omega_0 = (2k_0^2 n_1)^{-1/4}$, we can rewrite Eq.(2) in a dimensionless form as

$$\mathrm{i}\frac{\partial U}{\partial Z} + \frac{1}{2}\frac{\partial^2 U}{\partial X^2} - \frac{\mathrm{i}G}{2}U + \frac{1}{2}X^2 U + \sigma|U|^2 U = 0. \tag{3}$$

Here $\sigma = \pm 1$, with the upper (lower) sign corresponding to a self-focusing (self-defocusing) nonlinearity of the waveguide.

In order to obtain some new exact self-similar solitary waves of Eq.(3), firstly we present an appropriate transformation

$$\begin{aligned} &U(X,Z) = \frac{q(\xi,\eta)}{\sqrt{2}}\exp\left(\frac{G+1}{2}Z - \frac{\mathrm{i}X^2}{2}\right), \\ &\xi = X\exp(Z), \\ &\eta = \frac{1}{4}\exp(2Z). \end{aligned} \tag{4}$$

Then substituting Eq.(4) into Eq.(3) and setting $G = 1$, we find that the Eq.(3) is changed into an isospectral NLSE. Instead of considering the Eq.(3), we only need to solve the standard NLSE (5):[21]

$$\mathrm{i}q_\eta + q_{\xi\xi} + \sigma|q|^2 q = 0, \tag{5}$$

where $q = q(\xi,\eta)$ and $\sigma = \pm 1$. Using the Hirota method,[22] we can obtain multi-soliton solutions. The first step in this method is to transform the equation (5) into the Hirota bilinear form. For this purpose we use the transformation, $q = h/f$,

$$\begin{aligned} &(\mathrm{i}D_\eta + D_\xi^2)h.f = 0, \\ &D_\xi^2 f.f = \sigma h h^*, \end{aligned} \tag{6}$$

where $*$ denotes the complex conjugate, D_η and D_ξ are Hirota bilinear operators, and $h = h(\xi,\eta)$ is a complex function, while $f = f(\xi,\eta)$ is a real function.

2.2. One-solitonlike self-similar waves

Case 1 σ=1, bright one-solitonlike:

The bright one-solitonlike solution to Eq.(6) is given by $h_1 = \mathrm{e}^{\delta_1}$, $f_1 = 1 + \mathrm{e}^{\delta_1+\delta_1^*+\theta_{13}}$, that is[21]

$$q = \frac{\mathrm{e}^{\delta_1}}{1+\mathrm{e}^{\delta_1+\delta_1^*+\theta_{13}}}, \tag{7}$$

where

$$\begin{aligned} &\mathrm{e}^{\theta_{13}} = \frac{1}{2(k_1+k_1^*)^2}, \\ &\delta_1 = k_1(\xi + \mathrm{i}k_1\eta) + \delta_1^{(0)}, \end{aligned} \tag{8}$$

where k_1, $\delta_1^{(0)}$ are arbitrary complex constants. Using the transformation (4) we can obtain the bright one-solitonlike self-similar wave solution of Eq.(3),

$$\begin{aligned} &U = \frac{1}{\sqrt{2}}\frac{\mathrm{e}^{Z-\frac{\mathrm{i}X^2}{2}+\delta_1}}{1+\mathrm{e}^{\delta_1+\delta_1^*+\theta_{13}}}, \\ &\delta_1 = k_1\mathrm{e}^Z\left(X + \frac{\mathrm{i}k_1\mathrm{e}^Z}{4}\right) + \delta_1^{(0)}. \end{aligned} \tag{9}$$

This solution can also be written in the more conventional form

$$\begin{aligned} U =& k_{1\mathrm{R}}\mathrm{e}^Z \mathrm{sech}\left(\delta_{1\mathrm{R}} + \frac{\theta_{13}}{2}\right) \\ &\times \exp\left[\mathrm{i}\left(-\frac{1}{2}X^2 + \delta_{1\mathrm{I}}\right)\right], \end{aligned} \tag{10}$$

where we have introduced the subscripts R and I for the real and imaginary parts of the quantity in question and θ_{13} is real.

If we introduce the variables $W_0 = \dfrac{1}{k_{1\mathrm{R}}}$, $B_0 = k_{1\mathrm{I}}$, $X_0 = \dfrac{k_{1\mathrm{I}}}{2} - \dfrac{\delta_{1\mathrm{R}}^{(0)}}{k_{1\mathrm{R}}} - \dfrac{\theta_{13}}{2k_{1\mathrm{I}}}$, $\Theta_B(Z) = \dfrac{k_{1\mathrm{R}}^{@} - k_{1\mathrm{I}}^2}{4}\mathrm{e}^{2Z} + \delta_{1\mathrm{I}}^{(0)}$, we can rewrite Eq.(10) in the following form

$$U = \frac{1}{W(Z)}\mathrm{sech}\left(\frac{X - X_{\mathrm{c}}(Z)}{W(Z)}\right)$$

$$\times \exp[\mathrm{i}(-X^2/2 + B(Z)X + \Theta_B(Z))], \quad (11)$$

where

$$W(Z) = W_0 \mathrm{e}^{-Z}, \quad B(Z) = B_0 \mathrm{e}^{Z}, \quad (12)$$

$$X_\mathrm{c}(Z) = X_0 \mathrm{e}^{-Z} + B_0 \sinh(Z), \quad (13)$$

where X_c is the position of the self-similar wave centre; $W(Z)$, $B(Z)$ and $\Theta_B(Z)$ are the width, phase-shift coefficient and phase factor of the self-similar wave, respectively; W_0, B_0 and X_0 are initial values of the corresponding parameters. The solution (11) has been obtained in Ref.[10]. This bright self-similar wave is completely determined by two parameters k and $\delta^{(0)}$, i.e. W_0, B_0, X_0 and Θ_{B0}. The self-similar solitary wave solution derived here corresponds to hyperbolic secant pulses with a linear chirp which propagate without change in shape, only a scaling of their width and amplitude. It is remarkable that the peak amplitude grows as $A(Z) = 1/W(Z) = k_{1\mathrm{R}}\mathrm{e}^{Z}$, but the width of the bright self-similar wave tends to zero as $W(Z) = \dfrac{1}{k_{1\mathrm{R}}}\mathrm{e}^{-Z}$ when $Z \to \infty$. So the solution given by Eq.(10) can be applied to the problem of pulse compression in optical fibre amplifiers. It must be emphasized, however, that before this level of compression is reached in a real amplifier, the pulse evolution would be affected by higher order terms in the NLSE which are neglected in deriving this solution.

Another interesting feature of the novel self-similar wave solution is associated with the nontrivial dynamics of the solitary wave centre. The analysis of Eq.(13) reveals that if sign(X_0)=sign(B_0) and $|X_0| > |B_0|$, the magnitude of the centre coordinate attains its minimum value at a certain distance; otherwise it increases monotonically. These features are shown in Fig.1.

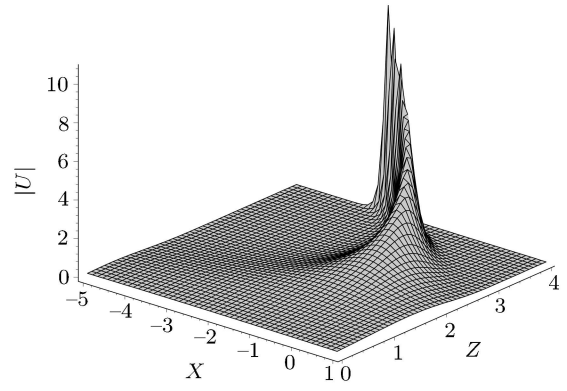


Fig.1. Evolution of a bright self-similar wave given by Eq.(9) with the parameter values $k_1 = 0.2 + 0.1I, \delta_1(0) = 0$.

Case 2 σ=–1, dark one-solitonlike:

Similarly, when $\sigma = -1$, we can obtain the dark one-solitonlike self-similar wave solution of Eq.(3),

$$\begin{aligned} U =& \frac{\alpha}{\sqrt{2}} \frac{1 + \mathrm{e}^{\delta_1 + 2\mathrm{i}\theta_1}}{1 + \mathrm{e}^{\delta_1}} \exp\left[Z - \frac{\mathrm{i}}{4}(2X^2 + \alpha^2 \mathrm{e}^{2Z}) \right], \\ \delta_1 =& k_1 X \mathrm{e}^{Z} + \frac{1}{4}\mathrm{e}^{2Z}\omega_1 + \delta_1^{'(0)}, \\ \omega_1 =& \pm k_1 \sqrt{2\alpha^2 - k_1^2}, \\ k_1 =& \sqrt{2}\alpha \sin(\theta_1), \end{aligned} \quad (14)$$

where θ_1 and $\delta_1^{'(0)}$ are arbitrary complex constants, α is a real constant. The properties of dark self-similar solitary wave expressed by Eq.(14) is similar to those of the bright self-similar wave.

2.3. Two-solitonlike self-similar waves and their collisions

Case 1 σ=1, bright two-solitonlike:

The bright two-solitonlike self-similar wave solution of Eq.(3) is

$$\begin{aligned} U =& \frac{\exp(Z - \frac{\mathrm{i}X^2}{2})}{\sqrt{2}} \frac{h_2}{f_2}, \\ h_2 =& \mathrm{e}^{\delta_1} + \mathrm{e}^{\delta_2} + \mathrm{e}^{\delta_1 + \delta_2 + \delta_1^* + \theta_{12} + \theta_{13} + \theta_{23}} \\ &+ \mathrm{e}^{\delta_1 + \delta_2 + \delta_2^* + \theta_{12} + \theta_{14} + \theta_{24}}, \\ f_2 =& 1 + \mathrm{e}^{\delta_1 + \delta_1^* + \theta_{13}} + \mathrm{e}^{\delta_1 + \delta_2^* + \theta_{14}} \\ &+ \mathrm{e}^{\delta_2 + \delta_1^* + \theta_{23}} + \mathrm{e}^{\delta_2 + \delta_2^* + \theta_{24}} \\ &+ \mathrm{e}^{\delta_1 + \delta_2 + \delta_1^* + \delta_2^* + \theta_{12} + \theta_{13} + \theta_{14} + \theta_{23} + \theta_{24} + \theta_{34}}, \end{aligned} \quad (15)$$

$$\begin{aligned} \delta_j =& k_j \mathrm{e}^{Z}\left(X + \frac{\mathrm{i}k_j \mathrm{e}^{Z}}{4} \right) + \delta_j^{(0)}, \quad (j = 1, 2), \\ \mathrm{e}^{\theta_{j,2+m}} =& \frac{1}{2(k_j + k_m^*)^2}, \quad (j, m = 1, 2), \\ \mathrm{e}^{12} =& 2(k_1 - k_2)^2, \\ \mathrm{e}^{34} =& 2(k_1^* - k_2^*)^2, \end{aligned} \quad (16)$$

where $k_1, k_2, \delta_1^{(0)}$ and $\delta_2^{(0)}$ are arbitrary complex constants. In Fig.2 we have a picture of a head-on collision with $k_1 = 0.1 + 0.1I, k_2 = 0.2 - 0.1I, \delta_1^{(0)} = 0$ and $\delta_2^{(0)} = 0$. After collision the bright two-solitonlike self-similar waves profiles remains unchanged and their amplitudes and widths simply scale with propagation distance, respectively.

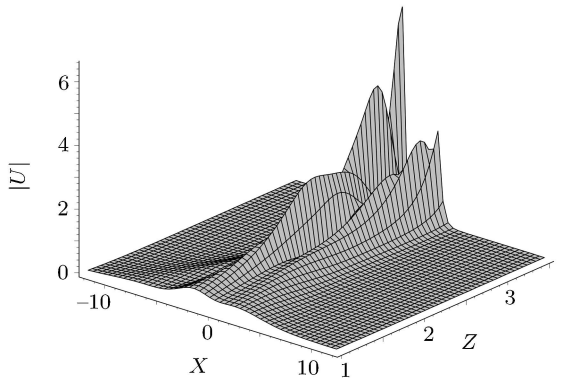


Fig.2. Head-on collision of bright two-solitonlike self-similar wave solution given by Eq.(15) with parameter values $k_1 = 0.1 + 0.1I, k_2 = 0.2 - 0.1I, \delta_1^{(0)} = 0, \delta_2^{(0)} = 0$.

Case 2 σ=–1, dark two-solitonlike:

The dark two-solitonlike self-similar wave solution of Eq.(3) is

$$
\begin{aligned}
U =& \frac{\alpha}{\sqrt{2}} \frac{1 + \mathrm{e}^{\delta_1 + 2\mathrm{i}\theta_1} + \mathrm{e}^{\delta_2 + 2\mathrm{i}\theta_2} + \mathrm{e}^{\delta_1 + \delta_2 + 2\mathrm{i}\theta_1 + 2\mathrm{i}\theta_2 + A_{12}}}{1 + \mathrm{e}^{\delta_1} + \mathrm{e}^{\delta_2} + \mathrm{e}^{\delta_1 + \delta_2 + A12}} \\
& \times \exp\left[Z - \frac{\mathrm{i}}{4}(2X^2 + \alpha^2 \mathrm{e}^{2Z}) \right], \\
\delta_j =& k_j X \mathrm{e}^Z + \frac{1}{4}\mathrm{e}^{2Z}\omega_j + \delta_j^{'(0)}, \\
\omega_j =& \pm k_j \sqrt{2\alpha^2 - k_j^2}, \\
k_j =& \sqrt{2}\alpha \sin(\theta_j), \ (j = 1, 2), \\
\mathrm{e}^{A_{12}} =& \left(\frac{\sin\left(\dfrac{\theta_1 - \theta_2}{2}\right)}{\sin\left(\dfrac{\theta_1 + \theta_2}{2}\right)} \right)^2 ,
\end{aligned} \tag{17}
$$

where θ_j and $\delta_j^{\prime(0)}, (j = 1, 2)$ are arbitrary complex constants, α is a real constant.

2.4. N-solitonlike self-similar waves

Case 1 σ=1, bright N-solitonlike:

The bright N-solitonlike self-similar wave solution of Eq.(3) is

$$
\begin{aligned}
U =& \frac{\exp\left(Z - \dfrac{\mathrm{i}X^2}{2} \right)}{\sqrt{2}} \frac{h_n}{f_n}, \\
h_n =& \sum_{\mu=0,1} A_2(\mu) \mathrm{e}^{\sum_{j=1}^{2n} \mu_j \delta_j + \sum_{1 \le j < m}^{2n} \mu_j \mu_m \theta_{jm}}, \\
f_n =& \sum_{\mu=0,1} A_1(\mu) \mathrm{e}^{\sum_{j=1}^{2n} \mu_j \delta_j + \sum_{1 \le j < m}^{2n} \mu_j \mu_m \theta_{jm}}, \\
\delta_j =& k_j \mathrm{e}^Z \left(X + \frac{\mathrm{i}k_j \mathrm{e}^Z}{4} \right) + \delta_j^{(0)}, \\
& (j = 1, 2, \cdots, n),
\end{aligned} \tag{18}
$$

where

$$
\begin{aligned}
& \delta_{n+j} = \delta_j^*, \ (j = 1, 2, \cdots, n), \\
& \mathrm{e}^{\theta_{j(n+m)}} = \frac{1}{2(k_j + k_m^*)^2}, \ (j, m = 1, 2, \cdots, n), \\
& \mathrm{e}^{jm} = 2(k_j - k_m)^2, \\
& \mathrm{e}^{\theta_{(n+j)(n+m)}} = 2(k_j^* - k_m^*)^2 = \mathrm{e}^{\theta_{jm}^*}, \\
& \quad (j < m = 2, 3, \cdots, n).
\end{aligned} \tag{19}
$$

$A_1(\mu)$ and $A_2(\mu)$ indicate that when $\mu_j = 0$ or $1 (j = 1, 2, \cdots, n)$ they must satisfy the condition as follows, respectively,

$$
\begin{aligned}
& \sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \mu_{n+j}, \\
& \sum_{j=1}^{n} \mu_j = \sum_{j=1}^{n} \mu_{n+j} + 1.
\end{aligned} \tag{20}
$$

Case 2 σ=–1, dark N-solitonlike:

The dark N-solitonlike self-similar wave solution of Eq.(3) is

$$
\begin{aligned}
U =& \frac{\alpha}{\sqrt{2}} \frac{\sum_{\mu=0,1} \mathrm{e}^{\sum_{j=1}^{n} \mu_j (\delta_j + 2\mathrm{i}\theta_j) + \sum_{1 \le j < m}^{n} \mu_j \mu_m A_{jm}}}{\sum_{\mu=0,1} \mathrm{e}^{\sum_{j=1}^{n} \mu_j \delta_j + \sum_{1 \le j < m}^{n} \mu_j \mu_m A_{jm}}} \\
& \times \exp\left[Z - \frac{\mathrm{i}}{4}(2X^2 + \alpha^2 \mathrm{e}^{2Z}) \right], \\
\delta_j =& k_j X \mathrm{e}^Z + \frac{1}{4}\mathrm{e}^{2Z}\omega_j + \delta_j^{'(0)}, \\
\omega_j =& \pm k_j \sqrt{2\alpha^2 - k_j^2}, \\
k_j =& \sqrt{2}\alpha \sin(\theta_j), \quad (j = 1, 2, \cdots, n), \\
\mathrm{e}^{A_{jm}} =& \left(\frac{\sin\left(\dfrac{\theta_j - \theta_m}{2}\right)}{\sin\left(\dfrac{\theta_j + \theta_m}{2}\right)} \right)^2 .
\end{aligned} \tag{21}
$$

3. Summary

In conclusion, we have found an appropriate transformation for the first time that the NLSE (3) with varying coefficients transform into standard NLSE (5). Using this transformation (4) and Hirota method, we have discovered and analytically described a wide class of spatial multi-solitonlike self-similar waves which can propagate in graded-index, nonlinear waveguide amplifiers. The bright and dark one-solitonlike solutions have been obtained in Ref.[10]. The two-solitonlike and N-solitonlike solutions are obtained for the first time. Then we further investigate the main features and collision of the bright solitary

wave solutions obtained by using direct computer simulations. Our results indicate that the profiles of these novel self-similar solitary wave solutions remain unchanged and their amplitudes and widths simply scale with propagation distance. The new self-similar solitary wave solutions obtained in this paper can be readily applied to pulse propagation in nonlinear optical fibre amplifiers, optical fibre compressors.

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