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The Lie symmetries and Noether conserved quantities of discrete mechanical systems with variable mass*

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This paper studies the Lie symmetries and Noether conserved quantities of discrete mechanical systems with variable mass. The discrete Euler–Lagrange equation and energy evolution equation are derived by using a total variational principle. The invariance of discrete equations under infinitesimal transformation groups is defined to be Lie symmetry. The condition of obtaining the Noether conserved quantities from the Lie symmetries is also presented. An example is discussed for applications of the results.

Keywords: discrete mechanics, variable mass system, Lie symmetry, Noether conserved quantity
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1. Introduction

Symmetries not only have an essential mathematical significance, but also they can provide profound insights into the behaviour of physical systems and play an important role in identifying invariant quantities in mechanical systems. Recently, the symmetries and conserved quantities have been thoroughly studied in constrained mechanical systems.[1−20]

It is well known that the variational principle plays an essential role in obtaining the equations of motion in mechanics. A number of authors have since proposed variational treatments of discrete mechanical systems: Lee[21,22] used time as a discrete dynamical variable, Guo *et al*[23−26] presented a discrete difference variational principle and introduced the concept of Euler–Lagrange cohomology, Marsden *et al* [27−29] developed a discrete integration algorithm that preserves the structure of a system when using the variational method, and Zhang *et al*[30−33] and Fu *et al*[34] studied the discrete variational principles of several constrained mechanical systems.

There exist two symmetries in seeking conserved quantities: Noether symmetry is an invariance of the Hamilton action functional under infinitesimal transformation groups; Lie symmetry is an invariance of the differential equations of motion under the infinitesimal transformation groups.[35,36] These two symmetries recently have been extended to the discrete mechanical systems: Dorodnitsyn[37] have adapted the Noether symmetry theory to discrete Lagrange system, Levi *et al*[38−40] have generalized the Lie symmetry method to the research on invariance of discrete difference and differential–difference equations of motion.

The purpose of this paper is to apply the symmetry method in the discrete variable mass mechanical system and find the corresponding conserved quantities. We deduce the discrete equations of motion by using the total variational technique and define the invariance of discrete equations to be the Lie symmetry. We also derive the criterion when Noether conserved quantities may be obtained from the Lie symmetry.

2. The discrete equations of motion of variable mass system

We consider a one-dimensional mechanical system denoted by its Lagrangian $L(t,q,\dot{q})$. The non-potential generalized force, generalized reverse-thrust force and variable mass of the system are indicated as $Q(t,q,\dot{q})$, $P(t,q,\dot{q})$ and $m(t,q,\dot{q})$ respectively. The

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Hamilton action functional is defined by

$$S=\int_{t_1}^{t_2}L(t,q,\dot{q})\mathrm{d}t. \tag{1}$$

The point transformation groups of time and generalized coordinates are

$$t^*=t+\Delta t=t+\varepsilon\tau(t,q), \tag{2}$$

$$q^*=q+\Delta q=q+\varepsilon\xi(t,q), \tag{3}$$

where ε is a group parameter, Δt and Δq are the variational of time and generalized coordinates, then τ and ξ are generators of the infinitesimal transformation groups. If Eqs.(2) and (3) are generalized quasi-symmetrical transformations, then they should satisfy the condition[1]:

$$\Delta S=S^*-S=-\int_{t_1}^{t_2}(Q+P)\delta q\mathrm{d}t. \tag{4}$$

where Δ indicates the total variational of S. Using the total variational relationship

$$\Delta q=\delta q+\dot{q}\Delta t, \tag{5}$$

we find that the equation (4) becomes

$$\Delta S+\int_{t_1}^{t_2}(Q+P)\Delta q\mathrm{d}t-\int_{q_1}^{q_2}(Q+P)\Delta t\mathrm{d}q=0. \tag{6}$$

In discrete mechanics, the discrete versions of L, Q, P, m are respectively $L_\mathrm{d}(t_k,\,t_{k+1},\,q_k,\,q_{k+1})$, $Q_\mathrm{d}(t_k,\,t_{k+1},\,q_k,\,q_{k+1})$, $P_\mathrm{d}(t_k,\,t_{k+1},\,q_k,\,q_{k+1})$ and $m_\mathrm{d}(t_k,\,t_{k+1},\,q_k,\,q_{k+1})$ $(k=0,1,\cdots,N-1)$. The corresponding version of equation (6) becomes

$$\begin{aligned}&\Delta S_\mathrm{d}+\sum_{k=0}^{N-1}(Q_\mathrm{d}+P_\mathrm{d})(t_{k+1}-t_k)\Delta q_k\\&-\sum_{k=0}^{N-1}(Q_\mathrm{d}+P_\mathrm{d})(q_{k+1}-q_k)\Delta t_k=0,\end{aligned} \tag{7}$$

where S_d is discrete action sum and denoted as

$$S_\mathrm{d}=\sum_{k=0}^{N-1}L_\mathrm{d}(t_k,t_{k+1},q_k,q_{k+1})(t_{k+1}-t_k). \tag{8}$$

Adopting the fixed boundary $\Delta t_0=\Delta t_N=0$ and $\Delta q_0=\Delta q_N=0$, we find that the prolongation of equation (7) is

$$\begin{aligned}&\Delta\sum_{k=0}^{N-1}L_\mathrm{d}(\phi_k)(t_{k+1}-t_k)\\&+\sum_{k=0}^{N-1}[Q_\mathrm{d}(\phi_k)+P_\mathrm{d}(\phi_k)](t_{k+1}-t_k)\Delta q_k\\&-\sum_{k=0}^{N-1}[Q_\mathrm{d}(\phi_k)+P_\mathrm{d}(\phi_k)](q_{k+1}-q_k)\Delta t_k\\=&\sum_{k=1}^{N-1}\{D_3L_\mathrm{d}(\phi_k)(t_{k+1}-t_k)+D_4L_\mathrm{d}(\phi_{k-1})(t_k-t_{k-1})\\&+[Q_\mathrm{d}(\phi_k)+P_\mathrm{d}(\phi_k)](t_{k+1}-t_k)\}\Delta q_k\\&+\sum_{k=1}^{N-1}\{D_1L_\mathrm{d}(\phi_k)(t_{k+1}-t_k)\\&+D_2L_\mathrm{d}(\phi_{k-1})(t_k-t_{k-1})+L_\mathrm{d}(\phi_{k-1})-L_\mathrm{d}(\phi_k)\\&-[Q_\mathrm{d}(\phi_k)+P_\mathrm{d}(\phi_k)](q_{k+1}-q_k)\}\Delta t_k,\end{aligned} \tag{9}$$

where D_j is the partial derivative of the discrete function with respect to the argument j and $\phi_k=(t_k,t_{k+1},q_k,q_{k+1})$, the terms Δt_k and Δq_k represent the variational in t_k and q_k respectively. With the requirement of total variational principle $\Delta S=0$, we obtain the discrete Euler–Lagrange equation

$$\begin{aligned}&D_3L_\mathrm{d}(\phi_k)(t_{k+1}-t_k)+D_4L_\mathrm{d}(\phi_{k-1})(t_k-t_{k-1})\\&+[Q_\mathrm{d}(\phi_k)+P_\mathrm{d}(\phi_k)](t_{k+1}-t_k)=0\end{aligned} \tag{10}$$

from the variational Δq_k, and the discrete energy evolution equation

$$\begin{aligned}&D_1L_\mathrm{d}(\phi_k)(t_{k+1}-t_k)+D_2L_\mathrm{d}(\phi_{k-1})(t_k-t_{k-1})\\&+L_\mathrm{d}(\phi_{k-1})-L_\mathrm{d}(\phi_k)\\&-[Q_\mathrm{d}(\phi_k)+P_\mathrm{d}(\phi_k)](q_{k+1}-q_k)=0\end{aligned} \tag{11}$$

from the variational Δt_k for discrete variable mass system.

3. The Lie symmetries of discrete variable mass system

Suppose that the infinitesimal point transformation groups of discrete time and generalized coordinates are

$$t_k^*=t_k+\Delta t_k=t_k+\varepsilon\tau_k(t_k,q_k), \tag{12}$$

$$q_k^*=q_k+\Delta q_k=q_k+\varepsilon\xi_k(t_k,q_k), \tag{13}$$

where τ_k, ξ_k are the discrete generators. The vector field of generators is

$$X_\mathrm{d}^{(0)}=\tau_k\frac{\partial}{\partial t_k}+\xi_k\frac{\partial}{\partial q_k}, \tag{14}$$

and its two- and three-point prolongation schemes are respectively

$$X_{\rm d}^{(1)} = X_{\rm d}^{(0)} + \tau_{k+1}\frac{\partial}{\partial t_{k+1}} + \xi_{k+1}\frac{\partial}{\partial q_{k+1}}, \tag{15}$$

$$X_{\rm d}^{(2)} = X_{\rm d}^{(1)} + \tau_{k-1}\frac{\partial}{\partial t_{k-1}} + \xi_{k-1}\frac{\partial}{\partial q_{k-1}}. \tag{16}$$

The discrete Euler–Lagrange equation (10) and the discrete energy evolution equation (11) can be rewritten as

$$E(\varphi_k) = 0, \tag{17}$$

$$F(\varphi_k) = 0, \tag{18}$$

where $\varphi_k = (t_{k-1},\ t_k,\ t_{k+1},\ q_{k-1},\ q_k,\ q_{k+1})$, $(k = 1, \cdots, N-1)$. Applying prolongation Eq.(16) to Eqs.(17)–(18), we obtain

$$X_{\rm d}^{(2)}[E(\varphi_k)] = 0, \tag{19}$$

$$X_{\rm d}^{(2)}[F(\varphi_k)] = 0. \tag{20}$$

Definition If the infinitesimal generators τ_k and ξ_k satisfy Eqs.(19) and (20), then invariance of the discrete equations of motion Eqs.(10) and (11) is the Lie symmetry of the discrete variable mass system. Equations (19) and (20) are thus the determining equations of Lie symmetry.

By solving these equations, we can obtain generators for the infinitesimal Lie symmetry transformations.

4. The Noether conserved quantities of discrete variable mass system

We introduce the recursive and derivative operators of discrete systems for any discrete variable or function

$$R_{\pm}f(z_k) = f(z_{k\pm1}), \tag{21}$$

$$D_{\rm d}f(z_k) = \frac{R_+f(z_k) - f(z_k)}{t_{k+1} - t_k}. \tag{22}$$

Using Eqs.(21) and (22), Eqs.(10) and (11) can be represented as

$$\begin{aligned} &D_3L_{\rm d}(\phi_k) + C_tD_4[R_-L_{\rm d}(\phi_k)] \\ &+ Q_{\rm d}(\phi_k) + P_{\rm d}(\phi_k) = 0, \end{aligned} \tag{23}$$

$$\begin{aligned} &D_1L_{\rm d}(\phi_k) + C_tD_2[R_-L_{\rm d}(\phi_k)] - D_{\rm d}[R_-L_{\rm d}(\phi_k)] \\ &- [Q_{\rm d}(\phi_k) + P_{\rm d}(\phi_k)]D_{\rm d}(q_k) = 0, \end{aligned} \tag{24}$$

where $C_t = (t_k - t_{k-1})/(t_{k+1} - t_k)$.

Proposition If there exists a discrete gauge function $G_k(t_k, q_k)$ satisfy the identity

$$\begin{aligned} &L_{\rm d}D_{\rm d}(\tau_k) + X_{\rm d}^{(1)}(L_{\rm d}) + (Q_{\rm d} + P_{\rm d}) \\ &\times [\xi_k - D_{\rm d}(q_k)\tau_k] + D_{\rm d}(G_k) = 0, \end{aligned} \tag{25}$$

and, in addition, the Eqs.(23), (24) holds, then the discrete system with variable mass has the discrete Noether conserved quantities

$$\begin{aligned} &\tau_kR_-L_{\rm d}(\phi_k) + \tau_k(t_k - t_{k-1})D_2[R_-L_{\rm d}(\phi_k)] \\ &+\xi_k(t_k - t_{k-1})D_4[R_-L_{\rm d}(\phi_k)] + G_k = {\rm const} \end{aligned} \tag{26}$$

and Eq.(25) is called the discrete Noether identity.

Proof Using Eqs.(15), (21) and (22), we find that the extension of Noether identity (25) gives

$$\begin{aligned} &L_{\rm d}D_{\rm d}(\tau_k) + X_{\rm d}^{(1)}(L_{\rm d}) + (Q_{\rm d} + P_{\rm d}) \\ &\times [\xi_k - D_{\rm d}(q_k)\tau_k] + D_{\rm d}(G_k) \\ =&\tau_k\{D_1L_{\rm d}(\phi_k) + C_tD_2[R_-L_{\rm d}(\phi_k)] - D_{\rm d}[R_-L_{\rm d}(\phi_k)] \\ &-[Q_{\rm d}(\phi_k) + P_{\rm d}(\phi_k)]D_{\rm d}(q_k)\} \\ &+ \xi_k\{D_3L_{\rm d}(\phi_k) + C_tD_4[R_-L_{\rm d}(\phi_k)] \\ &+ Q_{\rm d}(\phi_k) + P_{\rm d}(\phi_k)\} \\ &+ D_{\rm d}\{\tau_kR_-L_{\rm d}(\phi_k) + \tau_k(t_k - t_{k-1})D_2[R_-L_{\rm d}(\phi_k)] \\ &+ \xi_k(t_k - t_{k-1})D_4[R_-L_{\rm d}(\phi_k)] + G_k\} = 0. \end{aligned} \tag{27}$$

Considering Eqs.(23), (24), we have

$$\begin{aligned} &D_{\rm d}\{\tau_kR_-L_{\rm d}(\phi_k) + \tau_k(t_k - t_{k-1})D_2[R_-L_{\rm d}(\phi_k)] \\ &+\xi_k(t_k - t_{k-1})D_4[R_-L_{\rm d}(\phi_k)] + G_k\} = 0. \end{aligned} \tag{28}$$

We therefore obtain the discrete Noether conserved quantities (26). It is now straightforward to acquire the criterion for obtaining the Noether conserved quantities from the Lie symmetries based on the discussion above.

Criterion If the Lie symmetry generators τ_k and ξ_k satisfy the discrete Noether identity Eq.(25), then the Lie symmetries of the discrete variable mass system lead to the discrete Noether conserved quantities Eq.(26).

5. An example

Let us take an example to illustrate this procedure. The discrete Lagrangian of the system and its recursive operator are

$$L_{\rm d}(\phi_k) = \frac{1}{2}m_{\rm d}(\phi_k)\left(\frac{q_{k+1} - q_k}{t_{k+1} - t_k}\right)^2 \tag{29}$$

and

$$R_-[L_{\rm d}(\phi_k)] = L_{\rm d}(\phi_{k-1}) = \frac{1}{2}m_{\rm d}(\phi_{k-1})\left(\frac{q_k - q_{k-1}}{t_k - t_{k-1}}\right)^2. \quad (30)$$

The non-potential generalized force, generalized reverse-thrust force and variable mass of the discrete system are respectively

$$Q_{\rm d}(\phi_k) = \beta\frac{q_{k+1} - q_k}{t_{k+1} - t_k}, \quad (31)$$

$$P_{\rm d}(\phi_k) = \lambda\frac{q_{k+1} - q_k}{t_{k+1} - t_k}, \quad (32)$$

$$m_{\rm d}(\phi_k) = m_0\left(1 - \frac{1}{2}\alpha t_k - \frac{1}{2}\alpha t_{k+1}\right). \quad (33)$$

Substituting Eqs.(29)–(33) into Eqs.(10) and (11) we obtain the discrete Euler–Lagrange equation and energy evolution equation

$$E = - m_{\rm d}(\phi_k)\frac{q_{k+1} - q_k}{t_{k+1} - t_k} + m_{\rm d}(\phi_{k-1})\frac{q_k - q_{k-1}}{t_k - t_{k-1}} + (\beta + \lambda)(q_{k+1} - q_k), \quad (34)$$

$$\begin{aligned} F =& \frac{1}{2}m_{\rm d}(\phi_k)\left(\frac{q_{k+1} - q_k}{t_{k+1} - t_k}\right)^2 \\ & - \frac{1}{2}m_{\rm d}(\phi_{k-1})\left(\frac{q_k - q_{k-1}}{t_k - t_{k-1}}\right)^2 \\ & - \frac{1}{4}\alpha m_0\left[\frac{(q_{k+1} - q_k)^2}{t_{k+1} - t_k} + \frac{(q_k - q_{k-1})^2}{t_k - t_{k-1}}\right] \\ & - (\beta + \lambda)\frac{(q_{k+1} - q_k)^2}{t_{k+1} - t_k}. \end{aligned} \quad (35)$$

Substituting Eqs.(34) and (35) into Eqs.(19) and (20) we obtain the determining equations of Lie symmetry

$$\begin{aligned} & \tau_{k-1}D_1E(\varphi_k) + \tau_kD_2E(\varphi_k) + \tau_{k+1}D_3E(\varphi_k) \\ & + \xi_{k-1}D_4E(\varphi_k) + \xi_kD_5E(\varphi_k) \\ & + \xi_{k+1}D_6E(\varphi_k) = 0, \end{aligned} \quad (36)$$

$$\begin{aligned} & \tau_{k-1}D_1F(\varphi_k) + \tau_kD_2F(\varphi_k) + \tau_{k+1}D_3F(\varphi_k) \\ & + \xi_{k-1}D_4F(\varphi_k) + \xi_kD_5F(\varphi_k) \\ & + \xi_{k+1}D_6F(\varphi_k) = 0. \end{aligned} \quad (37)$$

The representations of $D_jE(\varphi_k)$ and $D_jF(\varphi_k)$ in (36) and (37) can be calculated from Eqs.(34) and (35). Solving determining equations (36) and (37), we will obtain three groups of the generators of Lie symmetry transformation groups:

$$\tau_k = 0, \qquad \xi_k = 0, \quad (38)$$

$$\tau_k = 0, \qquad \xi_k = 1, \quad (39)$$

$$\tau_k = 0, \qquad \xi_k = q_k. \quad (40)$$

Substituting Eqs.(38) and (39) into the discrete Noether identity Eq.(25), we obtain the gauge function

$$G_k(t_k, q_k) = 0 \quad (41)$$

corresponding to the Lie symmetry generators Eq.(38), so the system has a conserved quantity

$$I_{\rm N} = 0, \quad (42)$$

then the gauge function

$$G_k(t_k, q_k) = -(\beta + \lambda)q_k \quad (43)$$

corresponding to the generators Eq.(39), this time the system has a conserved quantity

$$\begin{aligned} I_{\rm N} =& m_0(1 - \frac{1}{2}\alpha t_{k-1} - \frac{1}{2}\alpha t_k)\frac{q_k - q_{k-1}}{t_k - t_{k-1}} - (\beta + \lambda)q_k \\ =& \text{const} \end{aligned} \quad (44)$$

according to the discrete Noether conserved quantities Eq.(26), the representations of $D_2[R_-L_{\rm d}(\phi_k)]$, $D_4[R_-L_{\rm d}(\phi_k)]$ in Eq.(26) can be calculated from Eq.(30). As for the generators Eq.(40), there exists no gauge function satisfying the discrete Noether identity Eq.(25).

6. Conclusions

There are four main results in this paper: (i) the discrete total variational representation Eq.(7) with non-conservative terms; (ii) the discrete Euler–Lagrange equation (10) and the discrete energy evolution equation (11) of discrete variable mass system; (iii) the determining equations (19) and (20) of discrete system with variable mass; (iv) the discrete Noether identity Eq.(25) and Noether conserved quantities Eq.(26). Using the results in the paper, we can investigate other discrete mechanical systems such as the non-holonomic mechanical system, for not only the Lie symmetries but also the Noether conserved quantities.

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