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Kac–Moody–Virasoro symmetry algebra of a (2+1)-dimensional bilinear system*

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Based on some known facts of integrable models, this paper proposes a new (2+1)-dimensional bilinear model equation. By virtue of the formal series symmetry approach, the new model is proved to be integrable because of the existence of the higher order symmetries. The Lie point symmetries of the model constitute an infinite dimensional Kac–Moody–Virasoro symmetry algebra. Making use of the infinite Lie point symmetries, the possible symmetry reductions of the model are also studied.

Keywords: general symmetries, Kac–Moody–Virasoro symmetry algebra, symmetry reduction

PACC: 0220, 0230, 1130L

1. Introduction

To begin with the paper, firstly let us review two well-known integrable bilinear (2+1)-dimensional sine–Gordon model equations.

The first one has the form

$$D_yD_xD_tF\cdot G=0,\quad D_xD_tF\cdot G=0, \tag{1}$$

which is the bilinear form of the well known isotropic integrable sine-Gordon system[1−4]

$$\left(\frac{\phi_x}{\sin\theta}\right)_x-\left(\frac{\phi_y}{\sin\theta}\right)_y+\frac{\phi_y\theta_x-\phi_x\theta_y}{\sin^2\theta}=0, \tag{2}$$

$$\left(\frac{\phi'_x}{\sin\theta}\right)_x-\left(\frac{\phi'_y}{\sin\theta}\right)_y+\frac{\phi'_y\theta_x-\phi'_x\theta_y}{\sin^2\theta}=0. \tag{3}$$

Equations (2) and (3) are the special cases of the Loewner–Konopelchenko–Rogers (LKR) system derived from the extension of a gas dynamics context,[5] where Hirota's bilinear operators D_x, D_y, D_t are defined as

$$\begin{aligned}&D_x^mD_y^nD_t^kF\cdot G\\ &=\partial_{\epsilon_1}^m\partial_{\epsilon_2}^n\partial_{\epsilon_3}^kF(x+\epsilon_1,y+\epsilon_2,t+\epsilon_3)\\ &\quad\cdot G(x-\epsilon_1,y-\epsilon_2,t-\epsilon_3)|_{\epsilon_1=0,\epsilon_2=0,\epsilon_3=0},\end{aligned}$$

in which F and G are arbitrary functions of x, y and t.

The second one

$$(D_t+D_x^2)F\cdot G=0,\quad D_y(D_t+D_x^2)F\cdot G=0, \tag{4}$$

is a subcase of the bilinear form discussed in Refs.[11, 12] of the non-isotropic integrable sine–Gordon (or sinh–Gordon) system[13]

$$[\alpha\phi_{yt}+\mathrm{e}^{-2\phi}(\mathrm{e}^{2\phi}\phi_{xt})_x]_y=-(s_x\mathrm{e}^{2\phi})_{xx}, \tag{5}$$

$$\begin{aligned}&\left[\mathrm{e}^{2\phi}\left(\phi_{xt}-\frac{1}{2}C_1\mathrm{e}^{2\phi}+\frac{1}{2}C_2\mathrm{e}^{-2\phi}\right)\right]_x\\ &+\alpha\mathrm{e}^{2\phi}\phi_{yt}+\alpha\mathrm{e}^{2\phi}\left(\mathrm{e}^{2\phi}s\right)_x=0,\end{aligned} \tag{6}$$

where C_1, C_2 and α are arbitrary constants.

From the above two special examples, an interesting idea related to higher dimensional integrable models jumps into our minds: Is the model

$$(D_t+D_x^n)F\cdot G=0\quad D_y(D_t+D_x^n)F\cdot G=0 \tag{7}$$

integrable for some suitable n in addition to $n=2$?

The main purpose of this paper is to prove the integrability of the model (7) for $n=3$, i.e.,

$$(D_t+D_x^3)F\cdot G=0\quad D_y(D_t+D_x^3)F\cdot G=0 \tag{8}$$

via the symmetry analysis. This paper is organized as follows. In Section 2, by means of the well established formal series symmetry approach,[6−10] we study the infinitely many symmetries and the related Kac–Moody–Virasoro type symmetry algebra of the new model Eq.(8). Then, whereby the general symmetries obtained in Section 2, symmetry reductions are researched in Section 3. Finally, Section 4 is devoted to a short summary and some discussions.

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2. General symmetries and the Kac–Moody–Virasoro Lie symmetry algebra

A symmetry, σ, of an arbitrary equation

$$H(u) = 0$$

is defined as a solution of

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon} H(u + \epsilon\sigma) \mid_{\epsilon=0} = 0 \tag{9}$$

in Refs.[17, 18]. According to symmetry definition Eq.(9), the symmetries of the new model Eq.(8) follow

$$(D_y + D_x^3)F \cdot \sigma_G - (D_y + D_x^3)G \cdot \sigma_F = 0, \tag{10}$$

$$D_t(D_y + D_x^3)F \cdot \sigma_G + D_t(D_y + D_x^3)G \cdot \sigma_F = 0, \tag{11}$$

where $\sigma_F \equiv \sigma_F(x, y, t)$ and $\sigma_G \equiv \sigma_G(x, y, t)$, the symmetry of the new (2+1)-dimensional model Eq.(8). Next, do the formal series expansion of symmetries σ_F and σ_G in terms of variable y:

$$\sigma_F = \sum_{k=0}^{\infty} f^{(n-k)} \sigma_{1n}[k], \tag{12}$$

$$\sigma_G = \sum_{k=0}^{\infty} f^{(n-k)} \sigma_{2n}[k], \tag{13}$$

where f is an arbitrary function of the spatial variable y, $f^{(n-k)}$ denotes the $(n-k)$-th order derivatives of f, and that $\sigma_{1n}[k]$ and $\sigma_{2n}[k]$ are arbitrary functions of x, t, F, G and arbitrary order derivatives of F and G, but do not explicitly depend on the spatial variable y.

Note: both the mastersymmetry approach[14] and the recursion operator method[15] for obtaining higher order symmetries of an integrable model are difficult to apply in high dimensions. So, the formal series symmetry approach, a method well established by one author (Lou) of this paper, is taken to explore generalized symmetries of the new (2+1)-dimensional model.

The direct substitution of Eqs.(16) and (17) into the symmetry definition equations (10) and (11) yields

$$\sum_{k=1}^{\infty} f^{(n+1-k)}(D_y + D_x^3)(F \cdot \sigma_{2n}[k-1] - G \cdot \sigma_{1n}[k-1]) + \sum_{k=0}^{\infty} f^{(n+1-k)}(-F \cdot \sigma_{2n}[k] + G \cdot \sigma_{1n}[k]) = 0, \tag{14}$$

$$\sum_{k=1}^{\infty} f^{(n+1-k)}[D_t(D_y + D_x^3)(F \cdot \sigma_{2n}[k-1] + G \cdot \sigma_{1n}[k-1])] - \sum_{k=0}^{\infty} f^{(n+1-k)} D_t(F \cdot \sigma_{2n}[k] + G \cdot \sigma_{1n}[k]) = 0. \tag{15}$$

Because of the independence of arbitrary function f on F and G, all the coefficients of $f^{(n+1-k)}$ in Eqs.(14) and (15) have to be zeros for arbitrary k:

$$-F \cdot \sigma_{2n}[0] + G \cdot \sigma_{1n}[0] = 0, \quad -D_t F \cdot \sigma_{2n}[0] - D_t G \cdot \sigma_{1n}[0] = 0, \quad (k = 0) \tag{16}$$

$$(D_y + D_x^3)(F \cdot \sigma_{2n}[k-1] - G \cdot \sigma_{1n}[k-1]) - F \cdot \sigma_{2n}[k] + G \cdot \sigma_{1n}[k] = 0,$$

$$D_t(D_y + D_x^3)(F \cdot \sigma_{2n}[k-1] + G \cdot \sigma_{1n}[k-1]) - D_t(F \cdot \sigma_{2n}[k] + G \cdot \sigma_{1n}[k]) = 0, \tag{17}$$

where

$$k \geq 1.$$

Directly solving equations (16) and (17) with computer algebra Maple yields the following solutions

$$\sigma_{1n}[0] = g_{n0}(x, y)F, \quad \sigma_{2n}[0] = g_{n0}(x, y)G, \tag{18}$$

$$\sigma_{1n}[k] = -\frac{a}{G} + F \int \left(\frac{G_t a}{FG^2} + \frac{b - a_t}{2FG} \right) \mathrm{d}t,$$

$$\sigma_{2n}[k] = -G \int \left(\frac{G_t a}{FG^2} + \frac{b - a_t}{2FG} \right) \mathrm{d}t, \tag{19}$$

where

$$\begin{aligned} a =\ & (D_y + D_x^3)F \cdot \sigma_{2n}[k-1] \\ & -(D_y + D_x^3)G \cdot \sigma_{1n}[k-1], \end{aligned} \tag{20}$$

$$\begin{aligned} b =\ & D_t(D_y + D_x^3)F \cdot \sigma_{2n}[k-1] \\ & +D_t(D_y + D_x^3)G \cdot \sigma_{1n}[k-1]. \end{aligned} \tag{21}$$

To see symmetries Eq.(19) more clearly, we write $\sigma_{1n}[k]$, $\sigma_{2n}[k]$ in another equivalent form

$$\begin{pmatrix}\sigma_{1n}[k]\\ \sigma_{2n}[k]\end{pmatrix}=\begin{pmatrix}-\frac{1}{G}+F\partial_t^{-1}\left(\frac{G_t}{FG^2}-\frac{\partial_t}{2FG}\right) & -F\partial_t^{-1}\frac{1}{2FG}\\ -G\partial_t^{-1}\left(\frac{G_t}{FG^2}-\frac{\partial_t}{2FG}\right) & -G\partial_t^{-1}\frac{1}{2FG}\end{pmatrix}^k\begin{pmatrix}a\\ b\end{pmatrix}, \tag{22}$$

where

$$\begin{pmatrix}a\\ b\end{pmatrix}=\begin{pmatrix}(D_y+D_x^3)F & -(D_y+D_x^3)G\\ D_t(D_y+D_x^3)F & D_t(D_y+D_x^3)G\end{pmatrix}^k\begin{pmatrix}g_{n0}(x,y)G\\ g_{n0}(x,y)F\end{pmatrix}, \tag{23}$$

where we have replaced $\sigma_{1n}[0]$ and $\sigma_{2n}[0]$ with Eq.(18).
For example,

$$\sigma_{1n}[1] = -6F_xg_{n0}(x,y)_{xx}+Fg_{n1}(x,y), \tag{24}$$

$$\sigma_{2n}[1] = -6G_xg_{n0}(x,y)_{xx}+Gg_{n1}(x,y), \tag{25}$$

$$\begin{aligned}\sigma_{1n}[2] = {}& -6F_xg_{n1}(x,y)_{xx}+6F_xg_{n0}(x,y)_{xxxxx}+18F_{xx}g_{n0}(x,y)_{xxxx}+6F_xg_{n0}(x,y)_{xxy}\\ & -18F_yg_{n0}(x,y)_{xxx}+Fg_{n2}(x,y),\end{aligned} \tag{26}$$

$$\begin{aligned}\sigma_{2n}[2] = {}& -6G_xg_{n1}(x,y)_{xx}+6G_xg_{n0}(x,y)_{xxxxx}+18G_{xx}g_{n0}(x,y)_{xxxx}+6G_xg_{n0}(x,y)_{xxy}\\ & -18G_yg_{n0}(x,y)_{xxx}+Gg_{n2}(x,y),\end{aligned} \tag{27}$$

$$\sigma_{1n}[3] = -\frac{\sigma_{1n}[2]}{G}+F\int\left(\frac{G_t\sigma_{1n}[2]}{FG^2}+\frac{\sigma_{2n}[2]-\sigma_{1nt}[2]}{2FG}\right)\mathrm{d}t, \tag{28}$$

$$\sigma_{2n}[k] = -G\int\left(\frac{G_t\sigma_{1n}[2]}{FG^2}+\frac{\sigma_{2n}[2]-\sigma_{1nt}[2]}{2FG}\right)\mathrm{d}t, \tag{29}$$

where $g_{n1}(x,y)$ and $g_{n2}(x,y)$ are arbitrary integral functions.

Thus, by virtue of Eqs.(22) and (23), we now obtain the formal series symmetry as follows according to Eqs.(12) and (13):

$$\begin{aligned}\begin{pmatrix}\sigma_F\\ \sigma_G\end{pmatrix} = {}& \sum_{k=0}^{\infty}f^{(n-k)}\cdot\begin{pmatrix}-\frac{1}{G}+F\partial_t^{-1}\left(\frac{G_t}{FG^2}-\frac{\partial_t}{2FG}\right) & -F\partial_t^{-1}\frac{1}{2FG}\\ -G\partial_t^{-1}\left(\frac{G_t}{FG^2}-\frac{\partial_t}{2FG}\right) & -G\partial_t^{-1}\frac{1}{2FG}\end{pmatrix}^k\\ & \cdot\begin{pmatrix}(D_y+D_x^3)F & -(D_y+D_x^3)G\\ D_t(D_y+D_x^3)F & D_t(D_y+D_x^3)G\end{pmatrix}^k\cdot\begin{pmatrix}g_{n0}(x,y)G\\ g_{n0}(x,y)F\end{pmatrix}.\end{aligned} \tag{30}$$

What interests us here is the truncated symmetries of the formal series symmetry Eq.(30) at a proper truncation point n. Now we pay attention to looking for the truncated symmetries. If there exists a set of $\sigma_{1n}[M_n]$ and $\sigma_{2n}[M_n]$, which satisfies

$$\begin{aligned}&(D_y+D_x^3)F\sigma_{2n}[M_n]\\ &-(D_y+D_x^3)G\sigma_{1n}[M_n]=0,\end{aligned} \tag{31}$$

$$\begin{aligned}&D_t(D_y+D_x^3)F\sigma_{2n}[M_n]\\ &+D_t(D_y+D_x^3)G\sigma_{1n}[M_n]=0,\end{aligned} \tag{32}$$

we say symmetries $\sigma_{1n}(f)$ and $\sigma_{2n}(f)$ are truncated ones, which become

$$\begin{aligned}\sigma_{1n}(f) &= \sum_{k=0}^{M_n}f^{(n-k)}\sigma_{1n}[k],\\ \sigma_{2n}(f) &= \sum_{k=0}^{M_n}f^{(n-k)}\sigma_{2n}[k],\end{aligned} \tag{33}$$

because of

$$\sigma_{1n}[k]=0,\quad \sigma_{2n}[k]=0$$

for any

$$k>M_n,$$

according to recursion relation Eq.(22). Complicated but careful calculations show us that Eqs.(31) and (32) are true for the integral functions being taken as

$$g_{n0}(x,y)=\frac{x^n}{n!},\quad (n=0,1,2,3,\ldots) \tag{34}$$

and

$$M_0=0,\quad M_n=n-1,\quad (n=1,2,3,\ldots). \tag{35}$$

For a clear understanding of truncated symmetries $\sigma_{1n}(f)$ and $\sigma_{2n}(f)$, we present the first four terms in the following with the help of Eq.(33) and Eqs.(24)–(29):

$$\sigma_0(f)=\begin{pmatrix} fF \\ fG \end{pmatrix}, \tag{36}$$

$$\sigma_1(f)=\begin{pmatrix} fxF \\ fxG \end{pmatrix}, \tag{37}$$

$$\sigma_2(f)=\begin{pmatrix} \frac{1}{2}f_yx^2F-6fF_x \\ \frac{1}{2}f_yx^2G-6fG_x \end{pmatrix}, \tag{38}$$

$$\sigma_3(f)=\begin{pmatrix} \frac{1}{6}x^3f_{yy}F-6f_yxF_x-18fF_y \\ \frac{1}{6}x^3f_{yy}G-6f_yxG_x-18fG_y \end{pmatrix}. \tag{39}$$

One can prove the correctness of the truncated symmetries Eqs.(36)–(39) by directly substituting them into symmetry equations (10) and (11).

Remarks

(a) Actually, combining the recursion relation Eq.(22) with Eq.(23) and the truncation conditions Eq.(34) with Eq.(35), we can obtain infinitely many truncated symmetries according to Eq.(33) in principle. However, the formidable difficulty in determining the integral functions makes it unpractical to obtain the higher-order truncated symmetries.

(b) Careful calculations also demonstrate us that truncated symmetries $\sigma_0(f)-\sigma_3(f)$ constitute a closed Kac–Moody–Virasoro Lie Algebra which reads

$$\begin{aligned}[\sigma_0(f),\sigma_1(f)] &= [\sigma_0(f),\sigma_2(f)]=[\sigma_0(f),\sigma_0(f)]\\ &=[\sigma_1(f),\sigma_1(f)]=[\sigma_2(f),\sigma_2(f)]\\ &=[\sigma_3(f),\sigma_3(f)]\\ &=0,\end{aligned} \tag{40}$$

$$[\sigma_0(f),\sigma_3(f)]=\begin{pmatrix} 18ff_yF \\ 18ff_yG \end{pmatrix}=\sigma_0(18ff_y), \tag{41}$$

$$[\sigma_1(f),\sigma_2(f)]=\begin{pmatrix} 6f^2F \\ 6f^2G \end{pmatrix}=\sigma_0(6f^2), \tag{42}$$

$$[\sigma_1(f),\sigma_3(f)]=\begin{pmatrix} 24xff_yF \\ 24xff_yG \end{pmatrix}=\sigma_1(24ff_y), \tag{43}$$

$$\begin{aligned}[\sigma_2,\sigma_3] &= \begin{pmatrix} 6(f_yf_{yyy}+f_{yy}^2)x^2F-72f_yf_{yy}F_x \\ 6(f_yf_{yyy}+f_{yy}^2)x^2G-72f_yf_{yy}G_x \end{pmatrix}\\ &=\sigma_2(12ff_y),\end{aligned} \tag{44}$$

where communicator is defined as

$$\begin{aligned}[A(u),B(v)] &= \frac{\partial}{\partial\epsilon}[A(u+\epsilon B(v))\\ &\quad -B(v+\epsilon A(u))]|_{\epsilon=0}.\end{aligned} \tag{45}$$

It is also known in Ref.[16] that if $A(u)$ and $B(v)$ are symmetries, so is $[A(u),B(v)]$ for the same model; this has been clearly reflected in the above Kac–Moody–Virasoro Lie algebra, namely Eqs.(40)–(44) literally. If we want to obtain infinitely dimensional Lie algebra in this way, a next higher order truncated symmetry as discussed about the KPE, IDLWE, NNVE models in Refs.[2–5] is needed. Fortunately, we can prove that

$$\begin{aligned}\sigma_4(f) &= \begin{pmatrix} \sigma_{14}(f) \\ \sigma_{24}(f) \end{pmatrix}=\sum_{k=0}^{k=3}f^{(4-k)}\begin{pmatrix} -\frac{1}{G}+F\partial_t^{-1}\left(\frac{G_t}{FG^2}-\frac{\partial_t}{2FG}\right) & -F\partial_t^{-1}\frac{1}{2FG} \\ -G\partial_t^{-1}\left(\frac{G_t}{FG^2}-\frac{\partial_t}{2FG}\right) & -G\partial_t^{-1}\frac{1}{2FG} \end{pmatrix}\\ &\cdot\begin{pmatrix} (D_y+D_x^3)F\cdot & -(D_y+D_x^3)G\cdot \\ D_t(D_y+D_x^3)F\cdot & D_t(D_y+D_x^3)G\cdot \end{pmatrix}\begin{pmatrix} \sigma_{13}[k] \\ \sigma_{23}[k] \end{pmatrix}\end{aligned} \tag{46}$$

is just the necessary higher order truncated symmetry that we need, which can be proved by substituting calculated result of Eq.(46) into the symmetry equations (10) and (11).

Thus now, a series of infinitely many truncated symmetries are obtained in the following recursive way

$$\sigma_{n+1}(f_2) = C_n[\sigma_4(f_1), \sigma_n(1)], \quad (n \geq 0) \tag{47}$$

where C_n is an n related constant, $f_2 = f_{1y}$ and f_1 is an arbitrary function of y. However, to get the concrete expressions of $\sigma_{n+1}(f_2)$ from Eq.(47) for large n is not an easy work because of the same difficulty in fixing the integrating functions from the symmetry definition equations (10) and (11). Actually, as in the cases of Refs.[2–5], we know that the higher order symmetries defined by Eq.(45) coincide with Eq.(33) up to some trivial constants C_n.

The existence of the infinitely many higher order generalized symmetries means that the model Eq.(8) is really integrable.

The function f in the formal series symmetry expansions Eqs.(12) and (13) may also be a function of t and/or x. However, the detailed calculations show us that the only truncated closed symmetries in these cases are

$$\sigma_{01}(h) = hF, \quad \sigma_{02}(h) = hG, \tag{48}$$

$$\sigma_{11}(h) = hF_t, \quad \sigma_{12} = hG_t, \tag{49}$$

where h is an arbitrary function of t. One can check the correctness of Eqs.(48) and (49) by direct substituting them into symmetry definition equations (10) and (11).

3. Symmetry reductions

When a symmetry, σ, is known, one can find the corresponding group invariant solutions of the model by solving the symmetry constraint condition

$$\sigma = 0 \tag{50}$$

and the original model at the same time. However, there is still no simple way to solve the symmetry constraint condition Eq.(50) for general higher order symmetries. Thus, here we just use the Lie point symmetries to get the related reductions[19] as usual.

Using the results of the last section, we find that the general Lie point symmetries of the new model Eq.(8) have the form

$$\begin{aligned}\sigma_F^1(x,y,t) =\ & (f + gx + j_y x^2 + a(t) + k_{yy}x^3)F \\ & -(12j + 36k_y x)F_x - 108kF_y \\ & +F_t v(t),\end{aligned} \tag{51}$$

$$\begin{aligned}\sigma_G^1(x,y,t) =\ & (f + gx + j_y x^2 + a(t) + k_{yy}x^3)G \\ & -(12j + 36k_y x)G_x - 108kG_y \\ & +G_t v(t),\end{aligned} \tag{52}$$

where f, g, j and k are all arbitrary functions of y, and $a(t)$ and $v(t)$ are arbitrary functions of t.

Next, some different cases have to be divided to discuss further.

Case 1 $k \neq 0$, $v(t) \neq 0$.

In this case, the new model system Eq.(8) is reduced to

$$\begin{aligned}&(D_\xi^3 - D_\eta)F_1 \cdot G_1 = 0, \\ &D_\eta(D_\eta - D_\xi^3)F_1 \cdot G_1 = 0,\end{aligned} \tag{53}$$

in which F_1 and G_1 are related to the original field F and G by

$$\begin{aligned}F &= F_1(\xi,\eta)\mathrm{e}^{p_1(x,y,t)}, \\ G &= G_1(\xi,\eta)\mathrm{e}^{p_1(x,y,t)},\end{aligned}$$

in which

$$\begin{aligned}p_1(x,y,t) =\ & -\frac{x}{k^{\frac{1}{3}}}\left(-9\int \frac{g}{k^{\frac{2}{3}}}\mathrm{d}y + 2\int \frac{j^2}{k^{\frac{5}{3}}}\mathrm{d}y + 18p_0\right) \\ & +\frac{jx^2 + k_y x^3}{108k} - \frac{q_1(y) + q_2(t)}{54},\end{aligned}$$

and

$$\begin{aligned}\xi &= \frac{x}{k^{\frac{1}{3}}} - \frac{1}{9}\int \frac{j}{k^{\frac{4}{3}}}\mathrm{d}y, \\ \eta &= -\int \frac{1}{k}\mathrm{d}y - 108\int \frac{1}{v(t)}\mathrm{d}t,\end{aligned}$$

where $q_1(y)$ and $q_2(t)$ are two new arbitrary functions of the indicated variables.

Case 2 $k = 0$, $v(t) \neq 0$

In this case, the general symmetry Eqs.{(51), (52)} is changed into

$$\begin{aligned}\sigma_F =\ & \left(f + gx + j_y x^2 + a(t)\right)F \\ & -12jF_x + v(t)F_t,\end{aligned} \tag{54}$$

$$\begin{aligned}\sigma_G =\ & \left(f + gx + j_y x^2 + a(t)\right)G \\ & -12jG_x + v(t)G_t.\end{aligned} \tag{55}$$

The corresponding reduction results of the model equation (8) are

$$(D_y + D_\eta)F_2 \cdot G_2 = 0, \tag{56}$$

$$\begin{aligned}&(-12j_y\eta - 6g)D_\eta^2 F_2 \cdot G_2 \\ &+12jD_\eta(D_\eta^3 + D_y)F_2 \cdot G_2 \\ &+12j_y(F_2 \cdot G_{2y} + F_{2y} \cdot G_2) \\ &\times(-2\eta g_y - 2\eta^2 j_{yy} - 2f_y)F_2 \cdot G_2 \\ &= 0,\end{aligned} \tag{57}$$

where F_2 and G_2 are related to F and G by

$$F = F_2(y,\eta)\mathrm{e}^{p_2(x,y,t)}, \quad G = G_2(y,\eta)\mathrm{e}^{p_2(x,y,t)}$$

in which

$$\begin{aligned} p_2(x,y,t) = & -48j_y j^2 \left(\int \frac{1}{v(t)}\mathrm{d}t\right)^3 \\ & -6j(g+2j_y x)\left(\int \frac{1}{v(t)}\mathrm{d}t\right)^2 \\ & -(gx+j_y x^2+f)\int \frac{1}{v(t)}\mathrm{d}t \\ & -\int \frac{a(t)}{v(t)}\mathrm{d}t, \end{aligned}$$

and

$$\eta = x + 12j\int \frac{1}{v(t)}\mathrm{d}t.$$

Case 3 $k \neq 0$, $v(t) = 0$.

In this special case, the general symmetries Eqs.(51) and (52) follow

$$\begin{aligned} \sigma_F = & \left(f + a(t) + gx + j_y x^2 + k_{yy}x^3\right)F \\ & -36xk_y F_x - 12jF_x - 108kF_y, \qquad (58) \\ \sigma_G = & \left(f + a(t) + gx + j_y x^2 + k_{yy}x^3\right)G \\ & -36xk_y G_x - 12jG_x - 108kG_y \qquad (59) \end{aligned}$$

and the corresponding reduction results are

$$\begin{aligned} & D_\xi^3 F_3 \cdot G_3 = 0, \\ & 54 D_t D_\xi^3 F_3 \cdot G_3 + \frac{\mathrm{d}a(t)}{\mathrm{d}t} F_3 \cdot G_3 = 0, \qquad (60) \end{aligned}$$

where F_3 and G_3 are related to the original F and G by

$$F = F_3(\xi,t)\mathrm{e}^{p_3(x,y,t)}, \quad G = G_3(\xi,t)\mathrm{e}^{p_3(x,y,t)}$$

in which

$$\begin{aligned} p_3(x,y,t) = & 81a(t)\int \frac{1}{k}\mathrm{d}y + \left(-9\int \frac{j}{k^{\frac{4}{3}}}\mathrm{d}y + \frac{81x}{k^{\frac{1}{3}}}\right) \\ & \times \int \frac{g}{k^{\frac{2}{3}}}\mathrm{d}y + 81\int \frac{f}{k}\mathrm{d}y \\ & + \left(2\int \frac{j}{k^{\frac{4}{3}}}\mathrm{d}y - \frac{18x}{k^{\frac{1}{3}}}\right)\int \frac{j^2}{k^{\frac{5}{3}}}\mathrm{d}y \\ & -2\int \frac{j^2}{k^{\frac{5}{3}}}\int \frac{j}{k^{\frac{4}{3}}}\mathrm{d}y\mathrm{d}y \\ & +9\int \frac{g}{k^{\frac{2}{3}}}\int \frac{j}{k^{\frac{4}{3}}}\mathrm{d}y\mathrm{d}y \\ & +\frac{81x^2(k_y x + j)}{k} \end{aligned}$$

and

$$\xi = \frac{x}{k^{\frac{1}{3}}} - \frac{1}{9}\int \frac{j}{k^{\frac{4}{3}}}\mathrm{d}y, \quad \eta = t.$$

Case 4 $k = 0$, $v(t) = 0$

In the last case, symmetries Eqs.(51) and (52) become

$$\begin{aligned} \sigma_F = & \left(f + gx + j_y x^2 + a(t)\right)F - 12jF_x, \\ \sigma_G = & \left(f + gx + j_y x^2 + a(t)\right)G - 12jG_x \end{aligned}$$

and the corresponding symmetry reduction reads

$$\begin{aligned} & D_y F_4 \cdot G_4 = 0, \\ & 12j^2 D_t D_y F_4 \cdot G_4 + \frac{\mathrm{d}a(t)}{\mathrm{d}t} g F_4 \cdot G_4 = 0, \qquad (61) \end{aligned}$$

where F_4 and G_4 are determined by

$$F = F_4(y,t)\mathrm{e}^{p_4(x,y,t)}, \quad G = G_4(y,t)\mathrm{e}^{p_4(x,y,t)}$$

in which

$$p_4(x,y,t) = \frac{x\left(3gx + 2j_y x^2 + 6f + 6a(t)\right)}{72j}.$$

4. Summary and discussion

In conclusion, in our paper, three main items of work have been done. Firstly, by the formal series symmetry approach, we have found generalized symmetries of the new model Eq.(8), viz. Eq.(33) with Eqs.(51) and (52), which demonstrates us that the new model Eq.(8) is integrable under the meaning that the model possesses infinitely many generalized symmetries. Secondly, the Kac–Moody–Virasoro type Lie point symmetry algebra of the new model Eq.(8) has been identified. Further, by means of the Lie point symmetries Eq.(33) investigated in Section 2, the symmetry reduction of the new model Eq.(8) is also explicitly obtained.

Thus far, we have proved that there are at least two integrable models in Eq.(7) for $n = 2$ and $n = 3$. Because of complicity of calculation, we have not yet found other integrable systems from Eq.(7).

Furthermore, reviewing the three integrable models, which are Eqs.(1), (4) and (8), namely

$$D_x D_t F \cdot G = 0, \qquad D_y D_x D_t F \cdot G = 0,$$

$$(D_t + D_x^2)F \cdot G = 0, \qquad D_y(D_t + D_x^2)F \cdot G = 0,$$

$$(D_t + D_x^3)F \cdot G = 0, \qquad D_y(D_t + D_x^3)F \cdot G = 0,$$

we may have the following more general ansatz: Ansatz: If

$$\begin{aligned} & P(D_t,\ D_x)F \cdot G = 0, \\ & Q(D_x)P(D_t,\ D_x)F \cdot G = 0 \qquad (62) \end{aligned}$$

is integrable, then

$$P(D_t,\ D_x)F\cdot G=0,$$
$$Q(S)P(D_t,\ D_x)F\cdot G=0, \tag{63}$$

where S is a bilinear operator related to an additional freedom which may be continuous, discrete and even supersymmetric, while $Q(S)$ and $Q(D_x)$ are bilinear polynomials of arbitrary operator S and bilinear operator D_x respectively, and $P(D_t,\ D_x)$ denotes a bilinear polynomial constituted by bilinear operator D_t and D_x.

Even if the ansatz is not correct, we believe that we can still obtain many new integrable models from the ansatz!

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