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The homotopic method of travelling wave solution for El Niño tropic sea–air coupled oscillator*

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The EI Niño and Southern Oscillation (ENSO) is an interannual phenomenon involved in the tropical Pacific sea–air interactions. In this paper, an asymptotic method of solving nonlinear equations for the ENSO model is proposed. And based on a class of oscillator of the ENSO model and by employing the method of homotopic mapping, the approximate solution of equations for the corresponding ENSO model is studied. It is proved from the results that homotopic method can be used for analysing the sea surface temperature anomaly in the equatorial Pacific of the sea–air oscillator for the ENSO model.

Keywords: El Niño-Southern oscillator model, nonlinear, homotopic mapping, sea–air oscillator
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1. Introduction

El Niño and Southern Oscillation (ENSO) is the abnormal events happening in the atmosphere and tropical Pacific ocean. The phenomenon of the ENSO is a very attractive object of research in the international academic circles.[1−9] Mo *et al* have considered a class of problems in atmospheric physics.[10−19] The oscillatory nature of ENSO involves both positive and negative sea–air feedbacks. And a positive sea surface temperature (SST) anomaly in the equatorial eastern Pacific has been considered. This anomaly reduces the zonal SST gradient and the strength of the southern oscillation circulation, resulting in weaker trade winds around the equator. The weaker trade winds in turn cause the ocean circulation to change and then reinforce the SST anomaly.

Recently, many scholars have investigated the nonlinear problem. Approximation methods have been developed and improved.[20−22] Using the asymptotic method, Mo *et al* considered a class of nonlinear problems in atmospheric physics and so on.[23−25] In this paper we study a class of ENSO model by using a simple and special method in approximate theory.

2. The model of sea–air oscillator of ENSO

In the ENSO oscillator model the variations of both the eastern and western Pacific anomaly patterns should be considered. Due to the complexity of ENSO, we study only an oscillator model i.e. a sea–air coupled oscillator model in this paper, By using the homotopic mapping method,[26−28] approximate solution of the equations for the model is obtained.

We now consider the following coupled equations, which describe the coupled effect between the atmosphere and ocean in the tropical Pacific:[8]

$$\frac{\partial u^{\mathrm{a}}}{\partial t}+\frac{\partial \phi}{\partial x}+Du^{\mathrm{a}}=0,$$
$$\frac{\partial \phi}{\partial t}+C_{\mathrm{a}}^{2}\frac{\partial u^{\mathrm{a}}}{\partial x}+D\phi=-Q,$$
$$\beta y u^{\mathrm{a}}+\frac{\partial \phi}{\partial y}=0,$$
$$\frac{\partial u^{\mathrm{s}}}{\partial t}+\frac{\partial \eta}{\partial x}+\mathrm{d}u^{\mathrm{a}}=\tau^{x},$$

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$$\frac{\partial \eta}{\partial t} + C_{\rm s}^2 \frac{\partial u^{\rm s}}{\partial x} + {\rm d}\eta = 0,$$
$$\beta y u^{\rm s} + \frac{\partial \eta}{\partial y} = 0,$$

where $u^{\rm a}$ and $u^{\rm s}$ are the temperature functions of atmosphere and ocean respectively; ϕ is the gravitational potential function of ocean; η is the mixed layer thickness of ocean; $C_{\rm a}$ and $C_{\rm s}$ are the gravitational wave speeds of atmosphere and ocean respectively; β is the Rossby parameter; Q is the calefaction rate of ocean for atmosphere; τ^x is the wind stress; D and d are Rayleigh and Newton cooling coefficients of ocean and atmosphere respectively. We take $Q = K_Q T$ and $\tau^x = K_S u^{\rm a}$, where T is the sea surface temperature, K_Q and K_S are rates of the sea surface temperature and the temperature functions of atmosphere respectively. And let $T = \kappa\eta$, where κ is a rate of the mixed layer thickness of ocean. Thus we have $Q = K_{\rm H}\eta$ and $K_{\rm H} = \kappa K_Q$. Through the dimensionless treatment:

$$x = L_x x', \quad t = (L_x/C_{\rm s})t',$$
$$y_{\rm a} = (C_{\rm a}/(2\beta))^{1/2} y',$$
$$y_{\rm s} = (C_{\rm s}/2\beta)^{1/2} y',$$
$$u^{\rm a} = C_{\rm a} u'^{\rm a}, \quad u^{\rm s} = C_{\rm s} u'^{\rm s},$$
$$\phi = C_{\rm a}^2 \phi', \quad \eta = C_{\rm s}^2 \eta',$$

where $L_x = C_{\rm s}/(2\beta)$, the original coupled equations can turn into the following coupled equations in the dimensionless form (the prime is omitted):

$$\frac{\partial u^{\rm a}}{\partial t} + \alpha^{-1}\frac{\partial \phi}{\partial x} + \tilde{D} u^{\rm a} = 0,$$
$$\frac{\partial \phi}{\partial t} + \alpha^{-1}\frac{\partial u^{\rm a}}{\partial x} + \tilde{D}\phi = -\tilde{K}_{\rm H}\eta, \tag{1}$$
$$\frac{1}{2} y u^{\rm a} + \frac{\partial \phi}{\partial y} = 0,$$
$$\frac{\partial u^{\rm s}}{\partial t} + \frac{\partial \eta}{\partial x} + \tilde{d} u^{\rm a} = \tilde{K}_S u^{\rm a}, \tag{2}$$
$$\frac{\partial \eta}{\partial t} + \frac{\partial u^{\rm s}}{\partial x} + \tilde{d}\phi = 0,$$
$$\frac{1}{2} y u^{\rm s} + \frac{\partial \eta}{\partial y} = 0, \tag{3}$$

where $\alpha = C_{\rm s}/C_{\rm a}$, $\tilde{D} = D/(2\beta C_{\rm a})$, $\tilde{d} = C_{\rm s}^2 d/(2\beta)$.

3. Homotopic mapping

Now we consider the coupled equations (1)–(3). Introducing a set of homotopic mappings[26−28] $H_i(W, U, V, p) : R^3 \times I \to R$, $(i = 1, 2, \ldots, 6)$, we have

$$\begin{aligned} H_i(u^{\rm a}, \phi, \eta, p) =& L_i(u^{\rm a}, \phi) - L_i(u_0^{\rm a}, \phi_0) \\ &+ p(L_i(u_0^{\rm a}, \phi_0) - N_i(\eta)), \\ &(i = 1, 2, 3), \end{aligned} \tag{4}$$

$$\begin{aligned} H_i(u^{\rm s}, \eta, u^{\rm a}, p) =& L_i(u^{\rm s}, \eta) - L_i(u_0^{\rm s}, \eta_0) \\ &+ p(L_i(u_0^{\rm s}, \eta_0) - N_i(u^{\rm a})), \\ &(i = 4, 5, 6), \end{aligned} \tag{5}$$

where $R = (-\infty, +\infty)$; $I = [0, 1]$; $(u_0^{\rm a}, u_0^{\rm s}, \phi_0, \eta_0)$ is an initial approximate solution of coupled equations; the operators L_i and N_i $(i = 1, 2, \ldots, 6)$ are

$$L_1(u^{\rm a}, \phi) = \frac{\partial u^{\rm a}}{\partial t} + \alpha^{-1}\frac{\partial \phi}{\partial x} + \tilde{D} u^{\rm a},$$
$$L_2(u^{\rm a}, \phi) = \frac{\partial \phi}{\partial t} + \alpha^{-1}\frac{\partial u^{\rm a}}{\partial x} + \tilde{D}\phi,$$
$$L_3(u^{\rm a}, \phi) = \frac{1}{2} y u^{\rm a} + \frac{\partial \phi}{\partial y},$$
$$L_4(u^{\rm s}, \eta) = \frac{\partial u^{\rm s}}{\partial t} + \frac{\partial \eta}{\partial x} + \tilde{d} u^{\rm s},$$
$$L_5(u^{\rm s}, \eta) = \frac{\partial \eta}{\partial t} + \frac{\partial u^{\rm s}}{\partial x} + \tilde{d}\eta$$
$$L_6(u^{\rm s}, \eta) = \frac{1}{2} y u^{\rm s} + \frac{\partial \eta}{\partial y},$$
$$N_i(\eta) = 0, \ \ (i = 1, 3), \quad N_2(\eta) = -\tilde{K}_{\rm H}\eta,$$
$$N_4(u^{\rm a}) = \tilde{K}_S u^{\rm a}, \ \ N_i(u^{\rm a}) = 0, \ \ (i = 5, 6).$$

Obviously, it follows from Eqs.(4) and (5) that $H_i(u^{\rm a}, \phi, \eta, 1) = 0$, $(i = 1, 2, 3)$ and $H_i(u^{\rm s}, \eta, u^{\rm a}, 1) = 0$ $(i = 4, 5, 6)$ are the same as coupled equations (1)–(3). Then the solution of coupled equations (1)–(3) is the same as the solution of $H_i = 0$, $(i = 1, 2, \ldots, 6)$ as $p \to 1$.

4. Approximate solution of coupled equations

In order to obtain the approximate solution of coupled Eqs.(1)–(3), we assume that the zeroth-order approximate solution for the system $L_i = 0$ $(i = 1, 2, \ldots, 6)$ in the homotopic mappings is $u_0^{\rm a}$, $u_0^{\rm s}$, ϕ_0, η_0. It is easy to see that from the first-order partial differential equations $L_i = 0$ $(i = 1, 2)$ and $L_i = 0$ $(i = 4, 5)$, we have

$$\frac{\partial(u_0^{\rm a} + \phi_0)}{\partial t} + \alpha^{-1}\frac{\partial(u_0^{\rm a} + \phi_0)}{\partial x} + \tilde{D}(u_0^{\rm a} + \phi_0) = 0, \tag{6}$$
$$\frac{\partial(u_0^{\rm a} - \phi_0)}{\partial t} - \alpha^{-1}\frac{\partial(u_0^{\rm a} - \phi_0)}{\partial x} + \tilde{D}(u_0^{\rm a} - \phi_0) = 0, \tag{7}$$
$$\frac{\partial(u_0^{\rm s} + \eta_0)}{\partial t} + \frac{\partial(u_0^{\rm s} + \eta_0)}{\partial x} + \tilde{d}(u_0^{\rm s} + \eta_0) = 0, \tag{8}$$

$$\frac{\partial(u_0^{\rm s}-\eta_0)}{\partial t}-\frac{\partial(u_0^{\rm s}-\eta_0)}{\partial x}+\tilde{d}(u_0^{\rm s}-\eta_0)=0. \tag{9}$$

The characteristic equations of Eq.(6) are

$$\frac{{\rm d}t}{1}=\frac{{\rm d}x}{\alpha^{-1}}=\frac{d(u_0^{\rm a}+\phi_0)}{\tilde{D}(u_0^{\rm a}+\phi_0)}. \tag{10}$$

It is not difficult to obtain the first integrals of Eq.(10), given as

$$x-\alpha^{-1}t=c_1 \quad \text{and} \quad \ln(u_0^{\rm a}+\phi_0)-\tilde{D}t=c_2,$$

where c_1 and c_2 are arbitrary constants. Thus the general solution for Eq.(6) is obtained as follows:

$$u_0^{\rm a}+\phi_0=\exp(\tilde{D}t+\Phi_{01}(x-\alpha^{-1}t,y)), \tag{11}$$

where Φ_{01} is an arbitrary function.

Similarly, we can obtain the general solution for Eqs.(7)–(9) as follows:

$$u_0^{\rm a}-\phi_0=\exp(\tilde{D}t+\Phi_{02}(x+\alpha^{-1}t,y)), \tag{12}$$

$$u_0^{\rm s}+\eta_0=\exp(\tilde{d}t+\Phi_{03}(x-t,y)), \tag{13}$$

$$u_0^{\rm a}-\eta_0=\exp(\tilde{d}t+\Phi_{04}(x+t,y)), \tag{14}$$

where Φ_{0i} $(i=2,3,4)$ are arbitrary functions. Considering $L_3(u_0^{\rm a},\phi_0)=L_6(u_0^{\rm s},\eta_0)=0$, we have

$$\begin{aligned}
\Phi_{01}(x-\alpha^{-1}t,y)&=\Psi_{01}(x-\alpha^{-1}t)-\frac{1}{4}y^2,\\
\Phi_{02}(x-\alpha^{-1}t,y)&=\Psi_{02}(x-\alpha^{-1}t)+\frac{1}{4}y^2,\\
\Phi_{03}(x-t,y)&=\Psi_{03}(x-t)-\frac{1}{4}y^2,\\
\Phi_{04}(x-t,y)&=\Psi_{04}(x-t)+\frac{1}{4}y^2,
\end{aligned}$$

where Ψ_{0i} $(i=1,2,3,4)$ are arbitrary functions too. And from expressions (11)–(14), we obtain the following zero-th order approximate travelling wave solution:

$$\begin{aligned}
u_0^{\rm a}=&\frac{1}{2}\bigg[\exp\bigg(\tilde{D}t+\Psi_{01}(x-\alpha^{-1}t)-\frac{y^2}{4}\bigg)\\
&+\exp\bigg(\tilde{D}t+\Psi_{02}(x+\alpha^{-1}t)+\frac{y^2}{4}\bigg)\bigg],\\
u_0^{\rm s}=&\frac{1}{2}\bigg[\exp\bigg(\tilde{d}t+\Psi_{03}(x-t)-\frac{y^2}{4}\bigg)\\
&+\exp\bigg(\tilde{d}t+\Psi_{04}(x+t)+\frac{y^2}{4}\bigg)\bigg],\\
\phi_0=&\frac{1}{2}\bigg[\exp\bigg(\tilde{D}t+\Psi_{01}(x-\alpha^{-1}t)-\frac{y^2}{4}\bigg)\\
&-\exp\bigg(\tilde{D}t+\Psi_{02}(x+\alpha^{-1}t)+\frac{y^2}{4}\bigg)\bigg],\\
\eta_0=&\frac{1}{2}\bigg[\exp\bigg(\tilde{d}t+\Psi_{03}(x-t)-\frac{y^2}{4}\bigg)\\
&-\exp\bigg(\tilde{d}t+\Psi_{04}(x+t)+\frac{y^2}{4}\bigg)\bigg].
\end{aligned}$$

Now we obtain a higher order approximate solution of coupled equations (1)–(3) by using the homotopic mappings (4) and (5) as shown below:

$$\begin{aligned}
u^{\rm a}&=\sum_{j=0}^{\infty}u_j^{\rm a}p^j, \quad u^{\rm s}=\sum_{j=0}^{\infty}u_j^{\rm s}p^j,\\
\phi&=\sum_{j=0}^{\infty}\phi_j p^j \ \text{ and } \ \eta=\sum_{j=0}^{\infty}\eta_j p^j.
\end{aligned} \tag{15}$$

Substituting expression (15) into $H_i(u^{\rm a},\phi,,p)=0$ $(i=1,2,3)$ and $H_i(u^{\rm s},\eta,,p)=0$ $(i=4,5,6)$, equating coefficients of terms with the same powers of p^j, we obtain

$$\begin{aligned}
&L_i(u_j^{\rm a},\phi_j)=N_i(\eta_{j-1}),\\
&(i=1,2,3) \ \text{and} \ (j=1,2,\ldots),
\end{aligned} \tag{16}$$

$$\begin{aligned}
&L_i(u_j^{\rm s},\eta_j)=N_i(u_{j-1}^{\rm a}),\\
&(i=4,5,6) \ \text{and} \ (j=1,2,\ldots).
\end{aligned} \tag{17}$$

From partial differential coupled equations $L_i(u_j^{\rm a},\phi_J)=N_i(\eta_{j-1})$ $(i=1,2)$ and $L_i(u_j^{\rm s},\eta_j)=N_i(u_{j-1}^{\rm a})$ $(i=4,5)$, of Eqs.(16) and (17), we can obtain

$$\begin{aligned}
&\frac{\partial(u_j^{\rm a}+\phi_j)}{\partial t}+\alpha^{-1}\frac{\partial(u_j^{\rm a}+\phi_j)}{\partial x}+\tilde{D}(u_j^{\rm a}+\phi_j)\\
&+\tilde{K}_{\rm H}\eta_{j-1}=0, \quad (j=1,2,\ldots),
\end{aligned} \tag{18}$$

$$\begin{aligned}
&\frac{\partial(u_j^{\rm a}-\phi_j)}{\partial t}-\alpha^{-1}\frac{\partial(u_j^{\rm a}-\phi_j)}{\partial x}+\tilde{D}(u_j^{\rm a}-\phi_j)\\
&-\tilde{K}_{\rm H}\eta_{j-1}=0, \quad (j=1,2,\ldots),
\end{aligned} \tag{19}$$

$$\begin{aligned}
&\frac{\partial(u_j^{\rm s}+\eta_j)}{\partial t}+\frac{\partial(u_j^{\rm s}+\eta_j)}{\partial x}+\tilde{d}(u_j^{\rm s}+\eta_j)\\
&-\tilde{K}_S u_{j-1}^{\rm a}=0, \quad (j=1,2,\ldots),
\end{aligned} \tag{20}$$

$$\begin{aligned}
&\frac{\partial(u_j^{\rm s}-\eta_j)}{\partial t}-\frac{\partial(u_j^{\rm s}-\eta_j)}{\partial x}+\tilde{d}(u_j^{\rm s}-\eta_j)\\
&-\tilde{K}_S u_{j-1}^{\rm a}=0, \quad (j=1,2,\ldots).
\end{aligned} \tag{21}$$

The characteristic equations of Eq.(18) are

$$\frac{{\rm d}t}{1}=\frac{{\rm d}x}{\alpha^{-1}}=\frac{{\rm d}(u_j^{\rm a}+\phi_j)}{\tilde{D}(u_j^{\rm a}+\phi_j)+\tilde{K}_{\rm H}\eta_{j-1}}, \quad (j=1,2,\ldots).$$

Similarly, we can obtain characteristic equations of Eqs.(19)–(21) and solve first integrals. From

$$\begin{aligned}
&L_3(u_j^{\rm a},\phi_J)=N_3(\eta_{j-1}) \quad \text{and}\\
&L_6(u_j^{\rm s},\eta_j)=N_6(u_{j-1}^{\rm a}), \quad (j=1,2,\ldots),
\end{aligned}$$

we can obtain travelling wave solutions of coupled equations (16) and (17) $u_j^{\rm a}$, $u_j^{\rm s}$, ϕ_j and η_j, $(j=1,2,\ldots)$ successively.

We can prove that the series in expression (15) are uniformly convergent on $p \in [0,1]$.[24−26] From the homotopic mappings (4) and (5), we obtain the mth order approximate solution of the original coupled equations as follows:

$$u^{\mathrm{a}} = \sum_{j=0}^{m} u_j^{\mathrm{a}}, \quad u^{\mathrm{s}} = \sum_{j=0}^{m} u_j^{\mathrm{s}},$$
$$\phi = \sum_{j=0}^{m} \phi_j, \quad \eta = \sum_{j=0}^{m} \eta_j.$$

5. Summary and discussion

In order to investigate the ENSO that is a very complicated natural phenomenon in atmospheric physics we need to develop a basic model for the sea–air oscillator. And we solve the nonlinear equations for the ENSO model by using the approximation method. The homotopic mapping method is a simple and valid method. Using the homotopic mapping method, the accuracy of solution is determined by using initial approximate solution of original coupled equations (1)–(3). In the homotopic mappings, we first assume the zeroth-order approximate solution to be u_0^{a}, u_0^{s}, ϕ_0, η_0, which is a set of solution for the coupled equations $L_i = 0$ $(i = 1, 2, \ldots, 6)$. Naturally, in this way, we can more quickly obtain the solution of the coupled equations (1)–(3). The homotopic mapping method is an approximate analytic method, which differs from the general numerical method. The expansion of solution using the homotopic mapping method can be made by continuously performing analytic operations. Thus, from approximate solutions, we can further study qualitatively and quantitatively the behaviours of the temperature anomaly in the equatorial Pacific, the thermocline anomaly, the zonal wind stress anomaly and so on, about which no discussion is given in this paper.

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