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Quantum communication in spin star configuration*

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This paper considers a generalized spin star system which can be solved exactly, with the central spin-$\frac{1}{2}$ system embedded in an outer ring of N spin-$\frac{1}{2}$ particles(denoted as spin bath). In this model, in addition to the central-outer interaction, each pair of nearest neighbour of the bath interacts within themselves. The general expressions of the eigenstates as well as the eigenvalues of the model are derived with the use of the symmetries of system. It analyses the quantum state transfer and the dynamical behaviour of entanglement created during quantum communication. It also analyses the efficiency of the configuration regarded as quantum phase covariant clone or decoherence model. Some interesting results are discovered concerning the properties of quantum communication in this model.

Keywords: quantum state transfer, phase covariant clone, entanglement dynamics
PACC: 7540F, 0367

1. Introduction

Recently, the studies of the spin star systems,[1−6] where a central spin 1/2 particle couples to spin bath of N spin 1/2 particles, have attracted a vast amount attention in the quantum community. As the spin star systems are appropriate models for solid-system, they bear actual practice relevance , especially in the quantum state transfer,[7] quantum cloning,[8] and quantum measurement.[9] The spin star configuration can also describe decoherence model[10] where a spin system surrounded by environment modelled as N spin-$\frac{1}{2}$ particles or, alternatively, a spin ring perturbed by environment modelled as a spin $\frac{1}{2}$ particle. In spite of the wide range of applications and its proven importance in quantum decoherence, the star configuration is rarely discussed systematically in the context of consideration the effect of interactions within the spin bath on the properties of the model. Motivated by this consideration, in this paper, we study a generalized spin star system which can be solved exactly. There exist couplings not only between the central spin and bath spins, but also between nearest neighbour spins within the bath. The goal of this paper is to analyse the properties in the process of quantum communication. Quantum state transfer is an important task in quantum communication. In the third part of this paper we focus on the quantum state transfer in this configuration and dynamical behaviour of entanglement generated in the process of quantum state transfer. The simple and universal expressions of instantaneous fidelity, as well as the average value, were calculated. The properties of the model were also analysed when the system modelled as quantum cloning and decoherence model. We examine the dynamical creation of entanglement as the system evolves under H. We show that for $\Delta = 0$ the transfer fidelity and the dynamical entanglement always reach optimal values as can be seen below. This implies that XX couplings prefer the fidelity of the quantum state transfer and entanglement creation.

This paper is organized as follows. In Section 2 we give the detail description of the model and derive the general expressions of the eigenstates and eigenvalues. Section 3 is devoted to the studies of quantum state transfer and entanglement dynamics during the quantum state transfer. We end the paper with a brief conclusion.

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2. The exact solutions of the generalized spin star system

Consider a system of $N+1$ spins with antiferromagnetic couplings arranged in a spin star configuration (See Fig.1): spin 0 is the middle one while spins 1 to N are the outer ones. A typical Hamiltonian which generalizes the star configuration may be written in the form

$$H = BS^z + \sum_{i=1}^{N} J_1(S_0^x S_i^x + S_0^y S_i^y + \Delta S_0^z S_i^z) + H_{\rm R}, \quad (1)$$

where $H_{\rm R} = \sum_{i=1}^{N} J_2\ \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1}$, with periodic boundary conditions $S_{N+1} = S_1$. Here $S^z = S_{\rm R}^z + S_0^z$ denotes the Z-component of the total spin of the system, and $S_{\rm R}^z = \sum_{i=1}^{n} S_i^z$ is the z-component of the total spin $S_{\rm R} = \sum_{i=1}^{n} S_i$ of the outer ring. Δ is the anisotropy parameter. Both of the couplings J_1 between the central spin and bath spins and the couplings J_2 within each pair of the nearest neighbours of the bath are assumed to be positive. For the model, two symmetries are essential for solving the system exactly. One is translational symmetry, i.e., the invariance of the Hamiltonian with exchange of any two bath spins. Another symmetry of this system is the rotational symmetry which leads to the conservation of the z-component of the total spin.

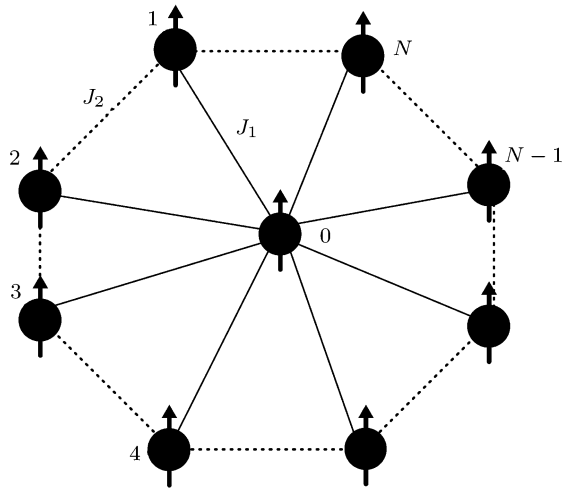


Fig.1. The figure depicts the general spin star configuration.

To calculate the energy eigenstates and the eigenvalues of the Hamiltonian, it is convenient to rewrite the Hamiltonian in raising and lowering operators in the following form

$$H = BS^z + J_1\left[\frac{1}{2}(S_0^+ S_{\rm R}^- + S_0^- S_{\rm R}^+) + \Delta S_0^z S_{\rm R}^z\right] + \sum_{i=1}^{N} J_2\left[\frac{1}{2}(S_i^+ S_{i+1}^- + S_i^- S_{i+1}^+) + S_i^z S_i^z\right], \quad (2)$$

where $S_{\rm R}^{\pm} = \sum_{i=1}^{N} S_i^{\pm}$, and $S_i^{\pm} \equiv S_i^x \pm i S_i^y$, is the usual raising(lowering) operator for the i'th outer spin or the central spin with $i = 0$. The standard basis for an individual qubit is chosen to be

$$|0\rangle_i = |\downarrow\rangle_i, \quad |1\rangle_i = |\uparrow\rangle_i, \quad i = 0, 1, 2, \ldots, N, \quad (3)$$

where $|\uparrow\rangle_i$ and $|\downarrow\rangle_i$ are the eigenvectors of the z-component of spin of an individual particle.

The following commutation relations are important to calculate the eigenstates and the eigenvalues of the star model:

$$[\,H,\ H_{\rm R}\,] = [\,H,\ S_{\rm R}^2\,] = [\,H,\ S^z\,] = 0, \quad (4)$$

$$[\,H_{\rm R},\ S_{\rm R}^2\,] = [\,H_{\rm R},\ S_{\rm R}^z\,] = [\,S_{\rm R}^2,\ S_{\rm R}^z\,] = 0. \quad (5)$$

Besides commutation relations, the relation $s = r \pm \frac{1}{2} \geq 0$ must be fulfilled, where s and r are the quantum numbers for S^2 and $S_{\rm R}^2$. Then, the commutation rules allows us to construct the energy eigenstates as follows:

$$|\psi\rangle = \alpha|1\rangle_0|\phi^N_{E_{\rm R}^r,r,m-\frac{1}{2}}\rangle_{\rm R} + \beta|0\rangle_0|\phi^N_{E_{\rm R}^r,r,m+\frac{1}{2}}\rangle_{\rm R}, \quad (6)$$

where the second ket in each term $|\phi^N_{E_{\rm R}^r,r,m\pm\frac{1}{2}}\rangle_{\rm R}$ stands for the eigenvectors of $H_{\rm R}, S_{\rm R}^2$ and $S_{\rm R}^z$ with corresponding eigenvalues $E_{\rm R}^r, r$ and $m \pm \frac{1}{2}$ of the outer ring. From the above eigenfunction and the relations we can immediately write down the following two functions:

$$\frac{J_1\beta\eta}{2} + \alpha\frac{J_1\Delta}{2}\left(m - \frac{1}{2}\right) + \alpha J_2 E_{\rm R}^r + \alpha B m = E\alpha, \quad (7)$$

$$\frac{J_1\alpha\eta}{2} - \beta\frac{J_1\Delta}{2}\left(m + \frac{1}{2}\right) + \beta J_2 E_{\rm R}^r + \beta B m = E\beta, \quad (8)$$

where $\eta = \sqrt{\left(r + m + \frac{1}{2}\right)\left(r - m + \frac{1}{2}\right)}$. To obtain the above expressions we have used the following relations:

$$S_0^+|0\rangle_0 = |1\rangle_0, \qquad S_0^-|1\rangle_0 = |0\rangle_0,$$

$$S_{\rm R}^+|\phi^N_{E_{\rm R}^r,r,m-\frac{1}{2}}\rangle_{\rm R} = \eta|\phi^N_{E_{\rm R}^r,r,m+\frac{1}{2}}\rangle_{\rm R},$$

$$S_{\rm R}^-|\phi^N_{E_{\rm R}^r,r,m+\frac{1}{2}}\rangle_{\rm R} = \eta|\phi^N_{E_{\rm R}^r,r,m-\frac{1}{2}}\rangle_{\rm R}.$$

If neither α nor β is equal to zero then we can get the energy eigenvalue

$$E = \frac{J_1}{2}\sqrt{\eta^2 + \Delta^2 m^2} - \frac{\Delta J_1}{4} + J_2 E_{\rm R}^r + Bm, \quad (9)$$

for $s = r + \frac{1}{2}$, $r = 1, 2, \ldots, \frac{N}{2}$ and

$$E = -\frac{J_1}{2}\sqrt{\eta^2 + \Delta^2 m^2} - \frac{\Delta J_1}{4} + J_2 E_{\mathrm{R}}^r + Bm, \quad (10)$$

for $s = r - \frac{1}{2}$, $r = 1, 2, \ldots, \frac{N}{2}$. Note that, for both cases, $m \neq \pm\left(\frac{1}{2} + r\right)$.

Using Eqs.(9) and (10), we can calculate the ratio $\frac{\alpha}{\beta}$,

$$\frac{\alpha}{\beta} = \frac{\pm\sqrt{\eta^2 + \Delta^2 m^2} + \Delta m}{\eta}, \quad m \neq \pm\left(\frac{1}{2} + r\right). \quad (11)$$

The above expressions for eigenstates and eigenvalues do not hold when either α or β equals to zero. There exist two special cases, namely, when $\alpha = 0$ and $\beta = 1$ or $\alpha = 1$ and $\beta = 0$.

In the case of $\alpha = 0, \beta = 1$, then the energy eigenstate becomes $|\downarrow\rangle_0 |\phi_{E_{\mathrm{R}}^r, r, -r}\rangle_{\mathrm{R}}$, and the corresponding eigenvalue is

$$E = \frac{1}{2} J_1 \Delta r + J_2 E_{\mathrm{R}}^r - B\left(r + \frac{1}{2}\right), \quad m = \frac{1}{2} + r, \quad (12)$$

for $r = 0, 1, 2, \ldots, \frac{N}{2}$. While in the case of $\beta = 0, \alpha = 1$, the eigenstate is $|\uparrow\rangle_0 |\phi_{E_{\mathrm{R}}^r, r, r}\rangle_{\mathrm{R}}$, and the corresponding eigenvalue is

$$\frac{1}{2} J_1 \Delta r + J_2 E_{\mathrm{R}}^r + B\left(r + \frac{1}{2}\right), \quad m = -\left(\frac{1}{2} + r\right), \quad (13)$$

for $r = 1, 2, \ldots, \frac{N}{2}$. The conditions of such states existing in the subspace of K-excitation subspace are given below. For $|\uparrow\rangle_0 |\phi_{E_{\mathrm{R}}^r, r, r}\rangle_{\mathrm{R}}$, when $r = r_m = \frac{1}{2}(K-1) + \left(-\frac{1}{2}\right)(N - K + 1) \geq 0$, i.e., $N \leq 2K - 2$, the kind of state can exist. For $|\downarrow\rangle_0 |\phi_{E_{\mathrm{R}}^r, r, r}\rangle_{\mathrm{R}}$, when $r = -r_m = -\left[\frac{1}{2}K + \left(-\frac{1}{2}\right)(N - K)\right] \geq 0$, i.e., $N \geq 2K$, the kind of state can exist. The more special properties of the model are determined by the properties of the surrounding outer ring.[11] When the solution for the Hamiltonian of the outer bath is known, then the total system is solvable.

3. Properties of quantum state transfer and entanglement dynamics

In this section, we turn to study the properties of quantum state transfer and the dynamical behaviour of entanglement created during quantum communication. The relevant quantity to quantify the quantum state transfer is given by transfer fidelity F. We can investigate the effect of the isotropic parameter Δ, the external control parameters B, J_1, J_2 and the number of bath spins N on the fidelity function, the central-bath entanglement (the entanglement between the central and individual spin of bath) and the in-bath entanglement (the entanglement between any two bath spins).

3.1. Quantum state transfer

In the context of the state transfer, only the dynamics of the first excitation sectors is relevant. So it suffices to restrict our attention in the N-dimensional single exciting subspace. The Hamiltonian matrix in the single excitation subspace has the following simple and symmetric form

$$\begin{pmatrix}
\mu & \frac{J_1}{2} & \frac{J_1}{2} & \frac{J_1}{2} & \frac{J_1}{2} & \ldots & \frac{J_1}{2} & \frac{J_1}{2} \\
\frac{J_1}{2} & \nu & \frac{J_2}{2} & 0 & 0 & \ldots & 0 & \frac{J_2}{2} \\
\frac{J_1}{2} & \frac{J_2}{2} & \nu & \frac{J_2}{2} & 0 & \ldots & 0 & 0 \\
\frac{J_1}{2} & 0 & \frac{J_2}{2} & \nu & \frac{J_2}{2} & \ldots & 0 & 0 \\
\frac{J_1}{2} & 0 & 0 & \frac{J_2}{2} & \nu & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{J_1}{2} & 0 & 0 & 0 & 0 & \ldots & \nu & \frac{J_2}{2} \\
\frac{J_1}{2} & \frac{J_2}{2} & 0 & 0 & 0 & \ldots & \frac{J_2}{2} & \nu
\end{pmatrix}. \quad (14)$$

Where $\mu = \frac{-J_1 \Delta N}{4} + \frac{J_2 N}{4} - \frac{B(N-1)}{2}$, and $\nu = \frac{\Delta J_1 (N-2)}{4} + \frac{J_2 (N-4)}{4} - \frac{B(N-1)}{2}$. In the single exciting subspace, the energy eigenstates and the corresponding eigen-energies can be calculated exactly with the following feature: the two eigenstates with the nonzero values of $\gamma_\pm$:

$$\gamma_\pm = \frac{\pm\sqrt{\zeta} + \Delta(1-N)}{2\sqrt{N}}, \quad (15)$$

where $\zeta = 4N + (N-1)^2 \Delta^2$.

The quantum state transfer protocol involves two steps: initialization and evolution. Initially we assume that the network in the state $|0\rangle_0 |\mathbf{0}\rangle_{\mathrm{R}}$. When it is time to use the configuration to transfer quantum state, a quantum state is sent, which is parameterized by the

angles (θ, φ) on the Bloch sphere as follows:

$$|\varphi\rangle_0 = \cos\frac{\vartheta}{2}|0\rangle_0 + \mathrm{e}^{\mathrm{i}\varphi}\sin\frac{\vartheta}{2}\beta|1\rangle_0, \tag{16}$$

this state is encoded at the central spin, then the total network is allowed to evolve under $U(t) = \mathrm{e}^{\mathrm{i}Ht}$ for a fixed time interval t. The state of the whole system at $t = 0$ is described as

$$|\varphi(t=0)\rangle = \cos\frac{\vartheta}{2}|0\rangle_0|\mathbf{0}\rangle_\mathrm{R} + \mathrm{e}^{\mathrm{i}\varphi}\sin\frac{\vartheta}{2}|1\rangle_0|\mathbf{0}\rangle_\mathrm{R}. \tag{17}$$

Rewrite the $|1\rangle_0|\mathbf{0}\rangle_\mathrm{R}$ as

$$\begin{aligned}&\frac{1}{\gamma_+ - \gamma_-}(\gamma_+ 1\rangle_0|\mathbf{0}\rangle_\mathrm{R} + |0\rangle_0\frac{1}{\sqrt{N}}\sum_{j=1}^{N}|\boldsymbol{j}\rangle_\mathrm{R})\\ &-\frac{1}{\gamma_+ - \gamma_-}(\gamma_-|1\rangle_0|\mathbf{0}\rangle_\mathrm{R} + |0\rangle_0\frac{1}{\sqrt{N}}\sum_{j=1}^{N}|\boldsymbol{j}\rangle_\mathrm{R}).\end{aligned} \tag{18}$$

Then the time evolution of $|1\rangle_0|\mathbf{0}\rangle_\mathrm{R}$, i.e. $\mathrm{e}^{-\mathrm{i}Ht}|1\rangle_0|\mathbf{0}\rangle_\mathrm{R}$ can be calculated as follows,

$$\begin{aligned}&\frac{\mathrm{e}^{-\mathrm{i}\varepsilon_1 t}}{\gamma_+ - \gamma_-}(\gamma_+|1\rangle_0|\mathbf{0}\rangle_\mathrm{R} + |0\rangle_0\frac{1}{\sqrt{N}}\sum_{j=1}^{N}|\boldsymbol{j}\rangle_\mathrm{R})\\ &-\frac{\mathrm{e}^{-\mathrm{i}\varepsilon_2 t}}{\gamma_+ - \gamma_-}(\gamma_-|1\rangle_0|\mathbf{0}\rangle_\mathrm{R} + |0\rangle_0\frac{1}{\sqrt{N}}\sum_{j=1}^{N}|\boldsymbol{j}\rangle_\mathrm{R}),\end{aligned} \tag{19}$$

where

$$\varepsilon_{1,2} = \pm\frac{J_1}{4}\sqrt{\zeta} - \frac{\Delta J_1}{4} + \frac{NJ_2}{4} + \frac{1-N}{2}B. \tag{20}$$

After the time evolution, the output state of the whole system is

$$|\varphi(t)\rangle = \chi_0|0\rangle_0|\mathbf{0}\rangle_\mathrm{R} + \chi_1|1\rangle_0|\mathbf{0}\rangle_\mathrm{R} + \frac{\chi_2}{\sqrt{N}}|0\rangle_0\sum_{j=1}^{N}|\boldsymbol{j}\rangle_\mathrm{R}, \tag{21}$$

here

$$\begin{aligned}\chi_0 &= \cos\frac{\vartheta}{2}\mathrm{e}^{-\mathrm{i}\varepsilon_0 t},\\ \chi_1 &= \mathrm{e}^{\mathrm{i}\varphi}\sin\frac{\vartheta}{2}\frac{(\gamma_+\mathrm{e}^{-\mathrm{i}\varepsilon_1 t} - \gamma_-\mathrm{e}^{-\mathrm{i}\varepsilon_2 t})}{\gamma_+ - \gamma_-},\\ \chi_2 &= \mathrm{e}^{\mathrm{i}\varphi}\sin\frac{\vartheta}{2}\frac{(\mathrm{e}^{-\mathrm{i}\varepsilon_1 t} - \mathrm{e}^{-\mathrm{i}\varepsilon_2 t})}{\gamma_+ - \gamma_-}.\end{aligned} \tag{22}$$

The reduced density matrix of any bath spin j in the outer ring after a time t, which is irrelevant to the site chosen for translational symmetry in the form, is given by

$$\rho_j(t) = \begin{pmatrix} 1 - \frac{1}{N}|\chi_2|^2 & \frac{1}{\sqrt{N}}\chi_0\chi_2^* \\ \frac{1}{\sqrt{N}}\chi_2\chi_0^* & \frac{1}{N}|\chi_2|^2 \end{pmatrix}. \tag{23}$$

A measure of the overlap between the input state and the output state is the fidelity. From the output state, we can calculate the fidelity of quantum transfer

$$\begin{aligned}F_j(t) = &\left(\cos\frac{\vartheta}{2}\right)^2\left(1 - \frac{2}{N}|\chi_2|^2\right) + \frac{1}{N}|\chi_2|^2\\ &+ 2\mathrm{Re}\left[\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}\mathrm{e}^{-\mathrm{i}\varphi}\chi_0\frac{1}{\sqrt{N}}\chi_2^*\right].\end{aligned} \tag{24}$$

Since the $|0\rangle_0|\mathbf{0}\rangle_\mathrm{R}$ is invariant under the evolution except for a global phase factor, it suffices to restrict our attention on the evolution of $|1\rangle_0|\mathbf{0}\rangle_\mathrm{R}$ of the initial state. Consequently, it is convenient to consider the transfer fidelity f when $\vartheta = \pi$,

$$f = \sqrt{\frac{1}{N}|\chi_2|^2} = \frac{2\sqrt{\sin\left(\frac{\sqrt{\zeta}}{4}J_1 t\right)^2}}{\sqrt{\zeta}}. \tag{25}$$

Immediately, we can see that the fidelity of quantum transfer is an even function of the anisotropic parameter Δ and a periodic function of J_1 and the time of evolution t. When the anisotropic parameter is equal to zero, the transfer fidelity reaches maximum

$$f^{\max} = f(\Delta = 0) = \frac{|\sin\left(\frac{\sqrt{N}}{2}J_1 t\right)|}{\sqrt{N}}. \tag{26}$$

Apparently, there is no influence of the coupling J_2 and the magnetic field B on the transfer fidelity.

Now we turn to estimate the efficiency of the model which is quantified by average value. The average fidelity $\bar{F}_j$ of quantum state transfer averaged over all input states in the Bloch sphere is

$$\begin{aligned}\bar{F}_j = &\frac{1}{2} + \frac{2}{3\zeta}\sin\frac{J_1\sqrt{\zeta}t}{4} - \frac{2}{3\sqrt{\zeta}}\\ &\times\cos\frac{J_1\sqrt{\zeta}t}{4}\cos\frac{(4B - J_1\Delta)t}{4}.\end{aligned} \tag{27}$$

In order to maximize the value of average fidelity as given in Eq.(27) we want to choose the parameters of magnetic field B and the coupling parameter J_1 appropriately. When $\frac{J_1 t}{4} = \frac{2k\pi + \pi/2}{\sqrt{\zeta}}$ and $Bt = \frac{2k\pi + \pi/2}{\sqrt{\zeta}}\Delta + 3\pi/2$, or $\frac{J_1 t}{4} = \frac{2k\pi + 3\pi/2}{\sqrt{\zeta}}$ and $Bt = \frac{2k\pi + 3\pi/2}{\sqrt{\zeta}}\Delta + \pi/2$, $k = 0, 1, 2, \ldots, \infty$, the average fidelity of the bath spin reaches maximum

$$\bar{F}_j^{\max} = \frac{1}{2} + \frac{2}{3\zeta} + \frac{2}{3\sqrt{\zeta}}. \tag{28}$$

Figure 2 shows the plot of the average fidelity of the bath spin as a function of the isotropic parameter Δ

and the number N of the bath spins (from top to bottom $N = 2, 3, \ldots, 20$). From Fig.2 we see that the average fidelity of the bath is even function of the anisotropic parameter Δ and decreases with N and the absolute value of Δ.

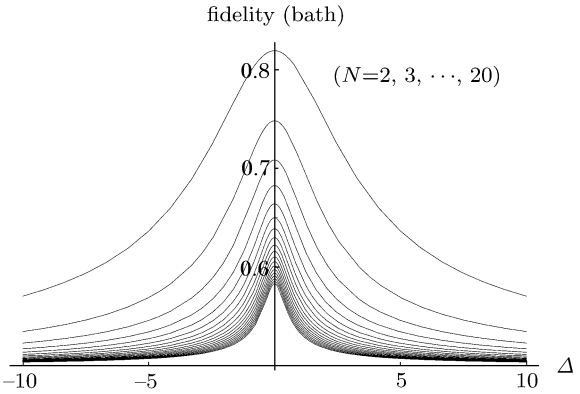


Fig.2. The dependence of the average fidelity of the bath spin on the isotropic parameter and the number of the bath spins.

To find out the optimal fidelity of quantum transfer, it is reasonable to focus here on the case of $\Delta = 0$. The optimal fidelity $\bar{F}_{\rm opt}$ in the model for various value of N are listed in Table 1.

Table 1. The optimal average fidelity $\bar{F}_{\rm opt}$ as a function of N.

N	2	3	4	5	6	7	∞
F	0.819	0.748	0.708	0.682	0.664	0.65	0.5

In limiting case of N, $\bar{F}_{\rm opt} = 0.5$.

The configuration can also be modelled as phase covariant clone[12] where the initial state is known to lie on the equator of the Bloch sphere with $\vartheta = \dfrac{\pi}{2}$. For arbitrary N we get the general expression of the maximal fidelity of the phase covariant clone:

$$F_{\rm clone} = \frac{1}{2} - \frac{\sin\left(\dfrac{1}{4}\sqrt{\zeta}J_1 t\right)\sin\left[\left(B - \dfrac{\Delta J_1}{4}\right)t\right]}{\sqrt{\zeta}}. \quad (29)$$

When $J_1 t = \dfrac{8k\pi + 2\pi}{\sqrt{\zeta}}$, and $Bt = \dfrac{(2k\pi + \pi/2)\Delta}{\sqrt{\zeta}} + \dfrac{3\pi}{2}$, the fidelity of the quantum phase covariant clone reaches maximal value

$$F_{\rm clone}^{\max} = \frac{1}{2} + \frac{1}{\sqrt{4N + (1-N)^2\Delta^2}}. \quad (30)$$

In order to compare our results with those obtained in Ref.[8], we consider two special cases of $\Delta = 0, 1$. It can immediately be seen that

$$F_{\rm clone}^{\max}(\Delta = 1) = \frac{1}{2} + \frac{1}{1+N}, \quad (31)$$

$$F_{\rm clone}^{\max}(\Delta = 0) = \frac{1}{2} + \frac{1}{2\sqrt{N}}. \quad (32)$$

These results coincide with the results in the Ref.[8] which is a special case of our model. Figure 3 shows the plot of the fidelity of the phase covariant clone as a function of the isotropic parameter Δ and the number N of the bath spins (from top to bottom N=2, 3, 4, 6, 8, 10, 20, 50, 100). From Fig.3 we see that the average fidelity of the bath is even function of the anisotropic parameter Δ and decreases with N and the absolute value of Δ.

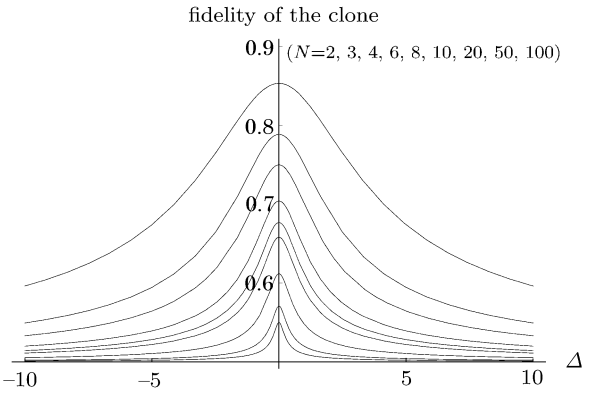


Fig.3. The dependence of the fidelity of the clone on the isotropic parameter and the number of the bath spins.

Similarly, we can calculate the reduced density matrix of the central spin

$$\rho_0(t) = \begin{pmatrix} 1 - |\chi_1|^2 & \chi_0\chi_1^* \\ \chi_1\chi_0^* & |\chi_1|^2 \end{pmatrix}. \quad (33)$$

And the fidelity of the central spin is

$$F_0(t) = \left(\cos\frac{\vartheta}{2}\right)^2 (1 - 2|\chi_1|^2) + |\chi_1|^2 + 2\mathrm{Re}\left[\cos\frac{\vartheta}{2}\sin\frac{\vartheta}{2}\mathrm{e}^{-\mathrm{i}\varphi}\chi_0\chi_1^*\right], \quad (34)$$

where

$$|\chi_1| = \frac{\sqrt{N\left(\dfrac{\zeta - 2N}{N} + 2\cos\left(\dfrac{\sqrt{\zeta}}{2}J_1 t\right)\right)}}{\sqrt{\zeta}}. \quad (35)$$

Note that when we model the bath spins as decoherence environment, the fidelity of the central spin is just the decohered factor which characterizes the efficiency of the central system under the influence of a decoherence environment. When $\dfrac{\sqrt{\zeta}}{2}J_1 t = 2k\pi$, $|\chi_1|$ reaches its maximal value 1 irrelevant to Δ and N. From this result we can conclude that the fidelity of

the central spin is equal to unit when $\frac{\sqrt{N}}{2}J_1 t = 2k\pi$ for arbitrary N.

Accordingly, we can calculate the average fidelity $\bar{F}_0$ of the central spin averaged over all input states in the Bloch sphere as follows:

$$\begin{aligned}\bar{F}_0 = \frac{1}{2} &+ \frac{\zeta-(N-1)\Delta}{\sqrt{\zeta}} \\ &\times \cos\left[\frac{J_1(4B-(N+1)\Delta+\sqrt{\zeta})}{2}\right] \\ &+ \frac{\zeta+(N-1)\Delta}{\sqrt{\zeta}} \\ &\times \cos\left[\frac{J_1(4B-(N+1)\Delta-\sqrt{\zeta})}{2}\right]. \end{aligned} \quad (36)$$

When $\frac{J_1 t}{4} = \frac{(k-k')\pi}{\sqrt{\zeta}}$ and $Bt = \frac{(k-k')\pi}{\sqrt{\zeta}}(\Delta + N\Delta + \sqrt{\zeta}) + 2k'\pi + \pi/2$, the average fidelity of the bath spin reaches its maximal value $\frac{5}{6}$ which is a constant irrelevant to the total number of spins. This is a somewhat surprising result which contrasts with the usually naive expectation that even small couplings of the surrounding environment will eventually result in complete decoherence of the central system. We can conclude that the fidelity of state transfer is only affected by J_1, Δ and B, yet is irrelevant to J_2. That is to say, the intra-bath couplings have no effect on the dynamics of quantum state transfer behaviour.

3.2. Entanglement dynamics

In this section, we turn to study the properties of dynamical creation of entanglement in this model. One can expect that the coupling parameter J_1 can control dynamical behaviour of entanglement, as the interaction is mediated by the centre one. Indeed, we can see that not only the coupling between the central spin and outer spins, but also the degree of rotation of the initial state of the central spin will control the entanglement distribution behaviour of the system. An natural measure to determine the two-party entanglement is the concurrence C. According to Ref.[13], the concurrence of two spin $\frac{1}{2}$ particles may be calculated from their density matrix via the formula

$$C(\rho) = \max[0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4], \quad (37)$$

where λ_i are the eigenvalues in the decreasing order of the hermitian matrix $\sqrt{\sqrt{\rho}\tilde{\rho}\sqrt{\rho}}$, and $\tilde{\rho} = (\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$.

The exchange symmetry of the bath spins implies that the entanglement between the central spin and any bath spin j is same. By tracing out ρ_{out} over $(N-2)$ spins of the system except the two selected spins, we can obtain the reduced density between the central spin and bath spin j. In the standard basis $|0\rangle|0\rangle, |0\rangle|1\rangle, |1\rangle|0\rangle, |1\rangle|1\rangle$, the reduced density ρ_{0j} can be expressed as

$$\begin{pmatrix} 1-|\chi_1|^2-\frac{1}{N}|\chi_2|^2 & \chi_0\chi_1^* & \chi_0\frac{1}{\sqrt{N}}\chi_2^* & 0 \\ \chi_1\chi_0^* & |\chi_1|^2 & \chi_1\frac{1}{\sqrt{N}}\chi_2^* & 0 \\ \frac{1}{\sqrt{N}}\chi_2\chi_0^* & \frac{1}{\sqrt{N}}\chi_2\chi_1^* & \frac{1}{N}|\chi_2|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (38)$$

From the above formula we can compute the central-bath entanglement. Direct calculation of the concurrence according to Eq.(37) yields the simple expression of concurrence:

$$C_{0j} = \frac{2}{\sqrt{N}}|\chi_1\chi_2^*| = \frac{4\left(\sin\frac{\vartheta}{2}\right)^2\left|\sin\left(\frac{\sqrt{\zeta}}{4}J_1 t\right)\right|\sqrt{\zeta-2N+2N\cos\left(\frac{\sqrt{\zeta}}{2}J_1 t\right)}}{\zeta}. \quad (39)$$

In the case of $\Delta = 0$,

$$C_{0j}^{\Delta=0} = \frac{\sin\left(\frac{\theta}{2}\right)^2|\sin(\sqrt{N}J_1 t)|}{\sqrt{N}}. \quad (40)$$

In this case, C_{0i} reaches maximum when $\sqrt{N}J_1 t = k\pi + \frac{\pi}{2}$. In the case of $\Delta = \pm 1$,

$$C_{0j}^{\Delta=\pm1} = \frac{4\sin\left(\frac{\theta}{2}\right)^2\left|\sin\left[\frac{(1+N)}{4}J_1 t)\right]\right|\sqrt{(1+N)^2+2N\cos\left[\frac{(1+N)}{2}J_1 t\right]}}{(1+N)^2}. \quad (41)$$

In the case, C_{0i} reaches maximum when $\sin\left(\frac{1+N}{4}J_1t\right) = \frac{N+1}{2\sqrt{2N}}$ for $2 \leq N \leq 5$ and $\sin\left(\frac{1+N}{4}J_1t\right) = 1$ for $N \geq 6$.

For illustration, we have displayed in Fig.4 the dependence of concurrence between the central spin and the bath spin on the ϑ when N=2, 4, 6, 8, 10, 20, 50 for anisotropic parameter $\Delta = 0, 1$ respectively. The central-bath entanglement goes down with the increase of the number of bath spins and reaches its maximal value when $\vartheta = \pi$.

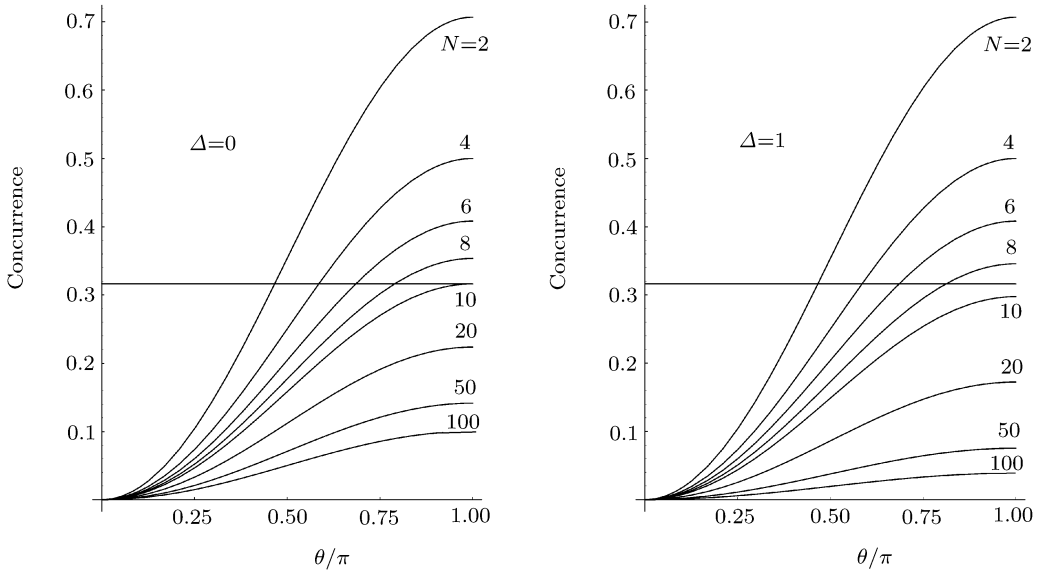


Fig.4. The concurrence of the central spin versus ϑ when $\Delta = 0$ and $\Delta = 1$, respectively, for various values of N(from top to bottom $N = 2, 4, 6, 8, 10, 20, 50$). In the left plot, $\sqrt{N}J_1t$ is set to be $k\pi + \frac{\pi}{2}$. In the right plot, $\sin\left(\frac{1+N}{4}J_1t\right) = \frac{N+1}{2\sqrt{2N}}$ when $2 \leq N \leq 5$ and $\sin\left(\frac{1+N}{4}J_1t\right) = 1$ when $N \geq 6$.

Similarly, we can consider the spins J and k in the bath where the choice of the two spins is irrelevant because of the nature of permutation symmetry of the Hamiltonian. Let ρ_{Kl} be the reduced density matrix. Their density matrix ρ_{Kl} in computation basis has the same form as follows:

$$\begin{pmatrix} 1-\frac{2}{N}|\chi_2|^2 & \chi_0\frac{1}{\sqrt{N}}\chi_2^* & \chi_0\frac{1}{\sqrt{N}}\chi_2^* & 0 \\ \frac{1}{\sqrt{N}}\chi_2\chi_0^* & \frac{1}{N}|\chi_2|^2 & \frac{1}{N}|\chi_2|^2 & 0 \\ \frac{1}{\sqrt{N}}\chi_2\chi_0^* & \frac{1}{N}|\chi_2|^2 & \frac{1}{N}|\chi_2|^2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (42)$$

According to Eq.(37), we can also calculate the in-bath entanglement:

$$C_{kl} = \frac{2}{N}|\chi_2|^2 = 8\left(\sin\frac{\vartheta}{2}\right)^2\frac{\left[\sin\left(\frac{\sqrt{\zeta}}{4}J_1t\right)\right]^2}{\zeta}. \quad (43)$$

The intra-bath entanglement C_{kl} reaches maximum when $\frac{J_1t}{4} = \frac{2k\pi+\frac{\pi}{2}}{\sqrt{\zeta}}$,

$$C_{kl}^{\max} = \frac{2}{N}|\chi_2|^2 = 8\left(\sin\frac{\vartheta}{2}\right)^2\frac{1}{\zeta}. \quad (44)$$

Figure 5 shows the dependence of the concurrence of the bath spins J and k on the ϑ for anisotropic parameter $\Delta = 0, 1$ respectively. The entanglement between the bath spins goes down on average with the increase of the total number of the particles and reaches its maximal value when $\vartheta = \pi$. Furthermore, it can be readily seen that the case of $\Delta = 0$ prefers entanglement creation. Increasing the absolute value of Δ significantly depresses the dynamical entanglement. A comparison between the two plots also illustrates that the in-bath entanglement decreases with the increase of N more rapidly in the case of $\Delta = 1$. In the thermodynamic limit $N \to \infty$, the entanglement is created, when either the cental-bath entanglement or the in-bath entanglement goes to zero.

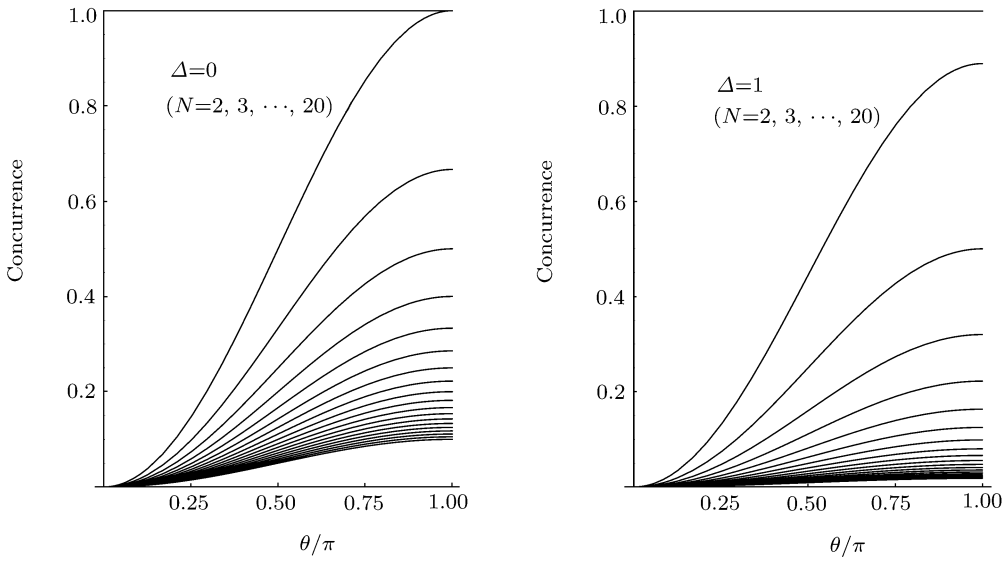


Fig.5. The concurrence of the bath spins J and k versus ϑ when $\Delta = 0$ and $\Delta = 1$, respectively, for various values of N (from top to bottom $N = 2, 3, \ldots, 20$). In the left plot, $\frac{N+1}{4}J_1 t = \frac{2k\pi+\pi}{2}$. In the right plot, $\frac{\sqrt{N}}{2}J_1 t = \frac{2k\pi+\pi}{2}$.

4. conclusion

In conclusion, in this paper a general spin star model were analysed and solved exactly. The prominent feature is that, besides considering XXZ couplings between the central spin and bath spins, here we also introduce the couplings within the bath spins. General expressions of the energy eigenstates and eigenvalues of the spin star model is calculated with the use of the symmetries of the system. We obtain the simple functions of the fidelity and the average fidelity of quantum state transfer. Especially, we also analyse the efficiency of the configuration regarded as quantum phase covariant clone or decoherence model. Some interesting results were discovered when we concern the properties of the quantum state transfer and dynamical behaviour of entanglement. The transfer fidelity, as well as the dynamical creation of entanglement, are even functions of the isotropic parameter. The maximum of average fidelity of the central spin is constant $\frac{5}{6}$ irrelevant to the anisotropic parameter and the number of bath spins. Furthermore we find out that the entanglement created in the quantum state transfer decreases with the total number of bath spins and the entanglement between bath spins reaches 1 when the total number of bath spins $N = 2$.

Finally, we want to stress that the results obtained in Refs.[1, 8], which are just special cases of our model when $J_2 = 0$ and $\Delta = 0, 1$, agree with the results obtained here.

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