Vol 17 No 2, February 2008
1674-1056/2008/17(02)/0543-07

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Gradient control and synchronization of Julia sets*

Zhang Yong-Ping(张永平)† and Liu Shu-Tang(刘树堂)

School of Control Science and Engineering, Shandong University, Jinan 250061, *China*

(Received 15 May 2007; revised manuscript received 31 May 2007)

This paper firstly introduces the control methods to fractals and give the definition of synchronization of Julia sets between two different systems. Especially, the gradient control method is taken on the classic Julia sets of complex quadratic polynomial $z_{n+1} = z_n^2 + c$, which realizes its Julia sets control and synchronization. The simulations illustrate the effectiveness of the method.

Keywords: Julia set, gradient control, synchronization
PACC: 0545 0555

1. Introduction

Since the emergence of fractals, it provides better explanations for lots of complicated phenomena because of its particular properties.[1,2] In 1918, the French mathematician Gaston Julia obtained an important notion in fractals theory, the Julia set, when he investigated the iteration of a complex function.[3] It has been extensively applied in physics, biology, engineering *et al* and it is an important question for study in nonlinear fields.[4−8] For example, based on the particle dynamics characteristics, the physical meaning of the generalized M-J set is discussed.[9] As a result, the fractal structure characteristics of the generalized M-J set could visually reflect the change rule of the particle's velocity. It is also pointed out that for arbitrary c of the system

$$z_{n+1} = z_n^q + c,$$

where c is a complex and $q > 1$, the generalized Julia set gives the closure of all the possible unstable orbit of the particle within the velocity space. When $c \to 0$, the absolute velocity of the particle is $v = 1$. It rotates unstably at the top of the potential barrier, as given in Fig.1.

It is noted that, in nonlinear fields, the nonlinear system described by the Julia set has very important nonlinear characteristics. According to the real world requirement, we usually should constrain the size of the nonlinear attractive region and the system should present different behaviour and performance. Thus, the effective control over the Julia set of nonlinear systems becomes very important. However, until now the study on the Julia set is only limited to its properties and the drawing of its graphics,[10,11] while the control over the Julia set is an entirely new area.



Fig.1. Unstable rotation of particles when $c \to 0$, $v = 1$.

In this paper, we take gradient control over the Julia set of the classic quadratic polynomial and also firstly give the definition of synchronization of Julia sets between two different parameters systems. The simulations illustrate the effectiveness of the method.

2. Basic theory

The Julia set $J(f)$, created by the iteration of a complex variable function f, is defined to be the closure of the repelling periodic points of f. By

*Project supported by the National Natural Science Foundation of China (Grant No 60472112) and a foundation for the author of National Excellent Doctoral Dissertation of China (FANEDD) (Grant No 200444).
†Corresponding author. E-mail: ypzhangsd@126.com

the Montel theorem, we have the following equivalent definition about the Julia set of the complex polynomial f: $J(f) = J_0(f) = \{z \in C :$ the function family $\{f^k\}_{k\geq 0}$ is nonnormal at $z\}$. For the complex polynomial f, its Julia set has the following properties.

(i) $J(f)$ is nonempty and bounded;

(ii) $J(f)$ is fully invariant, that is $J(f) = f(J(f)) = f^{-1}(J(f))$;

(iii) $J(f) = J(f^p)$ for any positive integer p;

(iv) If w is an attractive fixed point of f, then $\partial A(w) = J(f)$, where $A(w)$ is the attractive domain of the attractive fixed point w. That is $A(w) = \{z \in C : f^k(z) \to w, (k \to \infty)\}$. It is also the same as $w = \infty$.

From the definition of Julia set and its properties, we know that the iterative trajectory of the complex function is very important in the structure of its Julia set. Therefore, we can realize the Julia set control by the iterative trajectory control. Especially, we can consider the stability of the fixed point. That is we make the fixed point stable under control. Thus, we can control the trajectory of iterative points and achieve the objective of controlling Julia set.

In order to analyse the applications of the control method, we take the classic quadratic polynomial in the complex domain C

$$z_{n+1} = z_n^2 + c, \tag{1}$$

where c is a complex number and let $f_c(z) = z^2 + c$.

In fact, if $h(z) = \alpha z + \beta$ $(\alpha \neq 0)$, then $h^{-1}(f_c(h(z))) = (\alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c - \beta)/\alpha$. By choosing the proper value of α, β, c, we can make the expression be any desired quadratic function f. So $h^{-1} \circ f_c \circ h = f$, then $h^{-1} \circ f_c^m \circ h = f^m$ for any m. This predicates that the iterative sequence $\{f^m(z)\}$ of f at z is the image of the iterative sequence $\{f_c^m(h(z))\}$ of f_c at $h(z)$ under h^{-1}. Therefore the map h turns the dynamics f to the dynamics of f_c. Especially, z is a periodic point of f with period p if and only if $h(z)$ is a periodic point of f_c with period p. Thus, the Julia set of f is the image of the Julia set of f_c under h^{-1}. The transformation of h is called the conjugate between f and f_c. Every quadratic function is conjugative to f_c with some c. Therefore, we can effectively study the Julia sets of all quadratic polynomial by studying the Julia set of f_c with $c \in C$. Since h is a similitude transformation, the Julia set of any quadratic polynomial is similar in geometry to the Julia set of f_c with some $c \in C$.

3. Gradient control of Julia sets

From previous discussion, we will consider the stability of the fixed point $z^* = \dfrac{1 \pm \sqrt{1-4c}}{2}$ of system (1). For example, we will find a control item u_n such that the fixed point z^* of the controlled system

$$z_{n+1} = z_n^2 + c + u_n \tag{2}$$

is stable and $z_{n+1} \to z^*$, $n \to \infty$. Thus we realize the trajectory control of the map and then we can realize the Julia set control.

We will use the gradient control method to achieve the control purpose

$$z_n^2 + c + u_n \to z^*.$$

The control item u_n is designed by using the speed-gradient (SG) methodology reviewed in section IV.[12] This is a general technique for controlling nonlinear systems through an appropriate selection and dynamic minimization of a scalar goal function. Let $F(z, u) = z^2 + c + u$ and the goal of control be expressed as the limit relation $Q(z) \to 0$. In order to achieve the control goal we apply the SG algorithm in the finite form

$$u = -\varphi(\nabla_u \dot{Q}(z, u)),$$

where $\dot{Q} = \dfrac{\partial Q}{\partial z} F(z, u)$ is the speed of changing $Q(z)$, $\varphi(z)$ forms a sharp angle with z, i.e. $\varphi(z)\overline{z} > 0$ when $z \neq 0$. The first step of the gradient procedure is to calculate the speed $\dot{Q}$. The second step is to evaluate the gradient $\nabla_u \dot{Q}(z, u)$ with respect to controlling input u. Finally $\varphi(z)$ should be chosen to meet sharp angle condition. E.g. the choice $\varphi(z) = kz, k > 0$ yields the proportional (with respect to gradient) feedback

$$u = -k(\nabla_u \dot{Q}(z, u)).$$

In our case, the control objective is the fixed point z^* and we choose the objective function (to be minimized) as

$$Q(z) = \frac{1}{2}(z - z^*)^2$$

and write the general form of the gradient control algorithm as

$$u_n = -k\nabla_u Q(z_{n+1}),\ n > 0,$$

where ∇_u denote the gradient operated with respect to u, k is the control parameter. After some simple calculations, we obtain

$$u_n = -\frac{k}{1+k}[f_c(z_n) - z^*],$$

where $f_c(z) = z^2 + c$. Therefore, the controlled system (2) becomes

$$z_{n+1} = \frac{1}{1+k} f_c(z_n) + \frac{k}{1+k} z^*. \tag{3}$$

From the controlled system (3), it is seen that at each step the control actually decreases at a rate of $1+k$ times the uncontrolled value of the next step, and simultaneously adds a term which is independent of n but close to z^*. Obviously, z^* is also a fixed point for the controlled system (3). Denote

$$\begin{aligned} g(z) &= \frac{f_c(z)}{1+k} + \frac{k}{1+k} z^* \\ &= \frac{z^2 + c}{1+k} + \frac{k}{1+k} z^*. \end{aligned} \tag{4}$$

It is needed that $|g'(z^*)| < 1$ in order to make the fixed point x^* attractive. That is $\left|\frac{2z^*}{1+k}\right| < 1$. Then

$$k > |2z^*| - 1 = |1 \pm \sqrt{1-4c}| - 1.$$

We will discuss the structure of Julia set of the controlled system (3).

Consider the equation

$$\left|\frac{1}{1+k}\right| z^2 - z - \left|\frac{c+kz^*}{1+k}\right| = 0. \tag{5}$$

Obviously, Eq.(5) has a positive root

$$T = \frac{|1+k| + \sqrt{|1+k|^2 + 4|c+kz^*|}}{2}.$$

When $|z| > T$,

$$\begin{aligned} &|g(z)| \geq \frac{1}{|1+k|}||z^2| - |c+kz^*|| \\ &= \frac{1}{|1+k|}\left|\frac{|z^2|T^2}{T^2} - |c+kz^*|\right| \\ &= \left(\frac{|z|}{T}\right)^2 T + \frac{1}{|1+k|}\left(\left(\frac{|z|}{T}\right)^2 - 1\right)|c+kz^*| \\ &> \left(\frac{|z|}{T}\right)^2 T. \end{aligned}$$

Then $|g^n(z)| > \left(\frac{|z|}{T}\right)^{2^n} T \to \infty\ (n \to \infty)$. Therefore the Julia set of the controlled system (3) is located in the set $\{z : |z| \leq T\}$.

Using the property (iv) $J(f) = \partial A(z^*) = \partial A(\infty)$, we discuss the structure of Julia set of Eq.(3). Let C_0 to be a curve $|z| = T$. Obviously, it surrounds the attractive fixed point z^* of g. We know $(g^{-1}(z))^2 = (1+k)z - c - kz^*$, then $|g^{-1}(C_0)|^2 = \{|(1+k)Te^{i\theta} - c - kz^*| : 0 \leq \theta \leq 2\pi\}$. In view of

$$\begin{aligned} &|(1+k)T\mathrm{e}^{\mathrm{i}\theta} - c - kz^*| \\ &\leq |(1+k)T\mathrm{e}^{\mathrm{i}\theta}| + |c+kz^*| \\ &= |T\mathrm{e}^{\mathrm{i}\theta}|^2,\ 0 \leq \theta \leq 2\pi, \end{aligned}$$

we have $g^{-1}(C_0)$ is contained in the region surrounded by C_0. Let $C_1 = g^{-1}(C_0)$. For every θ, let $\varphi_1(\theta)$ be the end point on C_1 of the trajectory from C_0 (the curve beginning from C_0 and ending at C_1 perpendicularly is the trajectory). Let A_1 be the ring between C_0 and C_1, then $g^{-1}(A_1)$ is the ring A_2 that has inner loop $C_2 = g^{-1}(C_1)$ and outer loop C_1. The inverse image of the trajectory between C_0 and C_1 gives a bundle of trajectories connecting C_1 and C_2. Let $\varphi_2(\theta)$ be the end point of the trajectory beginning from C_1 and ended at C_2. Using the similar approach, we obtain a sequence C_k satisfying that C_{k+1} is contained in the region surrounded by C_k, $k = 0, 1, 2, \cdots$, and for each k the trajectory bundle links $\varphi_k(\theta)$ on C_k and $\varphi_{k+1}(\theta)$ on C_{k+1}. When $k \to \infty$, the curve C_k approaches to the border of the attractive region of z^*. This boundary is the Julia set $J(g)$. Since $\{\varphi_k(\theta)\}_{k=1}^{\infty}$ is contained in C_0 and so is bounded, there exists a subsequence satisfying $\lim\limits_{j\to\infty} \varphi_{k_j}^m = \varphi^m(\theta)$, and $\varphi^m(\theta)$ becomes the Julia set of the controlled system (3).

In addition, since

$$\begin{aligned} |g^{-1}(z)|^2 &= |(g^{-1}(z))^2| \\ &= |(1+k)z - c - kz^*| \\ &= |(1+k)(z-z^*) + z^* - c| \\ &\geq |1+k||z-z^*| - |z^*-c|, \end{aligned}$$

we know that the value of $|g^{-1}|$ increases with the increase of the value of $|1+k|$, which indicates that its inverse extends outwards in comparison with the original point. Therefore, its Julia set extends outwards in comparison with the original point from previous discussion. The following simulations illustrate our analysis and the effectiveness of this method. (The

boundary of the shaded region is the Julia set. The abscissa axis and ordinate axis of the coordinate system are the real part and imaginary part of the complex number respectively.)

In the above simulations, we take $c = -0.2 + 0.75\mathrm{i}$ in system (1). Then the fixed point of (1) is $z^* = 0.6277 + 0.9216\mathrm{i}$ and we can get $k > 0.1151$. Fig.2 is the original Julia set of system (1) when $c = -0.2 + 0.75\mathrm{i}$ and Fig.3 is the Julia set control with different values of k. From Fig.3, we know that the Julia set of the controlled system (3) is enlarging with the increase of k. This illustrates the effectiveness of this method.

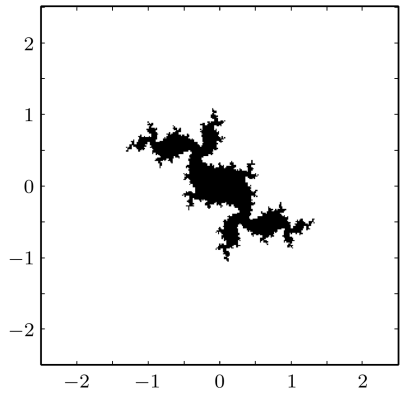


Fig.2. Original Julia set of system (1) when $c = -0.2 + 0.75\mathrm{i}$.

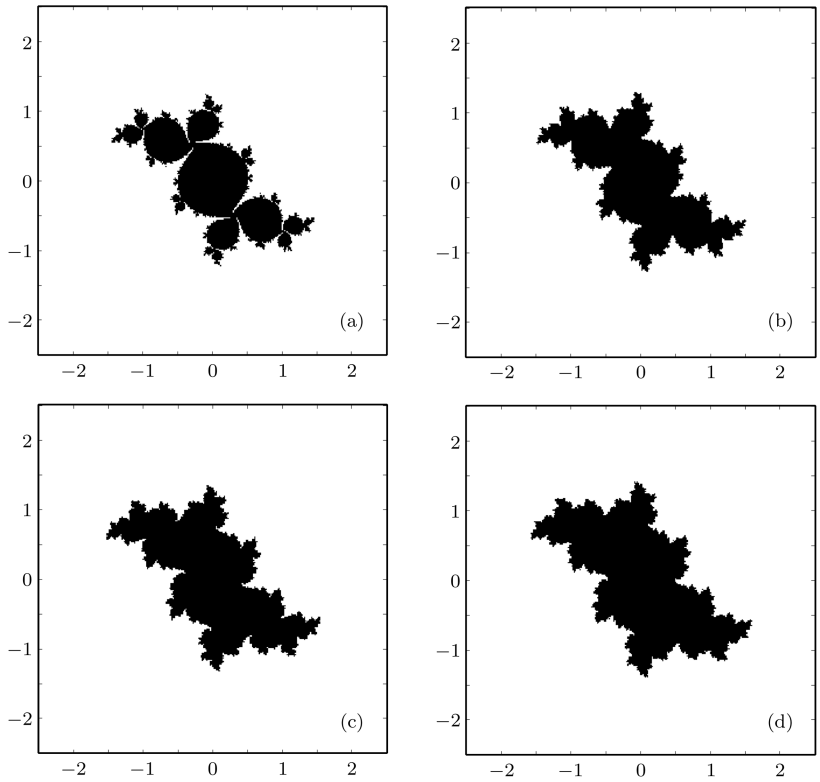


Fig.3. Julia set control with parameter k when $c = -0.2 + 0.75\mathrm{i}$. (a) $k = 0.15$; (b) $k = 0.2$; (c) $k = 0.25$; (d) $k = 0.3$.

4. Gradient synchronization of Julia sets

Chaos synchronization has been studied extensively. People have done abundant work and found various methods to realize chaos synchronization.[13−18] In general, consider two dynamical system $\dfrac{\mathrm{d}X}{\mathrm{d}t} = F(X;Y)$, $\dfrac{\mathrm{d}Y}{\mathrm{d}t} = G(X;Y)$, where t is the time, F, G could be different functions and $X \in R^m$, $Y \in R^m$. Let $X(t;t_0,X_0)$ and $Y(t;t_0,Y_0)$ be the solutions of the two systems respectively. Synchronization between $\dfrac{\mathrm{d}X}{\mathrm{d}t} = F(X;Y)$ and $\dfrac{\mathrm{d}Y}{\mathrm{d}t} = G(X;Y)$ was taken to occur if there are a subset $D(t_0) \subset R^m$ and X_0, $Y_0 \in D(t_0)$ such that $||X(t;t_0,X_0) - Y(t;t_0,Y_0)|| \to 0$ as $t \to \infty$. If $D(t_0) = R^m$, we call it the global synchronization; if $D(t_0)$ is a subset of R^m, we call it the local synchronization. We call $D(t_0)$ the synchronization domain. The global and local synchronization both belong to the conventional synchronization, that is $X(t;t_0,X_0) = Y(t;t_0,Y_0)$. When

$X(t;t_0,X_0)=\varphi(Y(t;t_0,Y_0))$ (where φ is some function), we call it the generalized chaos synchronization. Obviously, the conventional synchronization is a special case of the generalized synchronization.

However, synchronization of Julia sets which are closely related to chaos has never been studied before. In this section, we firstly give the definition of Julia sets synchronization and then we take the gradient method to achieve the Julia sets synchronization between two different systems. From the definition of Julia set, we know that the Julia set is determinate and unvaried once the parameters of the system are given. Therefore, the Julia sets synchronization certainly will be different from the chaos synchronization.

Consider two systems in the complex domain C

$$x_{n+1} = f(x_n), \tag{6}$$

$$y_{n+1} = g(y_n). \tag{7}$$

Their Julia sets are determinate when the parameters of the systems are given and denote them by J^* and J respectively. In order to make the Julia sets of Eqs.(6) and (7) associated, we couple systems (6) and (7). Add a coupling item in system (7)

$$y_{n+1} = g(y_n) + p(x_n, y_n; k), \tag{8}$$

where p is the coupling item about x_n, y_n, k is the coupling strength. It is obvious for every k, there exists a Julia set corresponded to it and denoted by J_k. Julia sets synchronization of Eqs.(6) and (7) was taken to occur if

$$\lim_{k\to k_0}(J_k \cup J^* - J_k \cap J^*) = 0 \tag{9}$$

for some k_0 (k_0 may be ∞).

It is easy to see that we take the limitation of the coupling strength k to achieve Julia sets synchronization. It is because Julia set $J(f)$ is close with the trajectory of f. In addition, we can not depict Julia sets synchronization as chaos synchronization in simulation since the Julia set is determined when parameters are given. But we can depict the progress of synchronization by taking different corresponding values of k, which can be seen in the following example. In this section, we realize Julia sets synchronization through the synchronization of trajectories. In the synchronization of trajectories, k illustrates the status of trajectories synchronization. It is just the synchronization status of the corresponding Julia sets J_k, where we take the same iterative original values in systems (6) and (7).

We now will discuss Julia set synchronization with the gradient method. From the discussion of the previous section, we take the coupled item $p(x_n, y_n; k) = -\dfrac{k}{1+k}(g(y_n) - f(x_n))$. Then the coupling system (8) becomes

$$y_{n+1} = g(y_n) - \frac{k}{1+k}(g(y_n) - f(x_n)). \tag{10}$$

From systems (6) and (10), we have

$$x_{n+1} - y_{n+1} = \frac{1}{1+k}(f(x_n) - g(y_n)). \tag{11}$$

By property (i) of the Julia set, we can obtain a Julia set by calculating the iteration of points in a bounded region D. Thus we only need to consider the iteration of points in D. In order to get the Julia set, we only consider the points whose trajectories are also in D since $z \notin J$ if there is an n_0 such that $f^{n_0}(z) \notin D$ from the property (ii). Since D is a bounded region, there is $M > 0$ such that $|z| < M$ for all $z \in D$.

We will discuss the synchronization of Julia sets created from the complex variable polynomial $z_{n+1} = z_n^2 + c$ when c takes different values. For example, we take $f(x_n) = x_n^2 + c_1, g(y_n) = y_n^2 + c_2$ in systems (6) and (7), where $c_1 \neq c_2$. From Eq.(11), we have

$$x_{n+1} - y_{n+1} = \tfrac{1}{1+k}[(x_n - y_n)(x_n + y_n) + c_1 - c_2].$$

Therefore,

$$\begin{aligned}
|x_{n+1} - y_{n+1}| &\le \frac{1}{|1+k|}(2M|x_n - y_n| + |c_1 - c_2|) = \frac{2M}{|1+k|}|x_n - y_n| + \frac{|c_1 - c_2|}{|1+k|} \\
&\le \frac{2M}{|1+k|}\left(\frac{2M}{|1+k|}|x_{n-1} - y_{n-1}| + \frac{|c_1 - c_2|}{|1+k|}\right) + \frac{|c_1 - c_2|}{|1+k|} \\
&= \left(\frac{2M}{|1+k|}\right)^2 |x_{n-1} - y_{n-1}| + \left(1 + \frac{2M}{|1+k|}\right)\frac{|c_1 - c_2|}{|1+k|} \\
&\le \cdots \\
&\le \left(\frac{2M}{|1+k|}\right)^n |x_1 - y_1| + \left(1 + \cdots + \left(\frac{2M}{|1+k|}\right)^{n-1}\right)\frac{|c_1 - c_2|}{|1+k|}
\end{aligned}$$

$$= \left[1 + \cdots + \left(\frac{2M}{|1+k|}\right)^n\right] \frac{|c_1 - c_2|}{|1+k|} = \frac{1 - \left(\frac{2M}{|1+k|}\right)^{n+1}}{1 - \frac{2M}{|1+k|}} \frac{|c_1 - c_2|}{|1+k|}. \tag{12}$$

When $\frac{2M}{|1+k|} < 1$, the right side of Eq.(12) goes to $\frac{|c_1 - c_2|}{|1+k| - 2M}$ as $n \to \infty$. Obviously, if $|1+k| \to \infty$ in this case, that is $k \to \infty$, we have

$$|x_{n+1} - y_{n+1}| \to 0.$$

So the synchronization of trajectories of systems (6) and (8) is achieved. From previous discussion, we know that the equation (9) is satisfied, that is Julia sets synchronization of systems (6) and (8) is achieved.

We can see the coupling degree of the coupling systems from the process $k \to \infty$. It is the degree of the trajectory synchronization and of course it is also the synchronization degree of Julia sets of the coupling systems. The synchronization status can be seen from the following simulations.

Figure 4 is the original Julia sets of systems (6) and (7). From the above simulation Fig.5, it is easy

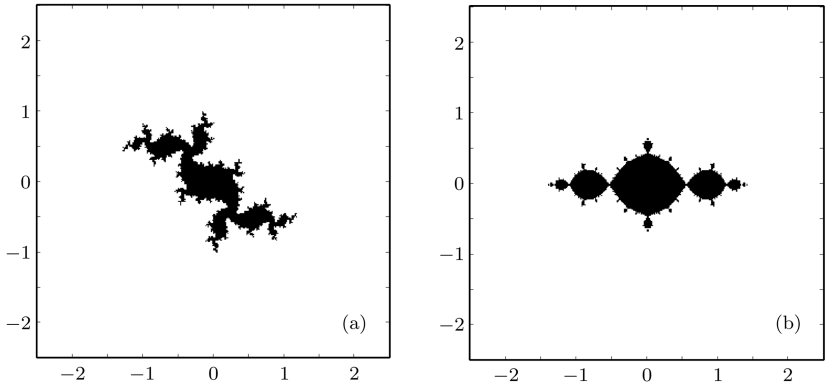


Fig.4. Original Julia sets of systems (6) and (7): (a) $c = -0.2 + 0.75\mathrm{i}$; (b) $c = -1$.

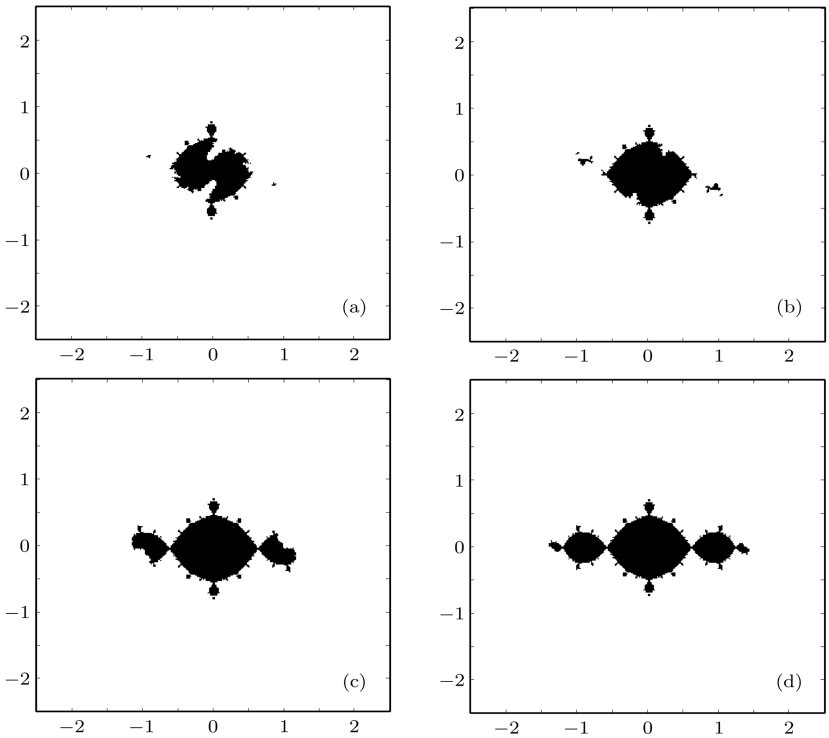


Fig.5. Status of Julia sets synchronization with parameters k. (a) $k = 0.05$; (b) $k = 0.1$; (c) $k = 0.3$; (d) $k = 0.8$.

to see that Julia sets synchronization has not been achieved when k is small. However, Julia set synchronization can be achieved when k is big enough, corresponding to the previous discussion. From the above simulation, it is easy to see that Julia set synchronization has not been achieved when k is small. However, Julia set synchronization can be achieved when k is big enough, corresponding to the previous discussion.

5. Conclusion

The Julia set, created by the iteration of a complex function, is the closure of the repelling periodic points. It is an important notion in fractals theory and has extensive applications in nonlinear fields. In this paper, we first introduce the control method over the Julia set and give the definition of Julia sets synchronization. Specially, we take the gradient control method over the classic Julia sets of the complex quadratic polynomial $z_{n+1} = z_n^2 + c$ and achieve the control and synchronization of Julia sets. The simulations illustrate the effectiveness of this method. This not only enriches the content of fractals theory but also helps us to understand the applications of Julia sets in various fields better and explain the corresponding complicated phenomena better.

References

[1] Halsey T C, Jensen M H, Kadanoff L P, Procaccia I and Shraiman B J 1986 *Phys. Rev.* A **33** 1141

[2] Li Y, Kong X M and Huang J Y 2002 *Acta Phys. Sin.* **51** 1346 (in Chinese)

[3] Falconer K 1990 *Fractals Geometry Mathematical Foundations and Applications* (New York: Wiley)

[4] Nauenberg M and Schellnhuber H J 1989 *Phys. Rev. Lett.* **62** 1807

[5] Bambi Hu and Bin Lin 1989 *Phys. Rev.* A **39** 4789

[6] Marmi S 1988 *J. Phys.* A **21** L961

[7] Rammal R 1984 *J. de Phys.* (Paris) **45** 191

[8] Biskup M, Borgs C, Chayes J T, Kleinwaks L J and Kotecky R 2000 *Phys. Rev. Lett.* **84** 4794

[9] Wang X Y and Meng Q Y 2004 *Acta Phys. Sin.* **53** 388

[10] Li S 2000 *5th International Conference on Signal Processing Proceedings* (Beijing: Publishing House Electronic Industry) 285

[11] Lakhtakia A 1987 *J. Phys.* A **20** 3533

[12] Fradkov A L and Pogromsky A Yu 1999 *Introduction to Control of Oscillations and Chaos* (Singapore: World Scientific)

[13] Zou Y L and Zhu J 2006 *Chin. Phys.* **15** 1965

[14] Min F H and Wang Z Q 2005 *Acta Phys. Sin.* **54** 4026 (in Chinese)

[15] Shan L, Li J and Wang Z Q 2006 *Acta Phys. Sin.* **55** 3950 (in Chinese)

[16] Liu Z Z, Tian Y T and Song Y 2006 *Acta Phys. Sin.* **55** 3945 (in Chinese)

[17] Zhou P 2007 *Chin. Phys.* **16** 1263

[18] Lü L, Guo Z A and Zhang C 2007 *Chin. Phys.* **16** 1603