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Stochastic resonance in an asymmetric bistable system driven by coloured correlated multiplicative and additive noise*

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This paper investigates the stochastic resonance (SR) phenomenon in an asymmetric system with coupling between multiplicative and additive noise when the coupling between two noise terms is coloured. The approximate expression of signal-to-noise ratio has been obtained by applying the two-state theory and SR exhibits in the bistable system. Moreover, the potential asymmetry r and cross-correlation strength λ can weaken the SR phenomenon, while the cross-correlation time τ can strengthen the SR phenomenon.

Keywords: stochastic resonance, asymmetry bistable system, signal-to-noise ratio
PACC: 0540, 0250

1. Introduction

Noise is generally associated with something which is irregular and cannot be perfectly controlled, and it is virtually unavoidable since it is impossible to isolate a system perfectly from its environment. However, during the last two decades, the discovery of stochastic resonance (SR) proved that the addition of the appropriate amount of noise can boost a signal and hence facilitate its detection in a noisy environment. The SR phenomenon has been investigated extensively both theoretically and experimentally due to its potential application.[1−12] The term 'SR' first appeared in Ref.[1], where it was proposed for almost periodic occurrences (approximately every 100 000 years) of ice ages on Earth during the last 700 000 years. Since then SR has grown into a rapidly developing topic, attracting interdisciplinary interest from physicists, biologists, engineers and medical doctors. The SR phenomenon appears in many fields, such as the periodic recurrence of ice ages, electronic and magnetic systems.[2,3] Recently, there have appeared some extensions of SR, such as stochastic multiresonance[13,14] and control of SR.[15,16]

Most of the studies focus on the symmetric bistable model. However, symmetry cannot be preserved in many real physical systems. Asymmetry has been introduced to fluxgate magnetometers and superconducting quantum interference devices to detect weak signals.[6,7] Wio[17] studied the role of potential asymmetry on SR in a bistable system driven by Gaussian white noise. Li[18] investigated the effect of asymmetry of the potential on SR for a bistable model driven by additive noise. Jin[19] studied SR in an asymmetric bistable system driven by white correlated multiplicative and additive noise. Meanwhile, only white correlation is considered in most of the analyses concerning the SR phenomenon. However, the coloured noise can also play an important part in the system. So in this paper, we study the effects of potential asymmetry and coloured correlation on SR driven by coloured correlated multiplicative and additive noise in a bistable system. We derive the expression of SNR by applying the two-stable theory and discuss the effects of potential asymmetry and coloured correlation on the SR.

2. SNR for the asymmetric bistable model

The model system we consider here corresponds to an overdamped bistable system driven by coloured correlated multiplicative and additive noise, which can

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be described by the Langevin equation:

$$\dot{x} = -U'(x) + x\xi(t) + A\cos\Omega t + \eta(t), \quad (1)$$

$$U(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4 + rx, \quad (2)$$

where r denotes the asymmetry of the potential. The potential $U(x)$ is an asymmetry bistable potential for $-2\sqrt{3}/9 < r < 2\sqrt{3}/9$. A is the amplitude of a periodic signal and Ω is the frequency. The noise terms $\xi(t)$ and $\eta(t)$ are correlated in the following form

$$\langle\xi(t)\rangle = \langle\eta(t)\rangle = 0, \quad (3)$$

$$\langle\xi(t)\xi(t')\rangle = 2\alpha\delta(t-t'),$$

$$\langle\eta(t)\eta(t')\rangle = 2D\delta(t-t'), \quad (4)$$

$$\langle\xi(t)\eta(t')\rangle = \langle\eta(t)\xi(t')\rangle = \frac{\lambda\sqrt{\alpha D}}{\tau}\exp\left[-\frac{1}{\tau}|t-t'|\right] \to 2\lambda\sqrt{\alpha D}, \quad \tau \to 0 \quad (5)$$

where α and D are the intensities of the multiplicative and additive noise respectively, $\lambda(|\lambda| \leq 1)$ is the strength of coupling between $\xi(t)$ and $\eta(t)$, τ is cross-correlation time of the multiplicative and additive noise.

According to Eq.(2), the potential $U(x)$ has two stable states x_{s1}, x_{s2} and an unstable state x_{un} for $-2\sqrt{3}/9 < r < 2\sqrt{3}/9$.

$$x_{s1} = -\frac{2\sqrt{3}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{3\sqrt{3}}{2}r\right) - \frac{\pi}{3}\right),$$

$$x_{s2} = \frac{2\sqrt{3}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{3\sqrt{3}}{2}r\right)\right),$$

$$x_{un} = -\frac{2\sqrt{3}}{3}\cos\left(\frac{1}{3}\arccos\left(-\frac{3\sqrt{3}}{2}r\right) + \frac{\pi}{3}\right). \quad (6)$$

Thus the corresponding Fokker–Planck equation of the asymmetric bistable system, Eq.(1), can be written as

$$\frac{\partial\rho(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{-U'(x) + D\left(Rx + \frac{\lambda\sqrt{R}}{1+c(\tau)}\right) + A\cos\Omega t\right\}\rho(x,t) + D\frac{\partial^2}{\partial x^2}\left\{Rx^2 + \frac{2\lambda\sqrt{R}}{1+c(\tau)}x + 1\right\}\rho(x,t), \quad (7)$$

where $R = \alpha/D$ denotes the ratio of multiplicative noise intensity to additive one. $c(\tau) = (3x_{s1}^2 - 1)\tau$ if $x \leq x_{un}$ and $c(\tau) = (3x_{s2}^2 - 1)\tau$ if $x > x_{un}$. $\rho_i(x,\lambda,r)$ satisfies

$$\frac{\partial\rho_i(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left\{-U'(x) + D\left(Rx + \frac{\lambda\sqrt{R}}{1+c_i(\tau)}\right) + A\cos\Omega t\right\}\rho_i(x,t) + D\frac{\partial^2}{\partial x^2}\left\{Rx^2 + \frac{2\lambda\sqrt{R}}{1+c_i(\tau)}x + 1\right\}\rho_i(x,t), \quad i = 1,2, \quad (8)$$

where $c_1(\tau) = (3x_{s1}^2 - 1)\tau$, $c_2(\tau) = (3x_{s2}^2 - 1)\tau$. Then the quasi-steady-state distribution function $\rho_i(x,\lambda,r)$ can be derived[20] from Eq.(8):

$$\rho_i(x,\lambda,r) = \frac{N_i}{\sqrt{Rx^2 + \frac{2\lambda\sqrt{R}}{1+c_i(\tau)}x + 1}}\exp\left[-\frac{\tilde{U}_i(x,\lambda,r)}{D}\right], \quad i = 1,2, \quad (9)$$

where N_i is the normalization constant. The generalized potential $\tilde{U}_i(x,\lambda,r)$ is given by

$$\tilde{U}_i(x,\lambda,r) = \frac{x^2}{2R} - \frac{2\theta x}{R\sqrt{R}} - \frac{R+(1-4\theta^2)}{2R^2}\ln(Rx^2 + 2\theta\sqrt{R}x + 1) + \frac{1}{\sqrt{R}(1-\theta^2)}\left[r + \frac{\theta}{\sqrt{R}} + \frac{\theta(3-4\theta^2)}{R\sqrt{R}}\right] \times \tan^{-1}\left(\frac{\theta+\sqrt{R}x}{\sqrt{1-\theta^2}}\right) - \frac{A\cos\Omega t}{\sqrt{R}(1-\theta^2)}\tan^{-1}\left(\frac{\theta+\sqrt{R}x}{\sqrt{1-\theta^2}}\right), \quad i = 1,2 \quad (10)$$

in which $\theta = \lambda/[1 + (3x_{s1}^2 - 1)\tau]$ if $x \leq x_{un}$ and $\theta = \lambda/[1 + (3x_{s2}^2 - 1)\tau]$ if $x > x_{un}$.

The approximative quasi-steady-state distribution function $\rho(x, \lambda, r)$ of Eq.(7) is

$$\rho(x,\lambda,r) = \begin{cases} a\rho_1(x,\lambda,r), & \text{if } x \leq x_{un}, \\ b\rho_2(x,\lambda,r), & \text{if } x < x_{un}, \end{cases} \tag{11}$$

where a, b satisfy $a\rho_1(x_{un}, \lambda, r) = b\rho_2(x_{un}, \lambda, r)$ and the normalization of $\rho(x, \lambda, r)$. The corresponding generalized potential is given by

$$\tilde{U}(x,\lambda,r) = \begin{cases} \tilde{U}_1(x,\lambda,r), & \text{if } x \leq x_{un}, \\ \tilde{U}_2(x,\lambda,r) + d, & \text{if } x > x_{un}, \end{cases} \tag{12}$$

where $d = \tilde{U}_1(x_{un}, \lambda, r) - \tilde{U}_2(x_{un}, \lambda, r)$.

Using the definition of mean first-passage time and steepest descent method,[21] one can obtain the expression of transition rates $W_\pm$ out of x_{s1}, x_{s2}

$$W_+ = \frac{\left|(-1+3x_{s1}^2)(-1+3x_{un}^2)\right|^{1/2}}{2\pi} \exp\left\{-\frac{1}{D}[\tilde{U}(x_{un},\lambda,r) - \tilde{U}(x_{s1},\lambda,r)]\right\}, \tag{13}$$

$$W_- = \frac{\left|(-1+3x_{s2}^2)(-1+3x_{un}^2)\right|^{1/2}}{2\pi} \exp\left\{-\frac{1}{D}[\tilde{U}(x_{un},\lambda,r) - \tilde{U}(x_{s2},\lambda,r)]\right\}, \tag{14}$$

in which x_{s1}, x_{s2}, x_{un} and $\tilde{U}(x, \lambda, r)$ were defined by Eqs.(6) and (12).

The system is subjected to a time dependent signal $A\cos\Omega t$, up to first order on its amplitude (assumed to be small) the transition rates can be expanded as follows by two-state model theory:

$$\begin{aligned} W_+ &= \mu_1 - \beta_1 A\cos\Omega t, \\ W_- &= \mu_2 + \beta_2 A\cos\Omega t. \end{aligned} \tag{15}$$

The master equation governing the evolution of $n_\pm(t)$ is written as follows:

$$\begin{aligned} \dot{n}_+ &= -\dot{n}_- \\ &= W_-(t)n_- - W_+(t)n_+ \\ &= W_-(t) - [W_+(t) + W_-(t)]n_+. \end{aligned} \tag{16}$$

Once Eq.(16) is integrated one can calculate the correlation function and the power spectrum,[17] the expression of signal-to-noise ratio (SNR) is given by

$$\tilde{R} = \frac{A^2\pi(\mu_1\beta_2 + \mu_2\beta_1)^2}{4\mu_1\mu_2(\mu_1+\mu_2)}, \tag{17}$$

in which

$$\begin{aligned} &\bar{\beta} = A\cos\Omega t, \quad \mu_1 = W_+|_{\bar{\beta}=0}, \\ &\mu_2 = W_-|_{\bar{\beta}=0}, \quad \beta_1 = -\frac{\mathrm{d}W_+}{\mathrm{d}\bar{\beta}}\Big|_{\bar{\beta}=0}, \\ &\beta_2 = \frac{\mathrm{d}W_-}{\mathrm{d}\bar{\beta}}\Big|_{\bar{\beta}=0}. \end{aligned}$$

3. Discussions and conclusion

According to Eq.(17), the SNR as function of different parameters of noise are plotted in Figs.1–9.

In Fig.1, we plot the curve of SNR as a function of additive noise intensity D with different potential asymmetry r. There is a single-peak in the curve of SNR with increasing D and SR phenomenon appears. The SNR decreases with increasing r. Furthermore, it can be appreciated that the value of the maximum of $\tilde{R}$ increases with the symmetry of the system, which is consistent with the result obtained in the asymmetric system only driven by additive white noise in Ref.[17].

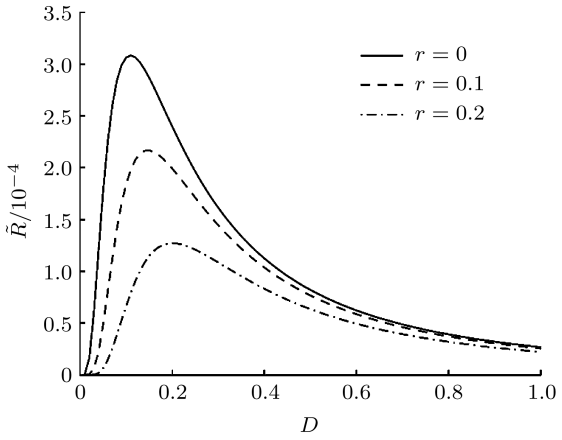


Fig.1. SNR as a function of additive noise intensity D with varied r for $A = 0.01$, $\alpha = 0.05, \lambda = 0.1, \tau = 0.1$.

Figure 2 shows the curves of SNR versus additive noise intensity D with varied cross-correlation strength λ. The SNR is decreased with increasing λ, the value of the maximum of $\tilde{R}$ increases with decreasing λ, which is consistent with the result obtained in the asymmetric system driven by white correlated multiplicative and additive noise in Ref.[19].

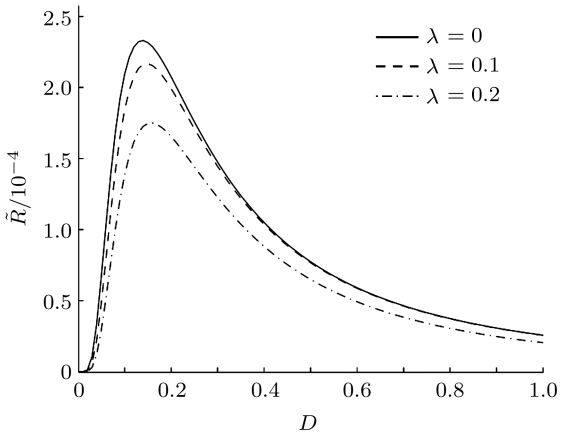


Fig.2. SNR as a function of additive noise intensity D with varied λ for $A = 0.01$, $\alpha = 0.05, r = 0.1, \tau = 0.1$.

Figure 3 shows the curves of SNR as functions of additive noise intensity D with varied cross-correlation time τ. The value of the maximum of $\tilde{R}$ increases with increasing τ. From Figs.2 and 3, we can see that the effect of cross-correlation strength λ and cross-correlation time τ on SNR are different for different values of the additive noise intensity D.

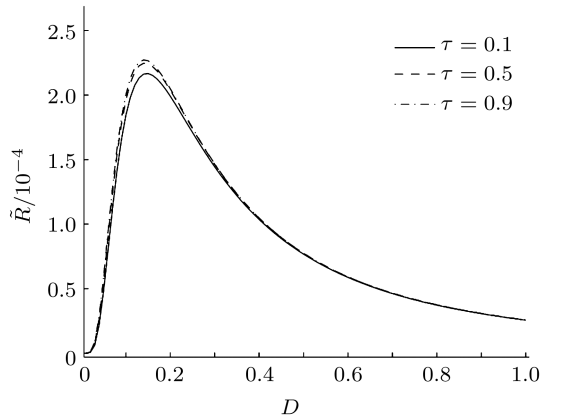


Fig.3. SNR as a function of additive noise intensity D with varied τ for $A = 0.01$, $\alpha = 0.05, r = 0.1, \lambda = 0.1$.

Figure 4 shows the curves of SNR $\tilde{R}$ versus multiplicative noise α with different potential asymmetry r. SNR increases with increasing α first, then decreases with increasing α. The curve of SNR appears a single-peak and SR exists for this case. The value of the maximum of $\tilde{R}$ increases with the symmetry of the system, which is consistent with the one shown in Fig.1.

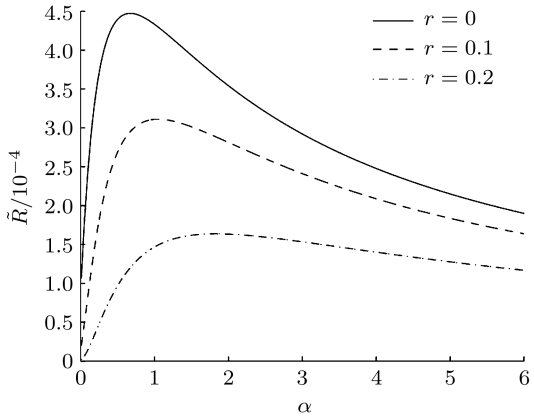


Fig.4. SNR as a function of multiplicative noise intensity α with varied r for $A = 0.01, D = 0.05, \tau = 0.1, \lambda = 0.1$.

Figure 5 shows the curves of SNR $\tilde{R}$ as functions of multiplicative noise intensity α with varied cross-correlation strength λ and SR exhibits for this case. The position of the peak shifts to the right with the increasing λ. The value of the maximum of $\tilde{R}$ increases with decreasing λ, this is consistent with the one shown in Fig.2.

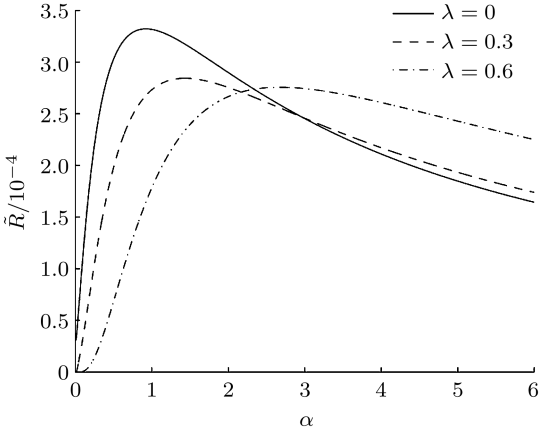


Fig.5. SNR as a function of multiplicative noise intensity α with varied λ for $A = 0.01, D = 0.05, r = 0.1, \tau = 0.1$.

Figure 6 shows the curve of SNR as a function of additive noise intensity α with varied cross-correlation time τ and SR exhibits in this case. The value of the maximum of $\tilde{R}$ increases with increasing τ. From Figs.5 and 6, we can see the effect of cross-correlation strength λ and cross-correlation time τ on SNR are different for different values of the multiplicative noise

intensity α, this is consistent with the one shown in Figs.2 and 3.

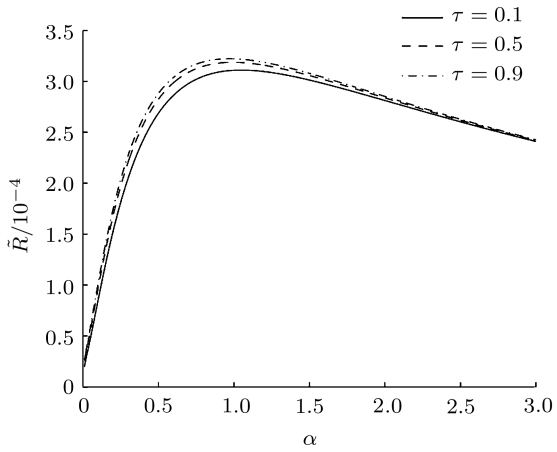


Fig.6. SNR as a function of multiplicative noise intensity α with varied τ for $A = 0.01, \alpha = 0.05, r = 0.1, \lambda = 0.1$.

The curves of SNR $\tilde{R}$ versus the ratio of multiplicative noise intensity to additive one R with different potential asymmetry r are shown in Fig.7. The curves of SNR exists a maximum with the increasing R and the value of the maximum increases with the symmetry of the system, this is consistent with the one shown in Fig.1. And the position of peak moves to the right with increasing r.

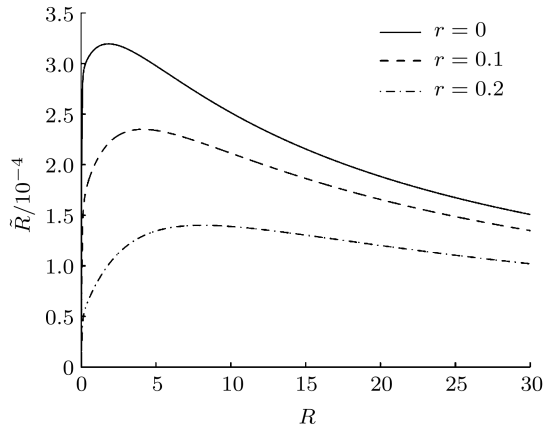


Fig.7. SNR as function of R with varied r for $A = 0.01, D = 0.1, \tau = 0.1, \lambda = 0.1$.

The curves of SNR $\tilde{R}$ as functions of cross-correlation strength λ with different potential asymmetry r are shown in Fig.8. There is a resonant peak appearing in the curves. If the non-monotonous behaviour of the SNR as a function of some characteristics of the noise is considered as the SR in a broad sense, then SR in a broad sense occurs for this case. The position of the peak moves to the left with increasing r and the peak is the broadest one in the case $r = 0$.

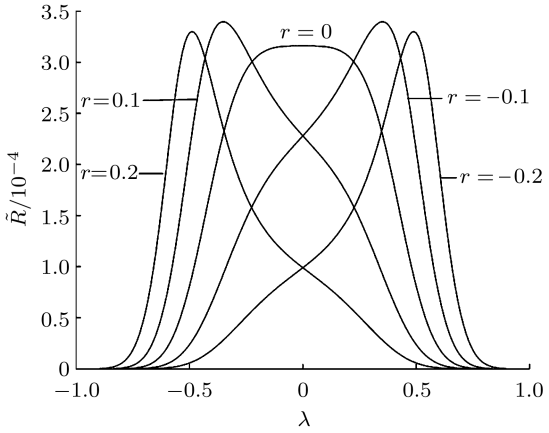


Fig.8. SNR as function of additive noise intensity λ with varied r for $A = 0.01,\ D = 0.1, \alpha = 0.1, \tau = 0.1$.

The curves of SNR $\tilde{R}$ as functions of cross-correlation strength λ with different cross-correlation time τ are shown in Fig.9. The SR exists in a broad sense. The position of the resonant peak shifts to the left with the increasing τ and the value the maximum of the peak decreases with increasing τ.

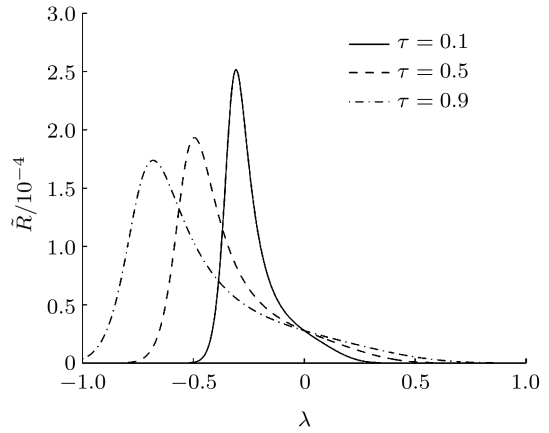


Fig.9. SNR as function of cross-correlation strength λ with varied τ for $A = 0.01,\ D = 0.03, \alpha = 0.02, \lambda = 0.1$.

In conclusion, SR phenomenon in an asymmetric bistable system driven by a weak periodic signal is investigated when the white multiplicative and additive noise are coupled with coloured correlation between two noise terms. Applying the method of the two – state theory, we derive the approximate expression of the SNR.

From Figs.1 and 4, we see that the potential asymmetry r can weaken the phenomenon of SR, which is consistent with the Refs.[17, 19]. From Figs.2 and 5, we know that the cross-correlation strength λ can weaken the phenomenon of SR. From Figs.3 and 6, we know that the cross-correlation time τ can strengthen the phenomenon of SR.

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