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An extended master-equation approach applied to aggregation in freeway traffic*

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We restudy the master-equation approach applied to aggregation in a one-dimensional freeway, where the decay transition probabilities for the jump processes are reconstructed based on a car-following model. According to the reconstructed transition probabilities, the clustering behaviours and the stochastic properties of the master equation in a one-lane freeway traffic model are investigated in detail. The numerical results show that the size of the clusters initially below the critical size of the unstable cluster and initially above that of the unstable cluster all enter the same stable state, which also accords with the nucleation theory and is known from the result in earlier work. Moreover, we have obtained more reasonable parameters of the master equation based on some results of cellular automata models.

Keywords: master equation, traffic jam, nucleation theory
PACC: 0500, 5110

1. Introduction

Traffic flows have attracted considerable attention,[1] for which there are two main reasons: the first one is the importance for traffic engineers who are concerned with the feasibility of attaining the limit capacity of traffic networks;[2] the second one is the interest of physicists who want to investigate the existence of various physical states and phase transitions in traffic systems so as to understand the fundamental principles governing the flow of vehicular traffic.

Over the past 50 years, many scientists, concerning the traffic problems, have developed various mathematical traffic models and traffic flow theories to describe the properties of traffic flows. In general, the models and the theories for studying traffic flows may be grouped into two kinds: one kind is to use the theories of mathematics and physics to describe 'macroscopic' properties of traffic flows, such as fluid-dynamical descriptions;[3−5] the other is to use microscopic simulations and in analogy to many-particle systems, where cars are considered as interacting 'elementary' particles and individual vehicles are focused on, such as car-following models[6−8] and cellular automata models.[9−12] These models and theories to explore the rules of traffic flows become more and more important for practical traffic engineering.

The spontaneous formation of traffic jams is one of the primary traffic problems. Many researchers, trying to describe it, have developed some theories. Kerner and Kinhäuser[13,14] have shown that, for high densities, a car cluster can spontaneously appear in which the average velocity of cars is considerably lower than in the initial flow and outside the cluster. Barlovic *et al*[15] studied the resolving probabilities, lifetimes of jam clusters and disturbances of a single-lane model based on velocity-dependent randomization model and the theory of random walk. The analytic solution to the master equation of aggregation in traffic has been obtained in the freeway traffic based on birth-death process.[16]

In analogy to the usual aggregation phenomena in nuclear physics, Mahnke and Pieret described the clustering behaviour based on a one-dimensional master equation and constructed the transition probabilities for the jump processes.[17] The stochastic master equation approach to describe traffic flow has been extended, which allows the situation that cars in a jam are not close to each other and that they are moving.[18] However, the decay transition probabilities for the jump processes are constant in the above two papers. It is the aim of the present paper to re-

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construct the decay transition probability and make it depend on some variable. Then we reinvestigate the clustering behaviours in a one-lane freeway traffic flow model based on our improvement and further numerical simulations.

2. Car-cluster model and the master equation

We will simply introduce a car-cluster model in the following. Traffic flow on a single-lane road is considered, and there is only the formation of a jam as a large car cluster arising on the road. Instead of dealing with a certain road part of length L and imposing some boundary conditions on its entries and exits, we examine a circular road of length L on which N cars are moving along. All the cars are assumed to be identical vehicles of effective length l_0. The cluster is specified by its size n, the number of aggregated cars.

Assume that there appears a size m of car cluster at time t, and consider that the probability of the attachment of a 'free' car unit to its upstream boundary appearing in $t \to t + \triangle t$ is $w_+(m,t)\triangle t$, and the rate of a car escaping from the cluster at its downstream front is $w_-(m,t)\triangle t$. Then the probability of no change in the unit number in $t \to t + \triangle t$ is $1 - (w_+(m,t) + w_-(m,t))\triangle t$. The probability of the change more than one unit appearing in $t \to t + \triangle t$ is zero.

Using the Markov theory,[19,20] we have the condition probability

$$\begin{aligned} & P_{1|1}(m,t|n,t+\triangle t) \\ = \; & w_+(m,t)\delta_{n,m+1}\triangle t - w_-(m,t) \\ & \times\delta_{n+1,m}\triangle t + (1 - (w_+(m,t) \\ & + w_-(m,t)))\delta_{n,m}\triangle t + O(\triangle t^2). \end{aligned} \tag{1}$$

The probability of taking n value at time t can be expressed as

$$P(n,t+\triangle t) = \sum_m P(m,t)P_{1|1}(m,t|n,t+\triangle t). \tag{2}$$

Substituting Eq.(1) into Eq.(2), we obtain the master equation of aggregation

$$\begin{aligned} \frac{\partial P(n,t)}{\partial t} = \; & w_+(n-1,t)P(n-1,t) + w_-(n+1,t) \\ & \times P(n+1,t) - (w_+(n,t) + w_-(n,t)) \\ & \times P(n,t), \end{aligned} \tag{3}$$

where $P(n,t)$ is the probability of finding a size -n cluster at time t and the size -n of the cluster satisfies the inequality $1 \le n \le (N-1)$. Then we obtain the same master equation as that given by Mahnke and Pieret.[17] We use the boundary conditions defined by Kühne:[21]

$$\frac{\partial P(0,t)}{\partial t} = w_-(1,t)P(1,t) - w_+(0,t)P(0,t), \tag{4}$$

$$\begin{aligned} \frac{\partial P(N,t)}{\partial t} = \; & w_+(N-1,t)P(N-1,t) \\ & - w_-(N,t)P(N,t), \end{aligned} \tag{5}$$

where Eqs.(4) and (5), respectively, describe the formation and dissolution of the cluster consisting of one car and the maximum possible cluster containing all the cars. Here function $P(0,t)$ is the probability of no cluster on the road. In order to deal with the master equation, we will need to choose the suitable transition rates based on phenomenological considerations.

3. Car-following model and the transition rates

Cars in a jam are not close to each other and they are moving, which has been extended by Mahnke.[18] The distance between the front bumpers of two neighbouring cars, in general, is $l_0 + \Delta x$, where Δx is understood as the headway between two 'effective' cars and l_0 is the identical length of vehicles, when the maximal velocity of each car is $v_{\max}$. Moreover, the desired velocity depends on the distance between two cars. The optimal velocity $v_{\rm opt}$ is given in dimensionless variables $w_{\rm opt} = v_{\rm opt}/v_{\max}$, and $\Delta y = \Delta x/l_0$ by the formula[18]

$$w_{\rm opt}(\Delta y) = \frac{(\Delta y)^2}{d^2 + (\Delta y)^2}, \tag{6}$$

where the parameter $d = D/l_0$ denotes a dimensionless parameter. D is the interaction distance between two cars corresponding to the velocity value $v_{\max}/2$. According to the expression (6), values of the dimensionless optimal velocity $w_{\rm opt}$ range from 0, corresponding to zero distance between cars, to 1, corresponding to an infinitely large distance or absence of interaction between cars.

Moving jams of highest density always occur in freeway traffic. The space distance $\Delta x_{\rm clust} = l_0 \Delta y_{\rm clust}$ denotes the distance between two neighbouring jammed cars, and it is a given measured quantity. The cases of $\Delta x_{\rm clust} = 0$ and $\Delta x_{\rm clust} \neq 0$

have been considered in earlier work.[17,18] Therefore, we will borrow some definitions from them. The length of the cluster (jam) and the average distance $\Delta x_{\rm free} = l_0 \Delta y_{\rm free}$ between neighbouring cars outside the jam, both of which depend on the number of congested cars n, are given by[18]

$$L_{\rm clust} = l_0 n + (n-1)\Delta x_{\rm clust}, \quad (7)$$

$$\Delta y_{\rm free}(n) = \frac{L/l_0 - N - (n-1)\Delta y_{\rm clust}}{N-n+1}, \quad (8)$$

where N is the total number of cars on the road.

To deal with Eq.(3), we will find expressions for the growth transition rate w_+ and the decay transition rate w_-. A car attaches itself to the cluster as soon as the distance to the last car in the cluster reduces from $\Delta x_{\rm free}$ to $\Delta x_{\rm clust}$. So the attachment rate to the cluster of size $n > 1$ is[18]

$$w_+(n) = b\frac{w_{\rm opt}(\Delta y_{\rm free}(n)) - w_{\rm opt}(\Delta y_{\rm clust})}{\Delta y_{\rm free}(n) - \Delta y_{\rm clust}}, \quad (9)$$

where $b = v_{\max}/l_0$. In earlier works(see Refs.[17] and [18]), the head car, which comes free, of the jam needs the average time τ, and the corresponding transition rate of the cars escaping from the cluster is simply $w_-(n) = 1/\tau$, where τ is a constant.

In physics, an isolated many-particle system will eventually reach equilibrium, regardless of what its initial state is. The typical time-scale for this process is called the relaxation time, which depends in detail on the nature of the inter-particle interactions. Since cars have been considered as interacting particles, many traffic models are regarded as many-particle systems. Therefore, the relaxation time is also a very important parameter in many highway traffic models. It describes the driver's technical-psychological quickness of adapting to the actual downstream state of the flow. We assume that the relaxation time is the time for the first car of the jam adapting to equilibrium state of the system. In earlier works, the relaxation time was assumed as a constant τ. K*ü*hne *et al*[21] have put forward an ansatz where the decay transition rate depends on size n of the cluster. However, the ansatz was based on the real data for a multi-lane highway. To extend the car-cluster model, we will modify the decay transition probability that depends on the size -n of the cluster in the following. It should include two parts: one is the response time lag τ_1 that describes the response delay time, and the other is the accelerating time τ_2 at which the driver adapts his speed to the mean speed of the free flow. Considering that the detachment of cars from the jams occurs more or less continuously in a real case, a parameter k is introduced to describe the correlative degree among continuously accelerating cars that leave the jam. If $k = 1$, this shows that each car of the cluster will not accelerate until the head car of the cluster attains the average speed. If $k < 1$, this indicates that cars will accelerate synchronously. When the cluster of size $n \geq 1$, we can write the following ansatz for τ_2

$$\tau_2 = \begin{cases} \dfrac{k}{b}\dfrac{\Delta y_{\rm free}(n) - \Delta y_{\rm clust}}{w_{\rm opt}(\Delta y_{\rm free}(n)) - w_{\rm opt}(\Delta y_{\rm clust})}, & \Delta y_{\rm free} < \Delta y^*, \\ \tau_\infty, & \Delta y_{\rm free} \geq \Delta y^*, \end{cases} \quad (10)$$

where Δy^* is the critical constant, and τ_∞ is a constant. When $\Delta y_{\rm free} \geq \Delta y^*$, cars are considered as noninteracting particles and the accelerating time of each car is uniform, which is assumed as τ_∞. When $\Delta y_{\rm free} \leq \Delta y^*$, we can further prove Eq.(10) by car-following models in the following.

In the car-following theories, each individual vehicle is in analogy to each individual particle in a system consisting of interacting classical particles. We consider a modified version of the car-following model,[6] which is described by the equation of motion of the ith car

$$\ddot{x}_i(t+T) = \frac{\lambda}{x_{i+1}(t) - x_i(t)}[\dot{x}_{i+1}(t) - \dot{x}_i(t)], \quad (11)$$

where λ is a constant, T is the delay time, and $x_i(t)$ denotes the longitudinal position along the roadway of the ith car at time t. If we assume that the acceleration of the car keeps invariable and the maximal speed is $\dot{x}_{i+1}(t)$, the accelerating time is proportional to the headway $\Delta x = x_{i+1}(t) - x_i(t)$. However, we consider it unreasonable if the headway is small. In order to make the decay rate become more reasonable, we can

assume the relation of τ_2 and $\Delta y_{\rm free}(n)$ to be

$$\tau_2 = A\Delta y_{\rm free}(n) + \frac{B}{\Delta y_{\rm free}(n)}, \tag{12}$$

where A and B are constants. If $\Delta y_{\rm clust}$ gets zero, Eq.(10) is satisfied with the assumption of Eq.(12). Thus the rate of the cars escaping from the cluster at its downstream front can be rewritten as

$$w_-(n) = \frac{1}{\tau_1+\tau_2} = \begin{cases} \dfrac{1}{\tau_1 + \dfrac{k}{b}\dfrac{\Delta y_{\rm free}(n) - \Delta y_{\rm clust}}{w_{\rm opt}(\Delta y_{\rm free}(n)) - w_{\rm opt}(\Delta y_{\rm clust})}}, & \Delta y_{\rm free} < \Delta y^*; \\ \dfrac{1}{\tau_1+\tau_\infty}, & \Delta y_{\rm free} \geq \Delta y^*. \end{cases} \tag{13}$$

We will obtain the value of k and prove, by numerical results, that our descriptions are reasonable in Sec.4.

4. Numerical results and stochastic simulations

A cluster arising on the road depends on the ratio of $w_+(n)$ to $w_-(n)$. When the attachment rate is larger than the decay rate, the cluster size will grow. In the opposite case, the cluster size will disassemble, corresponding to the condition that the growing transition rate will always be less than the decay transition rate in low densities.

In the free-flow regime, all the vehicles move with a constant speed of $v_{\rm max}$ so as to be below the critical density based on cellular automaton models for freeway traffic and other fundamental diagrams.[9−12] When the critical density is reached, the average space headway h_{s_i} is $v_{\rm max}+l_0$ for the ith vehicle. The value for the critical density $\rho_{\rm c}$ in dimensionless is[22]

$$\rho_{\rm c} = \frac{1}{v_{\rm max}/l_0 + 1}. \tag{14}$$

Let $\Delta y^* = v_{\rm max}/l_0 = b$ in the expression (13). For the parameter τ_1 that describes the response time of the drivers, in general, the value of it is one second. Considering that the condition of stable state is $w_+ = w_-$ and $\tau_1 = 1\,$s, one can obtain the value k by using the following expression with the help of Eq.(13):

$$\frac{w_+(\Delta y^*)}{w_-(\Delta y^*)} = w_+(\Delta y^*) + k = 1. \tag{15}$$

Correspondingly, the value of τ_∞ can be gained.

It is necessary to choose reasonable parameters based on practical flow-density behaviours because calculations are made for a finite system. We choose the same parameter values as those in Ref.[17] so as to compare with the previous results. The parameters are $l_0 = 6\,$m, $D = 13\,$m, $v_{\rm max} = 34\,$m/s, and $\Delta x_{\rm clust} = 1\,$m.[17] The corresponding dimensionless parameters are $b = v_{\rm max}/l_0 = 5.67$, $d = D/l_0 = 2.167$, $\Delta y^* = v_{\rm max}/l_0 = 5.67$ and $l_0\Delta y_{\rm clust} = \Delta x_{\rm clust}$. Using the expressions (13), (14) and (15), we obtain $k = 0.108$ and $\tau_\infty = 0.112\,$s.

In Fig.1, we show the curves of w_+/w_- versus $\Delta x_{\rm free}(= l_0\Delta y_{\rm free})$ for different decay rates. In the curves, there are two values Δx_1 and Δx_2 for $w_+/w_- = 1$. When $k = 0.108$ and $\tau_\infty = 0.112\,$s, which are obtained based on the cellular automata model, curve I is obtained. Curve III shows the result

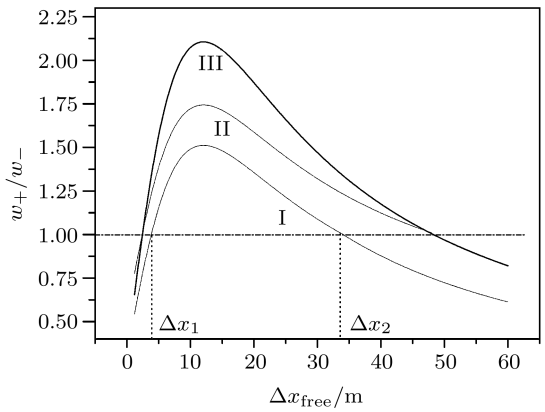


Fig.1. The plot of ratios of transition rates $w_+(\Delta x_{\rm free})/w_-(\Delta x_{\rm free})$ against $\Delta x_{\rm free}$ (the parameters of curve I are $k = 0.22, \tau_\infty = 0.28\,$s; those of curve II are $k = 0.334, \tau_\infty = 0.50\,$s; and the parameter of curve III is $\tau = 1.5\,$s).

of Mahnke,[18] with $\tau = 1.5\,\mathrm{s}$ being a const, and the condition $w_+(\Delta x)/w_-(\Delta x) = 1$ leads to two solutions $\Delta x_1 = 2.4\,\mathrm{m}$ and $\Delta x_2 = 48.5\,\mathrm{m}$. To compare with the result of Mahnke and Kaupužs,[18] we set $\Delta y^* = \Delta x_2/l_0 = 8.08$. With the use of Eqs.(13) and (15), we obtain curve II, corresponding to $\tau_\infty = 0.50\,\mathrm{s}$ and $k = 0.33$. It is surprising that the Δx_1 of curve II is the same as that of the curve III, which proves our assumption in Sec.3 being reasonable and feasible. Since the result of Mahnkes and Kaupužs is based on the multi-lane freeway,[18] we think that the result of ours which is based on cellular automata models is more reasonable for a single-lane freeway.

In low-density regime, the minimal value of Δx_{free} (homogeneous flow) is larger than Δx_2. For intermediate densities, the minimal value of Δx_{free} lies between Δx_1 and Δx_2. The minimal value of Δx_{free} in high densities is smaller than Δx_1. Hence, the density range is divided into three parts, which corresponds, respectively, to no jams, one equilibrium size and two equilibrium sizes.[18] We will illustrate the three parts in Fig.2. In Fig.2, the region II shows $w_+(c, c_1)/w_-(c, c_1) \geq 1$, where $c = N/N_{\max}$ and $c_1 = n/N_{\max}$; $n1(c)$ and $n2(c)$ show the two edges of the region II and correspond to $w_+(c, c_1)/w_-(c, c_1) = 1$. Therefore, the region $c \in [0, C_I]$ corresponds to the free traffic flow. In the region $c \in [C_{\mathrm{I}}, C_{\mathrm{II}}]$ there exists one cluster that coexists with a region of free traffic, and correspondingly there is only one equilibrium boundary $n1(c)$. There are two equilibrium sizes in the region $c \in [C_{\mathrm{II}}, 1]$ corresponding to two equilibrium boundaries $n1(c)$ and $n2(c)$. The cluster with a greater size n_{stab} is stable and that with smaller size n_{unst} is unstable. If the decay transition probabilities are defined and the case $\Delta x_{\mathrm{clust}} = 0$ is considered, then stationary solutions for the mean size of car clusters are obtained as follows:

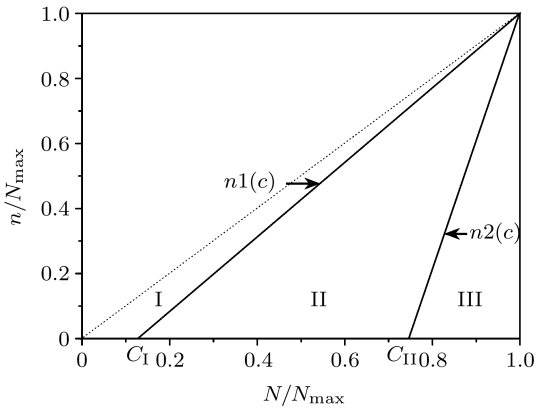


Fig.2. The plot of $n/N_{\max}$ as a function of $N/N_{\max}$. The region II shows $w_+(N/N_{\max}, n/N_{\max})/w_-(N/N_{\max}, n/N_{\max}) \geq 1$ (N denotes the number of cars moving on the road L, and n denotes the size of the cluster). The lines $n1(c)$ and $n2(c)$ are edges of region II, the values of which are 1. $N_{\max}$ denotes the maximal number of cars allowed on the road L.

$$n = \begin{cases} N + 1 - \dfrac{\left(\dfrac{L}{l_0} - N\right)(k-1)}{-b + \sqrt{b^2 + 4d^2 + 8d^2k - 4d^2k^2}}, & n \in n1(c), \\ N + 1 - \dfrac{\left(\dfrac{L}{l_0} - N\right)(k-1)}{-b - \sqrt{b^2 + 4d^2 + 8d^2k - 4d^2k^2}}, & n \in n2(c). \end{cases} \tag{16}$$

In analogy to the usual aggregation phenomena, Mahnke thought that there exists a nucleation barrier n_{bar} for the formation of the macroscopic cluster from an initially homogeneous state.[18] The size of nucleation barrier is the size of unstable cluster, i.e., $n_{\mathrm{bar}} = n_{\mathrm{unst}}$. A stable growth of the cluster can occur merely at $n > n_{\mathrm{unst}}$, whereas a stable dissolution of cluster can take place at $n < n_{\mathrm{unst}}$. The system can switch between the two macroscopic states due to the stochastic fluctuations. We reinvestigate the stochastic properties of the master equation. If the size of the cluster is initially below n_{unst}, we find that the cluster will dissolve and subsequently can grow until the stable state is reached (see Fig.3). Numerical experiments based on our studies prove that the bistable system finally evolves to the same stable state for different initial clusters, which is different from the result in Ref.[18]. These results satisfy the nucleation theory in physics, i.e., any metastable state will sooner or later enter stable state.[23] Comparing with the pio-

neer work,[18] we see that the extended works include the perturbations, and that there exists only a stable state.

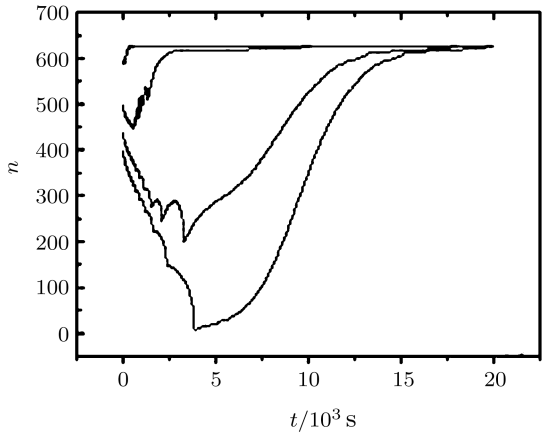


Fig.3. Stochastic trajectories starting with different initial sizes at a very high density 0.91 (total number of cars $N = 650$) together with the mean value ($N_{\max} = 714$ and $n_{\text{unst}} = 441$).

5. Conclusion

This paper first extends the stochastic master equation approach to describe the clustering behaviours in a one-lane freeway traffic. Based on the extended car-cluster model, we reconstruct the decay transition probabilities for jump processes in terms of a modified version of the car-following model, where the decay transition probabilities depend on the size of the cluster. The parameter values of Eq.(13) depend on the minimal noninteracting distance in dimensionless Δy^* between two neighbouring cars. In terms of the cellular automaton models, the decay transition probabilities are obtained in a one-lane traffic freeway traffic model (see Fig.1), and we believe that they are more reasonable than the results in earlier work. The density range, which is divided into three parts by two critical values, is shown clearly in Fig.2 : free flow, coexisting phases and over-crowd situation. Secondly, the stochastic properties of the master equation in a traffic system are investigated in detail. According to the first proposed Eq.(13), the numerical results have shown that the size of the cluster initially below the critical size of the unstable cluster will dissolve and afterwards can grow until the same stable states in which the size of the cluster initially above the critical size of the unstable cluster arrives in the high density region. As a result, our result accords with the nucleation theory, namely, any metastable state will finally enter stable state. In conclusion, our improvements are feasible, and numerical results of our researches prove that our results not only show that they are in agreement with the practical situations of freeway traffic, but also provide an example of the use of master equation to practical traffic problems.

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