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Entropy of quantum field in toroidal black hole without brick wall

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In this paper the entropy of a toroidal black hole due to a scalar field is investigated by using the DLM scheme. The entropy is renormalized to the standard Bekenstein–Hawking formula with a one-loop correction arising from the higher curvature terms of the gravitational action. For the scalar field, the renormalized Newton constant and two renormalized coupling constants in the toroidal black hole are the same as those in the Reissner–Nordstrom black hole except for other one.

Keywords: black hole, entropy renormalization, brick-wall model, scalar field
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1. Introduction

Since Bekenstein and Hawking's pioneering works[1−3] which provided that the entropy of a black hole is proportional to its horizon area, many efforts have been devoted to understanding the entropy of black holes.[4−6] One possible method is the brick-wall model put forward by 't Hooft,[6] in which the entropy of a black hole is identified with the entropy of a thermal gas of a quantum field excitations outside the event horizon. However, the density of states diverges as it approaches the horizon, and thus 't Hooft had to introduce a cutoff to eliminate this divergence. Subsequently, many authors give approaches to improve this model. For example, Li, Zhao and other authors presented a thin film brick-wall model by considering the entropy of a black hole only due to a thin layer quantum field outside near the event horizon.[7−9] In the other hand, Demers, Lafrance and Myers (DLM) applied a Pauli–Villars regulator to the study of the entropy of a black hole in order to avoid the ultraviolet divergence instead of the cutoff.[10] The (original and improved) brick-wall models have been successfully used in studies of the statistical-mechanical entropy of the black hole.[10−24]

In Ref.[10], DLM studied the entropy of a Reissner–Nordstrom black hole due to a scalar field. The original divergence in the brick wall is regulated by the very large mass of five fictitious regulator fields. The calculation of the brick-wall model gives the entropy as two terms: one is proportional to the horizon area and the other is a constant. Then they renormalized the entropy by introducing the Newton's constant and the entropy correction arising from the higher curvature effective gravitational action. Finally, they got the renormalized entropy whose leading term is proportional to the horizon area of the black hole and the others is precisely the one-loop correction arising from the higher curvature terms of the gravitational action. Their work confirmed the suggestion by Susskind and Uglum[25] that these divergences have the correct form to be absorbed in the Bekenstein–Hawking formula as a renormalization of Newton's constant.

People have some special interest in studying black holes with non-trivial topologies. For example, a toroidal black hole with horizon topology $S^1 \times S^1$ has been studied in Ref.[26–28]. Subsequently, its thermodynamical behaviour has been investigated by using the path-integral approach in York's formalism.[29] They gave that the entropy is one fourth of the horizon area, and thus satisfies the Bekenstein–Hawking entropy formula. Recently we also studied the thermodynamical properties of the toroidal black hole by using the thin film brick-wall model.[30] Our results in Ref.[30] is that the entropy of a toroidal black due to a scalar field is one fourth of the horizon area, and that due to a Dirac field is seven eighth of its area. These results are similar to that in black holes with horizon

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topology S^2.

In this paper, we investigate the entropy of the toroidal black hole by applying DLM's scheme. Firstly we calculate a scalar field entropy by using t'Hooft's brick model. By introducing five fictitious regulator fields, we then obtain the regulated entropy without a cutoff. Finally, the renormalized entropy of the black hole is given by considering that the entropy is the sum of the bare black hole entropy, the scalar field entropy and the entropy correction ΔS for higher curvature effective action. We find that the renormalized Newton's constant and two of three renormalized coupling constants in the toroidal black hole are the same as those in the Reissner–Nordstrom black hole except for one of them. These results give another evidence that confirms the suggestion by Susskind and Uglum.

2. Properties of a toroidal black hole

A generally statically plane-symmetric solution of the Einstein–Maxwell equations with cosmological constant Λ can be expressed as[31,32]

$$\begin{aligned} \mathrm{d}s^2 =& -\left(-\frac{1}{3}\Lambda r^2 - \frac{c}{r} + \frac{d^2}{r^2}\right)\mathrm{d}t^2 \\ &+ \left(-\frac{1}{3}\Lambda r^2 - \frac{c}{r} + \frac{d^2}{r^2}\right)^{-1}\mathrm{d}t^2 \\ &+ r^2\left(\mathrm{d}\varphi^2 + \mathrm{d}\psi^2\right), \end{aligned} \tag{1}$$

where c and d are two integration constants to be determined. By identifying $\varphi = 0$ with $\varphi = 2\pi$ and $\psi = 0$ with $\psi = 2\pi$ in Eq.(1), one can construct a toroidal solution, which describes a spacetime with a toroidal topology. The submanifold by setting t and r constant is a torus with area of $4\pi^2 r^2$, and its topology is $S^1 \times S^1$. In this case, one can further determine the integration constants c and d, and thus can express the metric (1) as[28]

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + f(r)^{-1}\,\mathrm{d}r^2 + r^2\left(\mathrm{d}\varphi^2 + \mathrm{d}\psi^2\right), \tag{2}$$

where

$$f(r) = -\frac{\Lambda r^2}{3} - \frac{2M}{\pi r} + \frac{4Q^2}{\pi r^2}, \tag{3}$$

M and Q are the mass and electric charge of the spacetime, respectively. The metric (2) becomes singular for $f(r) = 0$, which has two positive solutions r_+ and r_- as long as $0 \le Q^2 \le \frac{3}{8}\left(3M^4/2\pi|\Lambda|\right)^{1/3}$ for a negative $\Lambda < 0$. So the metric (2) describes a black hole with a Cauchy horizon r_- and an event horizon r_+ in the case of $\Lambda < 0$. When $Q^2 = \frac{3}{8}\left(3M^4/2\pi|\Lambda|\right)^{1/3}$, the event horizon and the Cauchy horizon coincide. For simplicity we only consider the case of $r_+ > r_-$. For convenience, we denote the other two negative roots of $f(r) = 0$ as $-a$ and $-b$, respectively, and then rewrite the Eq.(3) as

$$\begin{aligned} f(r) =& -\frac{\Lambda}{3}r^2\left(1 - \frac{r_+}{r}\right)\left(1 - \frac{r_-}{r}\right) \\ &\times \left(1 + \frac{a}{r}\right)\left(1 + \frac{b}{r}\right), \end{aligned} \tag{4}$$

where $-\Lambda > 0$, and r_+, r_-, a and b satisfy the following relations:

$$r_+ + r_- = a + b, \tag{5}$$

$$r_+ r_- - (r_+ + r_-)(a + b) + ab = 0, \tag{6}$$

$$r_+ r_-(a + b) - (r_+ + r_-)ab = \frac{6M}{\pi\Lambda}, \tag{7}$$

$$r_+ r_- ab = -\frac{12Q^2}{\pi\Lambda}. \tag{8}$$

The surface gravity κ of an event horizon can be given[33] as $\kappa^2 = -\frac{1}{2}\left(\nabla^\mu \chi^\nu\right)\left(\nabla_\mu \chi_\nu\right)$, where the evaluation is on the horizon and χ^μ is a Killing field that is normal to the horizon. Setting $\chi^\mu = \delta^\mu_0$, we can easily obtain $\kappa = \frac{1}{2}f'(r)$, where a prime stands for the derivative with respect to r. Therefore the inverse temperature of a toroidal black hole can be expressed as

$$\beta = \frac{2\pi}{\kappa} = \frac{4\pi}{f'(r_+)}. \tag{9}$$

3. Free energy of a scalar field

The Klein–Gordon equation in a curved spacetime can be written as

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial \Phi}{\partial x^\nu}\right) = m_0^2 \Phi, \tag{10}$$

where m_0 is the mass of a scalar field. According to the toroidal symmetry of the black hole, the wavefunction of the scalar field can be separately expressed as

$$\Phi = \mathrm{e}^{-\mathrm{i}Et}\mathrm{e}^{\mathrm{i}m\varphi}\mathrm{e}^{\mathrm{i}n\psi}R(r), \tag{11}$$

where $R(r)$ is only a function of r, E is the energy of the excitation state of the scalar field, and both m and n are integers for guarantying the periodic conditions of both φ and ψ, respectively. Substituting metric (2) into Eq.(10) and using Eq.(11), we can get the radial equation of the wavefunction

$$\begin{aligned} &f(r)\frac{\partial^2}{\partial r^2}R(r) + \left[2r^{-1}f(r) + f'(r)\right]\frac{\partial}{\partial r}R(r) \\ &+ \left[E^2 f^{-1}(r) - \left(m^2 + n^2\right)r^{-2} - m_0^2\right]R(r) = 0. \end{aligned} \tag{12}$$

Now we apply Wentzel–Kramers–Brillouin (WKB) approximation in Eq.(12). In this method $R(r)$ should be written as $R(r) \sim \mathrm{e}^{\mathrm{i}S(r)}$, and then substituting it into Eq.(12) we obtain the radial wave number

$$k_r = \frac{\partial S(r)}{\partial r} = f^{-1}(r)\sqrt{E^2 - [(m^2+n^2)r^{-2} + m_0^2]f(r)}. \quad (13)$$

For the semiclassical quantization condition, the radial wave number must satisfy the following relation

$$\int_{r_++h}^{L} k_r \mathrm{d}r = \pi n_r, \quad (14)$$

where n_r is a positive integer, h and L are two cutoff parameters introduced by 't Hooft.[6] Here h is a small number called the thickness of a brick wall.

According to the theory of a canonical ensemble, the free energy of a scalar field can be written as

$$\beta F = \sum_E \ln\left(1 - \mathrm{e}^{-\beta E}\right). \quad (15)$$

In semiclassical level, since energy E can be assumed to be continuous, the sum over E can be replaced by integration, i.e.

$$\begin{aligned}\beta E &= \int_0^{+\infty} g(E)\ln\left(1-\mathrm{e}^{-\beta E}\right)\mathrm{d}E \\ &= \int_0^{+\infty} \ln\left(1-\mathrm{e}^{-\beta E}\right)\mathrm{d}\Gamma(E) \\ &= \Gamma(E)\ln\left(1-\mathrm{e}^{-\beta E}\right)\Big|_0^{+\infty} \\ &\quad - \int_0^{+\infty} \frac{\Gamma(E)\,\mathrm{e}^{-\beta E}}{(1-\mathrm{e}^{-\beta E})}\beta \mathrm{d}E \\ &= -\beta\int_0^{+\infty} \frac{\Gamma(E)\,\mathrm{e}^{-\beta E}}{(\mathrm{e}^{\beta E}-1)}\mathrm{d}E, \end{aligned} \quad (16)$$

where $\Gamma(E)$ is the total number of microscopic states with energy less than E, $g(E) = \mathrm{d}\Gamma(E)/\mathrm{d}E$ is the density of states. In fact, according to the definition of $\Gamma(E)$, it can be written as

$$\begin{aligned}\Gamma(E) &= \sum_{m,n} n_r(E,m,n) \\ &= \int n_r(E,m,n)\,\mathrm{d}m\mathrm{d}n \\ &= \frac{1}{\pi}\int \mathrm{d}m \int \mathrm{d}n \int_{r_++h}^{L} f^{-1}(r) \\ &\quad \times \sqrt{E^2 - [(m^2+n^2)r^{-2} + m_0^2]f(r)}\mathrm{d}r, \end{aligned} \quad (17)$$

where Eqs.(13) and (14) have been used. The lower bound of the integrations for both m and n are zeros and upper bound which should be determined by guarantying the square root in the integrand is real. For solution of the integration, we transform (m,n) to (V,ξ) as $m = V\cos\xi$ and $n = V\sin\xi$, and then from above equation we have

$$\begin{aligned}\Gamma(E) =& \frac{2}{3}\int_{r_++h}^{L} r^2 f^{-2}(r) \\ &\times \left(E^2 - m_0^2 f(r)\right)^{3/2}\mathrm{d}r. \end{aligned} \quad (18)$$

By inserting Eq.(18) into Eq.(16), we get the free energy,

$$\begin{aligned}F =& -\frac{2}{3}\int_0^{+\infty}\frac{\mathrm{d}E}{\mathrm{e}^{\beta E}-1}\int_{r_++h}^{L} r^2 f^{-2}(r) \\ &\times \left(E^2 - m_0^2 f(r)\right)^{3/2}\mathrm{d}r, \end{aligned} \quad (19)$$

where the remaining integration is still taken for values where the square root is real. It is evident that the integration diverges as r approaches r_+ because of $f(r_+) = 0$. By introducing a new variable, $s = 1 - r_+/r_-$, the free energy can be given as

$$\begin{aligned}F =& -\frac{6}{(-\Lambda)^2 r_+}\int_0^{+\infty}\frac{\mathrm{d}E}{\mathrm{e}^{\beta E}-1} \\ &\times \int_{h'}^{L'} \frac{\left[E^2 - m_0^2(-\Lambda/3)r_+^2 s(1-s)^{-2}(1-u+us)(1+v-vs)(1+w-ws)\right]^{3/2}}{s^2(1-u+us)^2(1+v-vs)^2(1+w-ws)^2}\mathrm{d}s, \end{aligned} \quad (20)$$

where Eq.(4) has been used, and $u = r_-/r_+$, $v = a/r_+$, $w = b/r_+$, $L' = 1 - r_+/L$, $h' = h/(r_+ + h) \approx h/r_+$. For a small value of s, we have $\int_{h'} \mathrm{d}s/s^2 \approx -1/h'$, which diverges as $h' \to 0$. For this reason, 't Hooft introduced a cutoff to avoid the divergence. However, in the next section, we shall eliminate it by

using the Pauli–Villars regulator instead of the cutoff.

4. Entropy of the scalar field

Following DLM scheme,[10] we also adopt a Pauli–Villars regularization and thus introduce five fictitious regulator fields: ϕ_1, ϕ_2, ϕ_3, ϕ_4 and ϕ_5, with masses

$$m_{1,2} = \sqrt{\mu^2 + m_0^2}, \tag{21}$$

$$m_{3,4} = \sqrt{3\mu^2 + m_0^2}, \tag{22}$$

$$m_5 = \sqrt{4\mu^2 + m_0^2}, \tag{23}$$

where μ is the mass parameter. Here ϕ_1, ϕ_2 and ϕ_5 are three anticommuting fields, and ϕ_3 and ϕ_4 are two commuting fields. The original scalar field is ϕ_0 with mass m_0. Considering the five fields, we can repeat 't Hooft calculation. Each of the fields makes a contribution to the free energy, and the total free energy becomes

$$\bar{F} = -\frac{6}{(-\Lambda)^2 r_+}\sum_{i=0}^{5}\Delta_i\int_0^{+\infty}\frac{\mathrm{d}E}{\mathrm{e}^{\beta E}-1} \times\int_{h'}^{L'}\frac{\left[E^2 - m_i^2\,(-\Lambda/3)\,r_+^2 s\,(1-s)^{-2}\,(1-u+us)\,(1+v-vs)\,(1+w-ws)\right]^{3/2}}{s^2\,(1-u+us)^2\,(1+v-vs)^2\,(1+w-ws)^2}\mathrm{d}s, \tag{24}$$

where $\Delta_0 = \Delta_3 = \Delta_4 = +1$ for the commuting fields, and $\Delta_1 = \Delta_2 = \Delta_5 = -1$ for the anticommuting fields. These specially designated regulator fields are chosen in order to regulate the free energy to be finite as $h' \to 0$. The divergence of the revised free energy in Eq.(21) cancels out because of $\sum_{i=0}^{5}\Delta_i\int s^{-2}\mathrm{d}s = 0$ and $\sum_{i=0}^{5}\Delta_i m_i^2\int s^{-1}\mathrm{d}s = 0$. Integrating over s and E, and only considering the divergent contributions at the horizon, we obtain

$$\bar{F} \approx \frac{4\pi^2\eta}{5\,(-\Lambda)^2 r_+\,(1-u)^3\,(1+v)^3\,(1+w)^3}\frac{1}{\beta^4}A - \frac{\pi^2 r_+}{2\,(-\Lambda)\,(1-u)\,(1+v)\,(1+w)}\frac{1}{\beta^2}B, \tag{25}$$

where

$$A = \ln\frac{4\mu^2+m_0^2}{m_0^2} + 2\ln\frac{\mu^2+m_0^2}{3\mu^2+m_0^2}, \tag{26}$$

$$B = m_0^2\left(\ln\frac{m_0^2}{m_0^2+4\mu^2} + 2\ln\frac{m_0^2+3\mu^2}{m_0^2+\mu^2}\right) + \mu^2\left(2\ln\frac{m_0^2+3\mu^2}{m_0^2+\mu^2} + 4\ln\frac{m_0^2+3\mu^2}{m_0^2+4\mu^2}\right), \tag{27}$$

$$\eta = u + (v+w)\,(2u-1) + vw\,(3u-2) = 3\,(uvw-1)\,. \tag{28}$$

Here we have used Eqs.(5) and (6) in Eq.(28). The entropy is then given by

$$S = \beta^2\frac{\partial\bar{F}}{\partial\beta} = \frac{16\pi^4\eta}{5\,(-\Lambda)^2 r_+\,(1-u)^3\,(1+v)^3\,(1+w)^3}\frac{1}{\beta^3}A + \frac{\pi^2 r_+}{(-\Lambda)\,(1-u)\,(1+v)\,(1+w)}\frac{1}{\beta}B. \tag{29}$$

From Eq.(9) we have the inverse temperature $\beta = 12\pi\left[(-\Lambda)\,r_+\,(1-u)\,(1+v)\,(1+w)\right]^{-1}$. So the entropy can be expressed as

$$S = \frac{\overline{A}}{4}\frac{B}{12\pi} + \frac{\pi\eta\,(-\Lambda)\,r_+^2 A}{540}, \tag{30}$$

where $\overline{A} = 4\pi^2 r_+^2$ is the event horizon area of the toroidal black hole.

5. Effective action and entropy renormalization

The gravitational action with higher curvature terms can be expressed as

$$I_g = \int\mathrm{d}^4x\sqrt{-g}\Big[-\frac{\Lambda_B}{8\pi G_B} + \frac{R}{16\pi G_B} + \frac{\alpha_B}{4\pi}R^2 + \frac{\beta_B}{4\pi}R_{ab}R^{ab} + \frac{\gamma_B}{4\pi}R_{abcd}R^{abcd} + \cdots\Big], \tag{31}$$

where Λ_B is the cosmological constant, G_B is Newton's constant, while α_B, β_B and γ_B are dimensionless coupling constants for the interactions which are quadratic in the curvature. The subscript B indicates

that all of these constants are the bare coupling constants. The action for a minimally coupled neutral scalar field is

$$I_m = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[g^{ab}\nabla_a\phi\nabla_b\phi + m_0^2\phi^2\right]. \quad (32)$$

The one-loop effective action of the scalar field, with the DeWitt–Schwinger proper time representation, is given by[34,10]

$$W(g) = -\frac{1}{32\pi^2}\int \mathrm{d}^4x\sqrt{-g}\int_0^\infty \frac{\mathrm{d}s}{s^3} \times \sum_{n=0}^{\infty} a_n(x)(\mathrm{i}s)^n \mathrm{e}^{-\mathrm{i}m_0^2 s}, \quad (33)$$

where the expansion coefficients $a_n(x)$ are constructed with the metric and the curvature tensor, and the first three coefficients are

$$a_0(x) = 1, \quad (34)$$

$$a_1(x) = \frac{1}{6}R, \quad (35)$$

$$a_2(x) = \frac{1}{180}R^{abcd}R_{abcd} - \frac{1}{180}R^{ab}R_{ab} + \frac{1}{30}\Box R + \frac{1}{72}R^2. \quad (36)$$

Obviously, the ultraviolet divergences arise as $s \to 0$ in the first three terms of the series(30). Using a Pauli–Villars regularization scheme and introducing the five fictitious regulator fields in previous section, one can divide the divergent part from $W(g)$ in the form

$$\begin{aligned} W_{\mathrm{div}} =& -\frac{1}{32\pi^2}\int \mathrm{d}^4\sqrt{-g}\int_0^\infty \frac{\mathrm{d}s}{s^3}\left[a_0(x) + \mathrm{i}sa_1(x) + (\mathrm{i}s)^2 a_2(x)\right] \\ & \times \left[\mathrm{e}^{-\mathrm{i}m_0^2 s} - 2\mathrm{e}^{-\mathrm{i}\left(\mu^2+m_0^2\right)s} + 2\mathrm{e}^{-\mathrm{i}\left(3\mu^2+m_0^2\right)s} - \mathrm{e}^{-\mathrm{i}\left(4\mu^2+m_0^2\right)s}\right] \\ =& \frac{1}{32\pi^2}\int \mathrm{d}^4x\sqrt{-g}\int_0^\infty \left[-Ca_0(x) + Ba_1(x) + Aa_2(x)\right]\mathrm{d}s, \end{aligned} \quad (37)$$

where

$$\begin{aligned} C =& \mu^4\left[8\ln\frac{3\mu^2+m_0^2}{4\mu^2+m_0^2} + \ln\frac{3\mu^2+m_0^2}{\mu^2+m_0^2}\right] + 2m_0^2\mu^2\left[\ln\frac{3\mu^2+m_0^2}{\mu^2+m_0^2} + 2\ln\frac{3\mu^2+m_0^2}{4\mu^2+m_0^2}\right] \\ & + \frac{m_0^4}{2}\left[\ln\frac{m_0^2}{4\mu^2+m_0^2} + 2\ln\frac{3\mu^2+m_0^2}{\mu^2+m_0^2}\right], \end{aligned} \quad (38)$$

and Eqs.(26) and (27) have been used. By combining the scalar one-loop action with the original bare action in Eq.(31), we get the effective gravitation action

$$\begin{aligned} I_{\mathrm{eff}} =& I_g + W \\ =& \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{8\pi}\left(\frac{\Lambda_B}{G_B} + \frac{C}{4\pi}\right) + \frac{R}{16\pi}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) + \frac{R^2}{4\pi}\left(\alpha_B + \frac{A}{576\pi}\right)\right. \\ & \left. + \frac{1}{4\pi}R_{ab}R^{ab}\left(\beta_B - \frac{A}{1440\pi}\right) + \frac{1}{4\pi}R_{abcd}R^{abcd}\left(\gamma_B + \frac{A}{1440\pi}\right) + \cdots\right]. \end{aligned} \quad (39)$$

From Eq.(39), one obtains the renormalized Newton's constant G_R and coupling constants α_R, β_R and γ_R,

$$\frac{1}{G_R} = \frac{1}{G_B} + \frac{B}{12\pi}, \quad (40)$$

$$\alpha_R = \alpha_B + \frac{A}{576\pi}, \quad (41)$$

$$\beta_R = \beta_B - \frac{A}{1440\pi}, \quad (42)$$

$$\gamma_R = \gamma_B + \frac{A}{1440\pi}. \quad (43)$$

Now we renormalize the entropy. According to the DLM scheme[10], the entropy of a black hole is the sum of a bare black hole entropy, $S_{\mathrm{BH}} = \overline{A}/(4G_B)$, and the scalar field entropy. So the entropy of the toroidal black hole should be expressed as

$$\begin{aligned} S_{\mathrm{total}} =& S_{\mathrm{BH}} + S \\ =& \frac{\overline{A}}{4}\left(\frac{1}{G_B} + \frac{B}{12\pi}\right) + \frac{\pi\eta(-\Lambda)r_+^2 A}{540} \\ =& \frac{\overline{A}}{4}\frac{1}{G_R} + \frac{\pi\eta(-\Lambda)r_+^2 A}{540}, \end{aligned} \quad (44)$$

where Eq.(40) have been used. Like the case of the Reissner–Nordstrom black hole,[10] the first contribution of the entropy (30) provides precisely the one-loop renormalization of the Bekenstein–Hawking entropy. For accounting for the second term proportional to A in Eq.(30), we should calculate the one-loop correction to entropy due to higher curvature terms in the gravitational action[10]

$$\Delta S = \oint \mathrm{d}^2x\sqrt{h}(2\alpha_B R + \beta_B g_\perp^{ab} R_{ab} - \gamma_B R^{abcd}\widehat{\varepsilon}_{ab}\widehat{\varepsilon}_{cd}), \quad (45)$$

where the integral is evaluated over a space-like cross section of the event horizon, $g_\perp^{ab}$ is the metric in the normal subspace to this cross section, and $\widehat{\varepsilon}_{ab}$ is the binormal to the cross section. For the toroidal black hole, by direct calculation we have

$$\Delta S = \frac{8\pi^2(-\Lambda)r_+^2}{3} \times \left[-12\alpha_B + (3+uvw)\beta_B + 4uvw\gamma_B\right]. \quad (46)$$

So the total black hole entropy is

$$S_{\text{total}} = S_{\text{BH}} + S + \Delta S = \frac{1}{4G_R}\overline{A} + \frac{8\pi^2(-\Lambda)r_+^2}{3} \times\left[-12\alpha_B + (3+uvw)\beta_R + 4uvw\gamma_R\right], \quad (47)$$

where Eqs.(5), (6), (40), (42) and (43) have been used. In this manner, the scalar field entropy (30) accounts for precisely the scalar one-loop renormalization of the full black hole entropy.

6. Conclusion

By using the t' Hooft model and a Pauli–Villars regulator, we obtain the entropy of a scalar field in a toroidal black hole. The entropy (30) has two terms: one is proportional to the horizon area and other is not. The first contribution proportional to B provides precisely the one-loop renormalization of the standard Bekenstein–Hawking entropy. The second contribution is absorbed into the one-loop entropy correction for higher curvature effective action ΔS. It is noted that in a toroidal black hole the renormalized Newton's constant G_R and the renormalized coupling constants β_R and γ_R are the same as those in Reissner–Nordstrom (RN) black hole, though their one-loop entropy corrections are different. It is natural since the same scalar field is considered in the two black holes. It is also worthwhile to note that the one-loop correction entropy in Eq.(47) contains the term with α_B while the corresponding terms in RN[10] and Schwarzschild black hole[7] do not. The reason is that Ricci scalars of RN and Schwarzschild black holes are zeros, but for toroidal spacetime Ricci scalar is not zero. But it is a problem to be further studied that why the naked coupling constant α_B appears in Eq.(47) instead of renormalized one α_R.

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