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Some new exact solutions of Jacobian elliptic function about the generalized Boussinesq equation and Boussinesq–Burgers equation*

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By using the modified mapping method, we find some new exact solutions of the generalized Boussinesq equation and the Boussinesq–Burgers equation. The solutions obtained in this paper include Jacobian elliptic function solutions, combined Jacobian elliptic function solutions, soliton solutions, triangular function solutions.

Keywords: generalized Boussinesq equation, Boussinesq–Burgers equation, Jacobian elliptic function, modified mapping method

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1. Introduction

For the nonlinear evolution equations, many methods for constructing the exact solitary wave solutions and periodic wave solutions have been proposed, such as the Bäcklund transformation, Hirota method, Darboux transformation, homogeneous balance method, tanh method, triangular function method, Weierstrass elliptic function method, Jacobian elliptic function method, F-expansion method, trial method, mapping method, homotopic analysis method etc.[1−31]. Recently, Li[18] presented a novel class of explicit exact solutions to the Gross-Pitaevskii equation with weak bias magnetic and time-dependent laser fields with the help of the mapping method. Taogetusang *et al* [25] used an auxiliary equation and through a function transformation to solve two kinds of KdV equation with variable coefficients and KdV equation with forcible term. The Darboux transformation was applied to solve the Boussinesq–Burgers equation in Ref.[27].

In this paper, the modified mapping method is used to solve the generalized Boussinesq equation and the Boussinesq–Burgers equation. We find some new exact solutions include Jacobian elliptic function solutions, combined Jacobian elliptic function solutions, soliton-like solutions, and triangular function solutions.

2. The modified mapping method

The modified mapping method[18,20] can be introduced briefly as follows: Given a nonlinear partial differential equation (PDE) with, say, two variables x, t,

$$F(u, u_t, u_x, uu_x, u^2u_x, u_{tx}, u_{tt}, u_{xx}, \cdots) = 0. \quad (1)$$

Step 1 Travelling wave transformation:

$$u = u(\xi), \quad \xi = kx - wt. \quad (2)$$

Substituting Eq.(2) into Eq.(1), Eq.(1) can be converted to an ordinary differential equation (ODE).

$$G(u, u', uu', u'', \cdots) = 0. \quad (3)$$

where k, w are unknown real parameters, "$'$" denotes $\mathrm{d}/\mathrm{d}\xi$.

Step 2 We assume that Eq.(3) has the following solutions

$$u(x,t) = \sum_{i=0}^{N} g_i f^i(\xi) + \sum_{i=1}^{N} h_i f^{-i}(\xi), \quad (4)$$

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where g_0, g_i, h_i, $(i = 1, \cdots, N)$ are unknown real parameters. The positive integer N can be found by balancing the highest order derivative term and the nonlinear term in Eq.(3). $f(\xi)$ satisfies the following equation

$$f'(\xi) = c + bf^2(\xi) + \frac{a}{2}f^4(\xi),$$
$$f''(\xi) = bf(\xi) + af^3(\xi), \tag{5}$$

where a, b, c are real parameters. We will obtain the solutions of Eq.(5) as follows:[18]
When $a = 2m^2$, $b = -(1 + m^2)$, $c = 1$, then

$$f(\xi) = \mathrm{sn}(\xi) \quad \text{or}$$
$$f(\xi) = \mathrm{cd}(\xi) = \mathrm{cn}(\xi)/\mathrm{dn}(\xi); \tag{6}$$

when $a = -2m^2$, $b = 2m^2 - 1$, $c = 1 - m^2$, then

$$f(\xi) = \mathrm{cn}(\xi); \tag{7}$$

when $a = -2$, $b = 2 - m^2$, $c = m^2 - 1$, then

$$f(\xi) = \mathrm{dn}(\xi); \tag{8}$$

when $a = 2$, $b = -(m^2 + 1)$, $c = m^2$, then

$$f(\xi) = \mathrm{ns}(\xi) = 1/\mathrm{sn}(\xi) \qquad \text{or}$$
$$f(\xi) = \mathrm{dc}(\xi) = \mathrm{dn}(\xi)/\mathrm{cn}(\xi); \tag{9}$$

when $a = 2(1 - m^2)$, $b = 2m^2 - 1$, $c = -m^2$, then

$$f(\xi) = \mathrm{nc}(\xi) = 1/\mathrm{cn}(\xi); \tag{10}$$

when $a = 2(m^2 - 1)$, $b = 2 - m^2$, $c = -1$, then

$$f(\xi) = \mathrm{nd}(\xi) = 1/\mathrm{dn}(\xi); \tag{11}$$

when $a = 2$, $b = 2 - m^2$, $c = 1 - m^2$, then

$$f(\xi) = \mathrm{cs}(\xi) = \mathrm{cn}(\xi)/\mathrm{sn}(\xi); \tag{12}$$

when $a = 2(1 - m^2)$, $b = 2 - m^2$, $c = 1$, then

$$f(\xi) = \mathrm{sc}(\xi) = \mathrm{sn}(\xi)/\mathrm{cn}(\xi); \tag{13}$$

when $a = 2m^2(m^2 - 1)$, $b = 2m^2 - 1$, $c = 1$, then

$$f(\xi) = \mathrm{sd}(\xi) = \mathrm{sn}(\xi)/\mathrm{dn}(\xi); \tag{14}$$

when $a = 2$, $b = 2m^2 - 1$, $c = m^2(m^2 - 1)$, then

$$f(\xi) = \mathrm{ds}(\xi) = \mathrm{dn}(\xi)/\mathrm{sn}(\xi) \tag{15}$$

when $a = -1/2$, $b = (1 + m^2)/2$, $c = -(1 - m^2)^2/4$, then

$$f(\xi) = m\mathrm{cn}(\xi) \pm \mathrm{dn}(\xi); \tag{16}$$

when $a = 1/2$, $b = (1 - 2m^2)/2$, $c = 1/4$, then

$$f(\xi) = \mathrm{ns}(\xi) \pm \mathrm{cs}(\xi); \tag{17}$$

when $a = (1-m^2)/2$, $b = (1+m^2)/2$, $c = (1-m^2)/4$, then

$$f(\xi) = \mathrm{nc}(\xi) \pm \mathrm{sc}(\xi); \tag{18}$$

when $a = 1/2$, $b = (m^2 - 2)/2$, $c = m^4/4$, then

$$f(\xi) = \mathrm{ns}(\xi) \pm \mathrm{ds}(\xi); \tag{19}$$

when $a = m^2/2$, $b = (m^2 - 2)/2$, $c = m^2/4$, then

$$f(\xi) = \mathrm{sn}(\xi) \pm \mathrm{icn}(\xi) \qquad \text{or}$$
$$f(\xi) = \frac{\mathrm{dn}(\xi)}{\sqrt{1 - m^2}\mathrm{sn}(\xi) \pm \mathrm{cn}(\xi)}; \tag{20}$$

when $a = 1/2$, $b = (1 - 2m^2)/2$, $c = 1/4$, then

$$f(\xi) = m\mathrm{sn}(\xi) \pm \mathrm{idn}(\xi) \quad \text{or}$$
$$f(\xi) = \frac{\mathrm{sn}(\xi)}{1 \pm \mathrm{cn}(\xi)}; \tag{21}$$

when $a = m^2/2$, $b = (m^2 - 2)/2$, $c = 1/4$, then

$$f(\xi) = \frac{\mathrm{sn}(\xi)}{1 \pm \mathrm{dn}(\xi)}; \tag{22}$$

when $a = (m^2-1)/2$, $b = (m^2+1)/2$, $c = (m^2-1)/4$, then

$$f(\xi) = \frac{\mathrm{dn}(\xi)}{1 \pm m\mathrm{sn}(\xi)}; \tag{23}$$

when $a = (1-m^2)/2$, $b = (m^2+1)/2$, $c = (1-m^2)/4$, then

$$f(\xi) = \frac{\mathrm{cn}(\xi)}{1 \pm \mathrm{sn}(\xi)}; \tag{24}$$

when $a = (1 - m^2)^2/2$, $b = (1 + m^2)/2$, $c = 1/4$, then

$$f(\xi) = \frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi) \pm \mathrm{cn}(\xi)}; \tag{25}$$

when $a = m^4/2$, $b = (m^2 - 2)/2$, $c = 1/4$, then

$$f(\xi) = \frac{\mathrm{cn}(\xi)}{\sqrt{1 - m^2} \pm \mathrm{dn}(\xi)}, \tag{26}$$

where $i^2 = -1$ and m is the modulus of the elliptic function. The Jacobian elliptic functions $\mathrm{sn}(\xi)$, $\mathrm{cn}(\xi)$, $\mathrm{dn}(\xi)$ are double periodic and possess properties of triangular functions, namely

$$\mathrm{sn}^2(\xi) + \mathrm{cn}^2(\xi) = 1,$$
$$\mathrm{dn}^2(\xi) + m^2\mathrm{sn}^2(\xi) = 1,$$
$$\frac{\mathrm{dsn}(\xi)}{\mathrm{d}\xi} = \mathrm{cn}(\xi)\mathrm{dn}(\xi),$$
$$\frac{\mathrm{dcn}(\xi)}{\mathrm{d}\xi} = -\mathrm{sn}(\xi)\mathrm{dn}(\xi),$$
$$\frac{\mathrm{ddn}(\xi)}{\mathrm{d}\xi} = -m^2\mathrm{cn}(\xi)\mathrm{sn}(\xi). \tag{27}$$

When $m \to 0$, the Jacobian elliptic functions degenerate to the trigonometric function solutions of the given nonlinear PDE (NLPDE) i.e.

$$\mathrm{sn}(\xi) \to \sin(\xi), \mathrm{cn}(\xi) \to \cos(\xi), \mathrm{dn}(\xi) \to 1; \quad (28)$$

When $m \to 1$, the Jacobian elliptic functions degenerate to the soliton solutions of the given NLPDE i.e.

$$\mathrm{sn}(\xi) \to \tanh(\xi), \mathrm{cn}(\xi) \to \mathrm{sech}(\xi), \mathrm{dn}(\xi) \to \mathrm{sech}(\xi). \quad (29)$$

Step 3 Substituting Eq.(4) along with Eq.(5) into Eq.(3), we can obtain a set of algebraic polynomials for $f^j(\xi)(j = -N, \cdots, N)$. Setting the coefficients of these terms $f^j(\xi)$ to zero, then we have a system of over-determined equations with respect to unknown real parameters k, w, g_0, g_i, h_i $(i = 1, \cdots, N)$. Solving this system by use of the symbolic computation system Mathematica, we can find the travelling wave solutions of Eq.(1).

3. The application of the modified mapping method

3.1. The generalized Boussinesq equation[14]

$$\left(\frac{\partial}{\partial t} + p\frac{\partial}{\partial x}\right)^2 u + q\frac{\partial^2 u}{\partial x^2} + r\frac{\partial^2 u^2}{\partial x^2} - s\frac{\partial^4 u}{\partial x^4} = 0, \quad (30)$$

where p, r, q, s are real constants. Equation (30) is an extended form of the famous Boussinesq equation and the second-order Benjamin–Ono equation. When $p = 0$, Eq.(30) is the Boussinesq equation

$$u_{tt} + qu_{xx} + r(u^2)_{xx} - su_{xxxx} = 0, \quad (31)$$

and when $q = 0$, Eq.(30) is the second-order Benjamin-Ono equation

$$u_{tt} + 2pu_{xt} + p^2u_{xx} + r(u^2)_{xx} - su_{xxxx} = 0. \quad (32)$$

Substituting Eq.(2) into Eq.(30), we obtain the following ODE:

$$\begin{aligned} &2k^2r(u')^2 + [k^2(p^2+q) - 2kpw \\ &+ w^2 + 2k^2ru]u'' - k^4u^{(4)} = 0. \end{aligned} \quad (33)$$

Balancing the highest order derivative term $u^{(4)}$ with the nonlinear term $(u')^2$ or uu'' in Eq.(33) yields $N = 2$. Therefore, by the use of the modified mapping method, we assume the solutions of Eq.(33) in the following forms:

$$\begin{aligned} u(\xi) =& g_0 + g_1 f(\xi) + g_2 f^2(\xi) \\ &+ h_1 f^{-1}(\xi) + h_2 f^{-2}(\xi). \end{aligned} \quad (34)$$

Substituting Eqs.(5), (34) into Eq.(33) and setting the coefficients of the terms $f^i(\xi)$ $(i = -6, \cdots, 6)$ to zero, we can obtain the following set of over-determined equations with respect to unknown real parameters$\{k, w, g_0, g_1, h_1, g_2, h_2\}$:

$$\begin{aligned}
&ch_2k^2(-h_2r + 6ck^2s) = 0,\\
&ch_1k^2(-h_2r + ck^2s) = 0,\\
&-3ch_2k^2p^2 - 3ch_2k^2q - 3ch_1^2k^2r - 6cg_0h_2k^2r - 8bh_2^2k^2r + 60bch_2k^4s + 6ch_2kpw - 3ch_2w^2 = 0,\\
&-ch_1k^2p^2 - ch_1k^2q - 2cg_0h_1k^2r - 2cg_1h_2k^2r - 9bh_1h_2k^2r + 10bch_1k^4s + 2ch_1kpw - ch_1w^2 = 0,\\
&-2bh_2k^2p^2 - 2bh_2k^2q - 2bh_1^2k^2r - 4bg_0h_2k^2r - 3ah_2^2k^2r + 8b^2h_2k^4s + 18ach_2k^4s + 4bh_2kpw - 2bh_2w^2 = 0,\\
&bh_1k^2p^2 + bh_1k^2q + 2bg_0h_1k^2r + 2bg_1h_2k^2r + 6ah_1h_2k^2r - b^2h_1k^4s - 6ach_1k^4s - 2bh_1kpw + bh_1w^2 = 0,\\
&2cg_2k^2p^2 + ah_2k^2p^2 + 2cg_2k^2q + ah_2k^2q + 2cg_1^2k^2r + 4cg_0g_2k^2r + ah_1^2k^2r + 2ag_0h_2k^2r\\
&-8bcg_2k^4s - 4abh_2k^4s - 4cg_2kpw - 2ah_2kpw + 2cg_2w^2 + ah_2w^2 = 0,\\
&bg_1k^2p^2 + bg_1k^2q + 2bg_0g_1k^2r + 12cg_1g_2k^2r + 2bg_2h_1k^2r - b^2g_1k^4s - 6acg_1k^4s - 2bg_1kpw + bg_1w^2 = 0,\\
&-bg_2k^2p^2 - bg_2k^2q - bg_1^2k^2r - 2bg_0g_2k^2r - 3cg_2^2k^2r + 4b^2g_2k^4s + 9acg_2k^4s + 2bg_2kpw - bg_2w^2 = 0,\\
&ag_1k^2p^2 + ag_1k^2q + 2ag_0g_1k^2r + 18bg_1g_2k^2r + 2ag_2h_1k^2r - 10abg_1k^4s - 2ag_1kpw + ag_1w^2 = 0,\\
&3ag_2k^2p^2 + 3ag_2k^2q + 3ag_1^2k^2r + 6ag_0g_2k^2r + 16g_2^2k^2r - 60abg_2k^4s - 6ag_2kpw + 3ag_2w^2 = 0,\\
&ag_1k^2(-2g_2r + ak^2s) = 0,\\
&ag_2k^2(-g_2r + 3ak^2s) = 0.
\end{aligned} \quad (35)$$

With the aid of Mathematica, we have the following results of the Eq.(35).

Case 1

$$g_0 = \frac{-k^2p^2 - k^2q + 4bk^4s + 2kpw - w^2}{2k^2r},$$
$$g_1 = 0, g_2 = a\gamma, h_1 = 0, h_2 = 2c\gamma.$$

Case 2

$$g_0 = \frac{-k^2p^2 - k^2q + 4bk^4s + 2kpw - w^2}{2k^2r},$$
$$g_1 = 0, g_2 = a\gamma, h_1 = 0, h_2 = 0.$$

Case 3

$$g_0 = \frac{-k^2p^2 - k^2q + 4bk^4s + 2kpw - w^2}{2k^2r},$$
$$g_1 = 0, g_2 = 0, h_1 = 0, h_2 = 2c\gamma,$$

where $\gamma = 3k^2s/r$.

Let us take example for case 1, the Jacobian elliptic function exact solutions are as follows:

$$u_{1,1} = g_0 + 2\gamma[m^2\mathrm{sn}^2(\xi) + \mathrm{sn}^{-2}(\xi)],$$
$$u_{2,1} = g_0 + 2\gamma[m^2\mathrm{cd}^2(\xi) + \mathrm{cd}^{-2}(\xi)],$$
$$u_{3,1} = g_0 + 2\gamma[-m^2\mathrm{cn}^2(\xi) + (1-m^2)\mathrm{cn}^{-2}(\xi)],$$
$$u_{4,1} = g_0 + 2\gamma[-\mathrm{dn}^2(\xi) + (m^2-1)\mathrm{dn}^{-2}(\xi)],$$
$$u_{5,1} = g_0 + 2\gamma[\mathrm{ns}^2(\xi) + m^2\mathrm{ns}^{-2}(\xi)],$$
$$u_{6,1} = g_0 + 2\gamma[\mathrm{dc}^2(\xi) + m^2\mathrm{dc}^{-2}(\xi)],$$
$$u_{7,1} = g_0 + 2\gamma[(1-m^2)\mathrm{nc}^2(\xi) - m^2\mathrm{nc}^{-2}(\xi)],$$
$$u_{8,1} = g_0 + 2\gamma[(m^2-1)\mathrm{nd}^2(\xi) - \mathrm{nd}^{-2}(\xi)],$$
$$u_{9,1} = g_0 + 2\gamma[\mathrm{cs}^2(\xi) + (1-m^2)\mathrm{cs}^{-2}(\xi)],$$
$$u_{10,1} = g_0 + 2\gamma[(1-m^2)\mathrm{sc}^2(\xi) + \mathrm{sc}^{-2}(\xi)],$$
$$u_{11,1} = g_0 + 2\gamma[m^2(m^2-1)\mathrm{sd}^2(\xi) + \mathrm{sd}^{-2}(\xi)],$$
$$u_{12,1} = g_0 + 2\gamma[\mathrm{ds}^2(\xi) + m^2(m^2-1)\mathrm{ds}^{-2}(\xi)],$$
$$u_{13,1} = g_0 - \frac{\gamma}{2}[(m\mathrm{cn}(\xi) \pm \mathrm{dn}(\xi))^2 + (1-m^2)^2(m\mathrm{cn}(\xi) \pm \mathrm{dn}(\xi))^{-2}],$$
$$u_{14,1} = g_0 + \frac{\gamma}{2}[(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))^2 + (\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))^{-2}],$$
$$u_{15,1} = g_0 + \frac{(1-m^2)\gamma}{2}[(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))^2 + (\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))^{-2}],$$
$$u_{16,1} = g_0 + \frac{\gamma}{2}[(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^2 + m^4(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^{-2}],$$
$$u_{17,1} = g_0 + \frac{m^2\gamma}{2}[(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^2 + (\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^{-2}],$$
$$u_{18,1} = g_0 + \frac{m^2\gamma}{2}\left[\left(\frac{\mathrm{dn}(\xi)}{\sqrt{1-m^2}\mathrm{sn}(\xi) \pm \mathrm{cn}(\xi)}\right)^2 + \left(\frac{\mathrm{dn}(\xi)}{\sqrt{1-m^2}\mathrm{sn}(\xi) \pm \mathrm{cn}(\xi)}\right)^{-2}\right],$$
$$u_{19,1} = g_0 + \frac{\gamma}{2}[(m\mathrm{sn}(\xi) \pm \mathrm{idn}(\xi))^2 + (m\mathrm{sn}(\xi) \pm \mathrm{idn}(\xi))^{-2}],$$
$$u_{20,1} = g_0 + \frac{\gamma}{2}\left[\left(\frac{\mathrm{sn}(\xi)}{1 \pm \mathrm{cn}(\xi)}\right)^2 + \left(\frac{\mathrm{sn}(\xi)}{1 \pm \mathrm{cn}(\xi)}\right)^{-2}\right],$$
$$u_{21,1} = g_0 + \frac{\gamma}{2}\left[m^2\left(\frac{\mathrm{sn}(\xi)}{1 \pm \mathrm{dn}(\xi)}\right)^2 + \left(\frac{\mathrm{sn}(\xi)}{1 \pm \mathrm{dn}(\xi)}\right)^{-2}\right],$$
$$u_{22,1} = g_0 + \frac{(m^2-1)\gamma}{2}\left[\left(\frac{\mathrm{dn}(\xi)}{1 \pm m\mathrm{sn}(\xi)}\right)^2 + \left(\frac{\mathrm{dn}(\xi)}{1 \pm m\mathrm{sn}(\xi)}\right)^{-2}\right],$$
$$u_{23,1} = g_0 + \frac{(1-m^2)\gamma}{2}\left[\left(\frac{\mathrm{cn}(\xi)}{1 \pm \mathrm{sn}(\xi)}\right)^2 + \left(\frac{\mathrm{cn}(\xi)}{1 \pm \mathrm{sn}(\xi)}\right)^{-2}\right],$$
$$u_{24,1} = g_0 + \frac{\gamma}{2}\left[(1-m^2)^2\left(\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi) \pm \mathrm{cn}(\xi)}\right)^2 + \left(\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi) \pm \mathrm{cn}(\xi)}\right)^{-2}\right],$$
$$u_{25,1} = g_0 + \frac{\gamma}{2}\left[m^4\left(\frac{\mathrm{cn}(\xi)}{\sqrt{1-m^2} \pm \mathrm{dn}(\xi)}\right)^2 + \left(\frac{\mathrm{cn}(\xi)}{\sqrt{1-m^2} \pm \mathrm{dn}(\xi)}\right)^{-2}\right].$$

The above Jacobian elliptic function exact solutions degenerate into the following triangle function solutions for $m = 0$:

$$
\begin{aligned}
u_{1,2} =& g_0 + 2\gamma \csc^2(\xi),\\
u_{2,2} =& g_0 + 2\gamma \sec^2(\xi),\\
u_{3,2} =& g_0 + 2\gamma[\cot^2(\xi) + \tan^2(\xi)],\\
u_{4,2} =& g_0 + \frac{\gamma}{2}[(\csc(\xi) \pm \cot(\xi))^2\\
&+ (\csc(\xi) \pm \cot(\xi))^{-2}],\\
u_{5,2} =& g_0 + \frac{\gamma}{2}[(\sec(\xi) \pm \tan(\xi))^2\\
&+ (\sec(\xi) \pm \tan(\xi))^{-2}],\\
u_{6,2} =& g_0 + \frac{\gamma}{2}\left[\left(\frac{\sin(\xi)}{1 \pm \cos(\xi)}\right)^2 + \left(\frac{\sin(\xi)}{1 \pm \cos(\xi)}\right)^{-2}\right],\\
u_{7,2} =& g_0 + \frac{\gamma}{2}\left[\left(\frac{\cos(\xi)}{1 \pm \sin(\xi)}\right)^2 + \left(\frac{\cos(\xi)}{1 \pm \sin(\xi)}\right)^{-2}\right].
\end{aligned}
$$

When $m = 1$, we can obtain the following soliton solutions:

$$
\begin{aligned}
u_{1,3} =& g_0 + 2\gamma[\coth^2(\xi) + \tanh^2(\xi)],\\
u_{2,3} =& g_0 - 2\gamma \mathrm{sech}^2(\xi),\\
u_{3,3} =& g_0 + 2\gamma \mathrm{csch}^2(\xi),\\
u_{4,3} =& g_0 + \frac{\gamma}{2}[(\coth(\xi) \pm \mathrm{csch}(\xi))^2\\
&+ (\coth(\xi) \pm \mathrm{csch}(\xi))^{-2}],\\
u_{5,3} =& g_0 + \frac{\gamma}{2}[(\tanh(\xi) \pm \mathrm{isech}(\xi)^2\\
&+ (\tanh(\xi) \pm \mathrm{isech}(\xi)^{-2}],\\
u_{6,3} =& g_0 + \frac{\gamma}{2}\left[\left(\frac{\tanh(\xi)}{1 \pm \mathrm{sech}(\xi)}\right)^2 + \left(\frac{\tanh(\xi)}{1 \pm \mathrm{sech}(\xi)}\right)^{-2}\right],
\end{aligned}
$$

where $g_0 = \dfrac{-k^2p^2 - k^2q + 4bk^4s + 2kpw - w^2}{2k^2r}$, $\gamma = 3k^2s/r$, $\xi = kx - wt$.

3.2. The Boussinesq–Burgers equation[27]

The Boussinesq–Burgers soliton equation is given by

$$
\begin{aligned}
&u_t + 2uu_x - \frac{1}{2}v_x = 0,\\
&v_t - \frac{1}{2}u_{xxx} + 2(uv)_x = 0,
\end{aligned}
\tag{36}
$$

which is a nonlinear long-wave equation. A good understanding of its solutions is very helpful for coastal and civil engineers to apply the nonlinear water wave model to harbor and coastal designs. Therefore, finding more new solutions of the Boussinesq–Burgers equation is of great value in fluid dynamics.

Substituting Eq.(2) into Eq.(36), we obtain the following ODEs:

$$
-wu' + 2kuu' - \frac{1}{2}kv' = 0, \tag{37}
$$

$$
-wv' + 2k(uv)' - \frac{1}{2}k^3u''' = 0. \tag{38}
$$

Integrating Eqs.(2) and (38), we can obtain

$$
-wu + ku^2 - \frac{1}{2}kv + r_1 = 0, \tag{39}
$$

$$
-wv + 2kuv - \frac{1}{2}k^3u'' + r_2 = 0, \tag{40}
$$

where r_1, r_2 are integral constants.

Suppose N_1 is the highest order number of u, and N_2 is the highest order number of v, then we can obtain in Eqs.(39) and (40):

$$
2N_1 = N_2, \quad N_1 + N_2 = N_1 + 2. \tag{41}
$$

So Eq.(41) means $N_1 = 1$ and $N_2 = 2$. We can assume the solutions of Eq.(38) in the following special forms:

$$
\begin{aligned}
u(x,t) =& g_0 + g_1 f(\xi) + h_1 f^{-1}(\xi),\\
v(x,t) =& d_0 + d_1 f(\xi) + e_1 f^{-1}(\xi)\\
&+ d_2 f^2(\xi) + e_2 f^{-2}(\xi).
\end{aligned}
\tag{42}
$$

Substituting Eq.(5) and Eq.(7) into Eq.(39) and Eq.(40) respectively, and setting the coefficients of the terms $f^i(\xi)$ to zero, we can obtain the following set of equations with respect to the unknown real parameters $\{k, w, g_0, g_1, h_1, d_0, d_1, e_1, d_2, e_2, r_1, r_2\}$:

$$
\begin{aligned}
&(e_2 - 2h_1^2)k = 0,\\
&-e_1k + 4g_0h_1k - 2h_1w = 0,\\
&-d_0k + 2g_0^2k + 4g_1h_1k + 2r_1 - 2g_0w = 0,\\
&d_1k - 4g_0g_1k + 2g_1w = 0,\\
&(d_2 - 2g_1^2)k = 0,\\
&h_1k(2e_2 - ck^2) = 0,\\
&2e_2g_0k + 2e_1h_1k - e_2w = 0,\\
&4e_1g_0k + 4e_2g_1k + 4d_0h_1k - bh_1k^3 - 2e_1w = 0,\\
&2d_0g_0k + 2e_1g_1k + 2d_1h_1k + r_2 - d_0w = 0,\\
&4d_1g_0k + 4d_0g_1k + 4d_2h_1k - bg_1k^2 - 2d_1w = 0,\\
&2d_2g_0k + 2d_1g_1k - d_2w = 0,\\
&g_1k(4d_2 - ak^2) = 0.
\end{aligned}
\tag{43}
$$

With the aid of Mathematica, we have the following results for Eq.(43):

Case 1

$$g_0 = \frac{w}{2k}, \quad g_1^2 = \frac{ak^2}{8},$$
$$h_1 = 0, \quad d_0 = \frac{bk^2}{4}, \quad d_1 = 0,$$
$$e_1 = 0, \quad d_2 = \frac{ak^2}{4}, \quad e_2 = 0,$$
$$r_1 = \frac{bk^4 + 2w^2}{8k}, \quad r_2 = 0.$$

Condition 1

$$a > 0.$$

Case 2

$$g_0 = \frac{w}{2k}, \quad g_1 = 0, \quad h_1^2 = \frac{ck^2}{4},$$
$$d_0 = \frac{bk^2}{4}, \quad d_1 = 0, \quad e_1 = 0,$$
$$d_2 = 0, \quad e_2 = \frac{ck^2}{2},$$
$$r_1 = \frac{bk^4 + 2w^2}{8k}, \quad r_2 = 0.$$

Condition 2

$$c > 0.$$

Case 3

$$g_0 = \frac{w}{2k}, \quad g_1^2 = \frac{ak^2}{8}, \quad h_1^2 = \frac{ck^2}{4},$$
$$d_0 = \frac{b - \sqrt{2ac}}{4}k^2, \quad d_1 = 0, \quad e_1 = 0,$$
$$d_2 = \frac{ak^2}{4}, \quad e_2 = \frac{ck^2}{2},$$
$$r_1 = \frac{(b - 3\sqrt{2ac})k^4 + 2w^2}{8k}, \quad r_2 = 0.$$

Condition 3

$$a > 0, \quad c > 0.$$

Let us take example for case 3, the Jacobian elliptic function exact solutions are as follows:

$$u_{1,1} = \frac{w}{2k} \pm \frac{k}{2}[m\mathrm{sn}(\xi) + \mathrm{sn}^{-1}(\xi)],$$
$$v_{1,1} = -\frac{(m+1)^2k^2}{4} + \frac{k^2}{2}[m^2\mathrm{sn}^2(\xi) + \mathrm{sn}^{-2}(\xi)],$$
$$u_{2,1} = \frac{w}{2k} \pm \frac{k}{2}[m\mathrm{cd}(\xi) + \mathrm{cd}^{-1}(\xi)],$$
$$v_{2,1} = -\frac{(m+1)^2k^2}{4} + \frac{k^2}{2}[m^2\mathrm{cd}^2(\xi) + \mathrm{cd}^{-2}(\xi)],$$
$$u_{3,1} = \frac{w}{2k} \pm \frac{k}{2}[\mathrm{ns}(\xi) + m\mathrm{ns}^{-1}(\xi)],$$
$$v_{3,1} = -\frac{(1+m)^2k^2}{4} + \frac{k^2}{2}[\mathrm{ns}^2(\xi) + m^2\mathrm{ns}^{-2}(\xi)],$$
$$u_{4,1} = \frac{w}{2k} \pm \frac{k}{2}[\mathrm{dc}(\xi) + m\mathrm{dc}^{-1}(\xi)],$$
$$v_{4,1} = -\frac{(1+m)^2k^2}{4} + \frac{k^2}{2}[\mathrm{dc}^2(\xi) + m^2\mathrm{dc}^{-2}(\xi)],$$
$$u_{5,1} = \frac{w}{2k} \pm \frac{k}{2}[\mathrm{cs}(\xi) + \sqrt{1-m^2}\mathrm{cs}^{-1}(\xi)],$$
$$v_{5,1} = \frac{k^2}{4}(2 - m^2 - 2\sqrt{1-m^2}) + \frac{k^2}{2}[\mathrm{cs}^2(\xi) + (1-m^2)\mathrm{cs}^{-2}(\xi)],$$
$$u_{6,1} = \frac{w}{2k} \pm \frac{k}{2}[\sqrt{1-m^2}\mathrm{sc}(\xi) + \mathrm{sc}^{-1}(\xi)],$$
$$v_{6,1} = \frac{k^2}{4}(2 - m^2 - 2\sqrt{1-m^2}) + \frac{k^2}{2}[(1-m^2)\mathrm{sc}^2(\xi) + \mathrm{sc}^{-2}(\xi)],$$

$$u_{7,1} = \frac{w}{2k} \pm \frac{k}{4}[(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi)) + (\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))^{-1}],$$
$$v_{7,1} = -\frac{k^2m^2}{4} + \frac{k^2}{8}[(\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))^2 + (\mathrm{ns}(\xi) \pm \mathrm{cs}(\xi))^{-2}],$$
$$u_{8,1} = \frac{w}{2k} \pm \frac{k\sqrt{1-m^2}}{4}[(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi)) + (\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))^{-1}],$$
$$v_{8,1} = \frac{k^2m^2}{4} + \frac{k^2(1-m^2)}{8}[(\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))^2 + (\mathrm{nc}(\xi) \pm \mathrm{sc}(\xi))^{-2}],$$
$$u_{9,1} = \frac{w}{2k} \pm \frac{k}{4}[(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi)) + m^2(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^{-1}],$$
$$v_{9,1} = -\frac{k^2}{4} + \frac{k^2}{8}[(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^2 + m^4(\mathrm{ns}(\xi) \pm \mathrm{ds}(\xi))^{-2}],$$
$$u_{10,1} = \frac{w}{2k} \pm \frac{km}{4}[(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi)) + (\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^{-1}],$$
$$v_{10,1} = -\frac{k^2}{4} + \frac{k^2m^2}{8}[(\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^2 + (\mathrm{sn}(\xi) \pm \mathrm{icn}(\xi))^{-2}],$$

$$u_{11,1}=\frac{w}{2k}\pm\frac{km}{4}\left[\left(\frac{\mathrm{dn}(\xi)}{\sqrt{1-m^2}\mathrm{sn}(\xi)\pm\mathrm{cn}(\xi)}\right)+\left(\frac{\mathrm{dn}(\xi)}{\sqrt{1-m^2}\mathrm{sn}(\xi)\pm\mathrm{cn}(\xi)}\right)^{-1}\right],$$

$$v_{11,1}=-\frac{k^2}{4}+\frac{k^2m^2}{8}\left[\left(\frac{\mathrm{dn}(\xi)}{\sqrt{1-m^2}\mathrm{sn}(\xi)\pm\mathrm{cn}(\xi)}\right)^{2}+\left(\frac{\mathrm{dn}(\xi)}{\sqrt{1-m^2}\mathrm{sn}(\xi)\pm\mathrm{cn}(\xi)}\right)^{-2}\right],$$

$$u_{12,1}=\frac{w}{2k}\pm\frac{k}{4}[(m\mathrm{sn}(\xi)\pm\mathrm{idn}(\xi))+(m\mathrm{sn}(\xi)\pm\mathrm{idn}(\xi))^{-1}],$$

$$v_{12,1}=-\frac{k^2m^2}{4}+\frac{k^2}{8}[(m\mathrm{sn}(\xi)\pm\mathrm{idn}(\xi))^2+(m\mathrm{sn}(\xi)\pm\mathrm{idn}(\xi))^{-2}],$$

$$u_{13,1}=\frac{w}{2k}\pm\frac{k}{4}\left[\left(\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{cn}(\xi)}\right)+\left(\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{cn}(\xi)}\right)^{-1}\right],$$

$$v_{13,1}=-\frac{k^2m^2}{4}+\frac{k^2}{8}\left[\left(\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{cn}(\xi)}\right)^{2}+\left(\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{cn}(\xi)}\right)^{-2}\right],$$

$$u_{14,1}=\frac{w}{2k}\pm\frac{k}{4}\left[m\left(\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{dn}(\xi)}\right)+\left(\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{dn}(\xi)}\right)^{-1}\right],$$

$$v_{14,1}=\frac{k^2(m^2-m-2)}{8}+\frac{k^2}{8}\left[m^2\left(\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{dn}(\xi)}\right)^{2}+\left(\frac{\mathrm{sn}(\xi)}{1\pm\mathrm{dn}(\xi)}\right)^{-2}\right],$$

$$u_{15,1}=\frac{w}{2k}\pm\frac{k\sqrt{1-m^2}}{4}\left[\left(\frac{\mathrm{cn}(\xi)}{1\pm\mathrm{sn}(\xi)}\right)+\left(\frac{\mathrm{cn}(\xi)}{1\pm\mathrm{sn}(\xi)}\right)^{-1}\right],$$

$$v_{15,1}=\frac{k^2m^2}{4}+\frac{k^2(1-m^2)}{8}\left[\left(\frac{\mathrm{cn}(\xi)}{1\pm\mathrm{sn}(\xi)}\right)^{2}+\left(\frac{\mathrm{cn}(\xi)}{1\pm\mathrm{sn}(\xi)}\right)^{-2}\right],$$

$$u_{16,1}=\frac{w}{2k}\pm\frac{k}{4}\left[\left(1-m^2\right)\left(\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)\pm\mathrm{cn}(\xi)}\right)+\left(\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)\pm\mathrm{cn}(\xi)}\right)^{-1}\right],$$

$$v_{16,1}=\frac{k^2m^2}{4}+\frac{k^2}{8}\left[(1-m^2)^2\left(\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)\pm\mathrm{cn}(\xi)}\right)^{2}+\left(\frac{\mathrm{sn}(\xi)}{\mathrm{dn}(\xi)\pm\mathrm{cn}(\xi)}\right)^{-2}\right],$$

$$u_{17,1}=\frac{w}{2k}\pm\frac{k}{4}\left[m^2\left(\frac{\mathrm{cn}(\xi)}{\sqrt{1-m^2}\pm\mathrm{dn}(\xi)}\right)+\left(\frac{\mathrm{cn}(\xi)}{\sqrt{1-m^2}\pm\mathrm{dn}(\xi)}\right)^{-1}\right],$$

$$v_{17,1}=-\frac{k^2}{4}+\frac{k^2}{8}[m^4\left(\frac{\mathrm{cn}(\xi)}{\sqrt{1-m^2}\pm\mathrm{dn}(\xi)}\right)^{2}+\left(\frac{\mathrm{cn}(\xi)}{\sqrt{1-m^2}\pm\mathrm{dn}(\xi)}\right)^{-2}].$$

The above Jacobian elliptic function exact solutions will degenerate into the following triangle function solutions for $m=0$:

$$u_{1,2}=\frac{w}{2k}\pm\frac{k}{2}\csc(\xi),$$

$$v_{1,2}=-\frac{k^2}{4}+\frac{k^2}{2}\csc^2(\xi),$$

$$u_{2,2}=\frac{w}{2k}\pm\frac{k}{2}\sec(\xi), v_{2,2}=-\frac{k^2}{4}+\frac{k^2}{2}\sec^2(\xi),$$

$$u_{3,2}=\frac{w}{2k}\pm\frac{k}{2}[\cot(\xi)+\tan(\xi)],$$

$$v_{3,2}=\frac{k^2}{2}[\cot^2(\xi)+\tan^2(\xi)],$$

$$u_{4,2}=\frac{w}{2k}\pm\frac{k}{4}[(\csc(\xi)\pm\cot(\xi))+(\csc(\xi)\pm\cot(\xi))^{-1}],$$

$$v_{4,2}=\frac{k^2}{8}[(\csc(\xi)\pm\cot(\xi))^2+(\csc(\xi)\pm\cot(\xi))^{-2}],$$

$$u_{5,2}=\frac{w}{2k}\pm\frac{k}{4}[(\sec(\xi)\pm\tan(\xi))+(\sec(\xi)\pm\tan(\xi))^{-1}],$$

$$v_{5,2}=\frac{k^2}{8}[(\sec(\xi)\pm\tan(\xi))^2+(\sec(\xi)\pm\tan(\xi))^{-2}]n,$$

$$u_{6,2}=\frac{w}{2k}\pm\frac{k}{4}\left[\left(\frac{\sin(\xi)}{1\pm\cos(\xi)}\right)+\left(\frac{\sin(\xi)}{1\pm\cos(\xi)}\right)^{-1}\right],$$

$$v_{6,2}=\frac{k^2}{8}\left[\left(\frac{\sin(\xi)}{1\pm\cos(\xi)}\right)^{2}+\left(\frac{\sin(\xi)}{1\pm\cos(\xi)}\right)^{-2}\right],$$

$$u_{7,2}=\frac{w}{2k}\pm\frac{k}{4}\left[\left(\frac{\cos(\xi)}{1\pm\sin(\xi)}\right)+\left(\frac{\cos(\xi)}{1\pm\sin(\xi)}\right)^{-1}\right],$$

$$v_{7,2}=\frac{k^2}{8}\left[\left(\frac{\cos(\xi)}{1\pm\sin(\xi)}\right)^{2}+\left(\frac{\cos(\xi)}{1\pm\sin(\xi)}\right)^{-2}\right].$$

When $m=1$, we can obtain the following soliton solutions:

$$u_{1,3} = \frac{w}{2k} \pm \frac{k}{2}[\coth(\xi) + \tanh(\xi)],$$
$$v_{1,3} = -k^2 + \frac{k^2}{2}[\coth^2(\xi) + \tanh^2(\xi)],$$
$$u_{2,3} = \frac{w}{2k} \pm \frac{k}{2}\mathrm{csch}(\xi),$$
$$v_{2,3} = \frac{k^2}{4} + \frac{k^2}{2}\mathrm{csch}^2(\xi),$$
$$u_{3,3} = \frac{w}{2k} \pm \frac{k}{4}[(\coth(\xi) \pm \mathrm{csch}(\xi)) + (\coth(\xi) \pm \mathrm{csch}(\xi))^{-1}],$$
$$v_{3,3} = -\frac{k^2}{4} + \frac{k^2}{8}[(\coth(\xi) \pm \mathrm{csch}(\xi)\Big)^2 + (\coth(\xi) \pm \mathrm{csch}(\xi))^{-2}],$$
$$u_{4,3} = \frac{w}{2k} \pm \frac{k}{4}[(\tanh(\xi) \pm \mathrm{isech}(\xi)) + (\tanh(\xi) \pm \mathrm{isech}(\xi))^{-1}],$$
$$v_{4,3} = -\frac{k^2}{4} + \frac{k^2}{8}[(\tanh(\xi) \pm \mathrm{isech}(\xi))^2 + (\tanh(\xi) \pm \mathrm{isech}(\xi))^{-2}],$$
$$u_{5,3} = \frac{w}{2k} \pm \frac{k}{4}\left[\left(\frac{\tanh(\xi)}{1 \pm \mathrm{sech}(\xi)}\right) + \left(\frac{\tanh(\xi)}{1 \pm \mathrm{sech}(\xi)}\right)^{-1}\right],$$
$$v_{5,3} = -\frac{k^2}{4} + \frac{k^2}{8}\left[\left(\frac{\tanh(\xi)}{1 + \mathrm{sech}(\xi)}\right)^2 + \left(\frac{\tanh(\xi)}{1 + \mathrm{sech}(\xi)}\right)^{-2}\right],$$

where $\xi = kx - wt$.

4. Summary

Based on the modified mapping method and the use of symbolic computation system Mathematica, we have obtained the Jacobian elliptic function solutions, combined Jacobian elliptic function solutions, triangular function solutions and soliton solutions to the nonlinear generalized Boussinesq equation and the Boussinesq–Burgers equation. At the same time, we have successfully recovered the exact solutions in Ref.[14] by using the obtaining quasi-solution step-by-step principle (the solutions in case1 and case3 are new solutions) and those in Ref.[27] by using the Darboux transformation method (the solutions in case2 and case3 are new solutions).

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