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A new expansion method of first order nonlinear ordinary differential equation with at most a sixth-degree nonlinear term and its application to mBBM model*

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Based on a first order nonlinear ordinary differential equation with at most a sixth-degree nonlinear term which is extended from a type of elliptic equation, and by converting it into a new expansion form, this paper proposes a new algebraic method to construct exact solutions for nonlinear evolution equations. Being concise and straightforward, the method is applied to modified Benjamin–Bona–Mahony (mBBM) model, and some new exact solutions to the system are obtained. The algorithm is of important significance in exploring exact solutions for other nonlinear evolution equations.

Keywords: nonlinear evolution equations, new expansion method, mBBM model, exact solutions
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1. Introduction

An increasing amount of effort has been spent on the construction of exact solutions to nonlinear evolution equations (NEEs), for their important role in understanding the nonlinear problems. Up to now, a wealth of powerful method have been proposed, such as the inverse scattering method,[1] Bäcklund transformation,[2] the Hirota's bilinear method,[3] the extended Painlevé analysis,[4] the truncated Painlevé expansion method,[5] the sine-cosine method,[6] the homogeneous balance method,[7] the tanh function expansion method,[8] the Jacobi elliptic function expansion method,[9] F-expansion method,[10] etc.

Due to many exact solutions of NEEs can be written as a polynomial in several elementary or special functions which satisfy a first order nonlinear ordinary differential equation with a sixth-degree nonlinear term, recently, after Fan's[11] unified algebraic method, which was based on a first order nonlinear ordinary differential equation with a fourth-degree nonlinear term, Huang and Zhang[12,13] further extend Fan's method to a more general form, which possesses a sixth-degree nonlinear term, and successfully derive more new exact solutions of NEEs. Furthermore, Taogetusang and Sirendaoreji,[14] in the near past time, had explicitly explored many new solutions to the first order nonlinear ordinary differential equation with a sixth-degree nonlinear term presented in Huang and Zhang's method.[12,13]

The aim of this paper, motivated by Refs.[12]–[15], is to perform a first order nonlinear ordinary differential equation with at most a sixth-degree nonlinear term which is, in nature, an extension of a type of elliptic equation,[16] into a new expansion form. As a consequence, a new algebraic method to seek exact solutions for NEEs is constructed and it is applied to the modified Benjamin–Bona–Mahony (mBBM) model.

2. Description of the method

Suppose that a NEE for $u(x,t)$ is given by

$$P(u, u_t, u_x, u_{tt}, u_{xx}, \ldots) = 0, \tag{1}$$

in which both nonlinear terms and higher-order derivatives of $u(x,t)$ are all involved. In general, the left-hand side of Eq.(1) is a polynomial in $u(x,t)$ and its various derivatives.

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We seek for the travelling wave solutions of Eq.(1) by taking

$$u = u(\xi), \quad \xi = k_1 x + k_4 t, \tag{2}$$

where $k_l (l = 1, 4)$ are wave parameters to be determined later.

Substituting Eq.(2) into Eq.(1) we get a nonlinear ordinary differential equation

$$K(u, \mathrm{d}u/\mathrm{d}\xi, \mathrm{d}^2u/\mathrm{d}\xi^2, \ldots) = 0. \tag{3}$$

Then, based on Ref.[13] and further extending the M to $-M$, we expand $u(\xi)$ into a new polynomial of $f(\xi)$ as follows:

$$u(\xi) = \sum_{i=-M}^{M} \left[A_i f^i(\xi) + B_i f^{i-1}(\xi) f'(\xi)\right], \tag{4}$$

where the prime '$'$' means $\mathrm{d}/\mathrm{d}\xi$; A_i and B_i are constants to be determined and M is an integer to be fixed while $f(\xi)$ satisfies the following first order nonlinear ordinary differential equation with at most a sixth-degree nonlinear term, which is, in effect, extended from a type of elliptic equation,[16] namely

$$(f')^2 = h_0 + h_1 f + h_2 f^2 + \cdots + h_N f^N, N \leq 6, \tag{5a}$$

i.e.

$$f' = e\sqrt{h_0 + h_1 f + h_2 f^2 + \cdots + h_N f^N}, N \leq 6, \tag{5b}$$

where $e = \pm 1$, and $h_j (j = 0, 1, \ldots, N)$ are the constants to be determined later, and with the different values of h_j, Eq.(5) has a series of different fundamental solutions.[11−14]

On the basis of Eqs.(4) and (5), it is not difficult to see that the highest degree of $\mathrm{d}^p u/\mathrm{d}\xi^p$ is

$$O\left(\frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\right) = M + \frac{N}{2}p - p, \quad p = 1, 2, 3, \ldots, \quad N = 0, \ldots, 6, \tag{6a}$$

and

$$O\left(u^q \frac{\mathrm{d}^p u}{\mathrm{d}\xi^p}\right) = (q+1)M + \frac{N}{2}p - p, \quad q = 0, 1, 2, \ldots, \quad p = 1, 2, 3, \ldots, N = 0, \ldots, 6. \tag{6b}$$

Therefore, by balancing the highest derivative term with the highest nonlinear term, we can obtain a relation between M and N, from which the different possible values of M and N can be clearly obtained.

For each pair of values of M and N, substituting Eqs.(4) and (5) into Eq.(3) and setting all the coefficients of

$$f^r \left(\sqrt{\sum_{j=0}^{N} h_j f^j}\right)^s \ (r = 0, -1, 1, -2, 2, \ldots; s = 0, 1) \tag{7}$$

to zero, then we may explicitly determine the coefficients A_i, B_i, h_j, k_l from the equations of their algebraic relations.

3. Application to modified Ben– jamin–Bona–Mahony (mBBM) model

The mBBM model has been extensively studied,[17] and it reads

$$u_t + c_0 u_x + \alpha u^2 u_x + \beta u_{xxt} = 0. \tag{8}$$

Substituting Eq.(2) into Eq.(8), we have

$$Au' + u^2 u' + Bu''' = 0, \tag{9}$$

where $A = (k_4 + c_0 k_1)/\alpha k_1$, $B = \beta k_1 k_4/\alpha$.

By balancing the highest-order derivative term with the nonlinear term of $u(\xi)$ in Eq.(9), we have

$$N - 2M - 2 = 0. \tag{10}$$

Thus, we can take two pairs of values M and N as follows:

$$M_1 = 1, N_1 = 4; \quad M_2 = 2, N_2 = 6. \tag{11}$$

For simplicity, we only consider the case $M_2 = 2, N_2 = 6$ in this paper. So the ansatz solution of Eq.(4) can be rewritten as

$$\begin{aligned} u(\xi) = {} & A_0 + A_1 f + A_2 f^2 + A_{-1} f^{-1} + A_{-2} f^{-2} \\ & + B_0 f^{-1} f' + B_1 f' + B_2 f f' + B_{-1} f^{-2} f' \\ & + B_{-2} f^{-3} f', \end{aligned} \tag{12}$$

where, accordingly, $f(\xi)$ satisfies

$$\begin{aligned} (f')^2 = {} & h_0 + h_1 f + h_2 f^2 + h_3 f^3 + h_4 f^4 \\ & + h_5 f^5 + h_6 f^6. \end{aligned} \tag{13}$$

Then, in this paper, we take $e = 1$ in Eq.(5b) for example.

Substituting Eq.(12) into Eq.(9) and using Eq.(13), we can couvert the left-hand side of Eq.(9) into a polynomial in $f^r \left(\sqrt{\sum_{j=0}^{6} h_j f^j}\right)^s (r =$

$-10,-9,\ldots,0,\ldots,13,14; s=0,1$). Setting the coefficients of the polynomial to zero we obtain a set of algebraic equations about coefficients $A_i, B_i (i=-2,-1,0,1,2)$ and $h_j (j=0,1,\ldots,6)$. Since they are very complex in the form, we omit them in this paper for simplicity.

Solving the algebraic equations with the aid of Mathematica, one can obtain three sets of solutions as follows:

Case 1

$$\begin{aligned} A_0 &= \pm h_4\sqrt{-\frac{3B}{2h_6}},\quad A_2=\pm 2\sqrt{-6Bh_6},\\ A_1 &= A_{-1}=A_{-2}=B_0=B_1=B_2\\ &= B_{-1}=B_{-2}=0,\ \ h_1=h_3=h_5=0,\\ h_0 &= h_0,\quad h_2=h_2, h_4=h_4,\quad h_6=h_6,\\ A &= B\left(\frac{3h_4^2}{2h_6}-4h_2\right),\quad Bh_6<0. \end{aligned} \tag{14}$$

Case 2

$$\begin{aligned} A_2 &= \sqrt{-6Bh_6},\ \ B_0=\pm\sqrt{-6B},\\ A_0 &= A_1=A_{-1}=A_{-2}=B_1=B_2\\ &= B_{-1}=B_{-2}=0,\\ h_1 &= h_3=h_5=0,\\ h_0 &= h_0,\quad h_2=h_2,\quad h_4=h_4,\quad h_6>0,\\ A &= 2Bh_2,\ \ B<0. \end{aligned} \tag{15}$$

Case 3

$$\begin{aligned} A_0 &= \frac{1}{6h_4}\sqrt{-\frac{6h_6}{B}}(A-2Bh_2),\\ A_2 &= -\sqrt{-6Bh_6},\ \ B_0=\pm\sqrt{-6B},\\ B_{-2} &= \mp\frac{1}{6h_4}\sqrt{-\frac{6}{B}}(2Bh_2-A),\\ A_1 &= A_{-1}=A_{-2}=B_1=B_2\\ &= B_{-1}=0, h_1=h_3=h_5=0,\\ h_0 &= 0,\quad h_2=h_2,\quad h_4=h_4,\quad h_6>0,\\ & 4ABh_2-A^2=4B^2h_2^2,\ \ B<0. \end{aligned} \tag{16}$$

Equations (14)–(16) with Eq.(12) are the solutions to the mBBM model (8) if the appropriate parameters $\{A=(k_4+c_0k_1)/\alpha k_1, B=\beta k_1k_4/\alpha\}$ are given. For example,[14] in Case 3, if $h_2>0$, we have

$$f(\xi)=\left\{\frac{h_2[-1+(\coth 2\sqrt{h_2}\xi\pm \mathrm{csch}2\sqrt{h_2}\xi)^2]}{h_4+2\sqrt{h_2h_6}(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)}\right\}^{1/2}$$

for Eq.(13), so

$$\begin{aligned} u(\xi) &= \frac{1}{6h_4}\sqrt{-\frac{6h_6}{B}}(A-2Bh_2)-\sqrt{-6Bh_6}\left(\frac{h_2[-1+(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)^2]}{h_4+2\sqrt{h_2h_6}(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)}\right)\\ &\pm\sqrt{-6B}\left\{\frac{h_2[-1+(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)^2]}{h_4+2\sqrt{h_2h_6}(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)}\right\}^{-1/2}\Omega\\ &\mp\frac{1}{6h_4}\sqrt{-\frac{6}{B}}(2Bh_2-A)\left\{\frac{h_2[-1+(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)^2]}{h_4+2\sqrt{h_2h_6}(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)}\right\}^{-3/2}\Omega, \end{aligned} \tag{17}$$

where

$$\begin{aligned} \Omega &= \frac{1}{2}\left\{\frac{h_2[-1+(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)^2]}{h_4+2\sqrt{h_2h_6}(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)}\right\}^{-1/2}\frac{1}{(h_4+2\sqrt{h_2h_6}(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi))^2}\\ &\times\Bigg[-4h_2\sqrt{h_2}\frac{(1\pm\cosh 2\sqrt{h_2}\xi)^2}{\sinh^4 2\sqrt{h_2}\xi}((\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)(h_4+2\sqrt{h_2h_6}(\coth 2\sqrt{h_2}\xi\\ &\pm\mathrm{csch}2\sqrt{h_2}\xi))-\sqrt{h_2h_6}(-1+(\coth 2\sqrt{h_2}\xi\pm\mathrm{csch}2\sqrt{h_2}\xi)^2))\Bigg]. \end{aligned}$$

The other specific solutions of Eq.(13) can be obtained from Refs.[12–14]. In the similar process discussed above, the solutions for the above three cases to the mBBM model can be explicitly separated. Case 1 can be obtained by Huang and Zhang's method,[12,13] while, to the best of the authors' knowledge, Case 2 and Case 3 are only, at present, derived by means of our proposed algorithm. Here, we omit the other specific form of solutions to the mBBM model for limited space.

4. Conclusion

In summary, we have presented a new algebraic method based on the first order nonlinear ordinary differential equation with at most a sixth-degree nonlinear term Eq.(5). This method has been applied to the mBBM model and we have obtained some new exact solutions of this nonlinear system. Obviously, the main advantages of this method are firstly, the range of i covers from $-M$ to M; secondly, by calculating $f(\xi)$ which is different solution satisfying relation (5), one can derive the concentrated formulae of the solutions to the equations considered; thirdly, from the concentrated formulae of the solutions, many periodic wave solutions can be simultaneously separated. It is clear that the method presented here greatly extend the applicability of the Huang and Zhang's method[12,13] in obtaining exact solutions of NEEs. We believe that if we can explore more new special solutions of Eq.(5), we will obtain more exact solutions of NEEs by means of our proposed algorithm, and it is very natural to apply this method to other NEEs that contain higher dimensions and derivative orders.

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