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Hojman conserved quantity deduced by weak Noether symmetry for Lagrange systems*

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This paper studies the Hojman conserved quantity, a non-Noether conserved quantity, deduced by special weak Noether symmetry for Lagrange systems. Under special infinitesimal transformations in which the time is not variable, its criterion is given and a method of how to seek the Hojman conserved quantity is presented. A Hojman conserved quantity can be found by using the special weak Noether symmetry.

Keywords: Lagrange system, special weak Noether symmetry, Hojman conserved quantity
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1. Introduction

Since Noether, a German woman scientist, published her famous paper 'Invariante Variationsprobleme' in 1918,[1] the theory of symmetries and conserved quantities has played an important role in mathematics and physics, especially in modern analytical mechanics. Since then, the study on Noether symmetry has gradually made progress and become more perfect.[2−12] Recently Mei brought forward a kind of new symmetry, i.e. a weak Noether symmetry, and presented a method to look for a Noether conserved quantity caused by weak Noether symmetry.[13] The weak Noether symmetry is derived from generalized accelerations in the Noether identity which must be expressed as functions of time, generalized coordinates and generalized velocities by using the differential equations. Apart from the fact that from the weak Noether symmetry one can obtain conserved quantities, it also discovered that it is more prevalent than the Noether symmetry. In this paper, we mainly investigate the Hojman conserved quantity led by the special weak Noether symmetry of Lagrange systems.

2. Criterion of special weak Noether symmetry

Consider a non-singular Lagrange system

$$E_s(L) = 0, \quad (s = 1, \ldots, n), \tag{1}$$

where

$$E_s = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s} - \frac{\partial}{\partial q_s} \tag{2}$$

is an Euler operator. From Eq.(1), one can find all generalized accelerations which can be denoted as follows:

$$\ddot{q}_s = \alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{3}$$

Choose the special infinitesimal transformations of Lie group with respect to time and coordinates in which time is invariable as

$$t^* = t, q_s^*(t^*) = q_s(t) + \varepsilon\xi_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \tag{4}$$

where ε is an infinitesimal parameter, ξ_s are infinitesimal generators.

From Eq.(1), under the infinitesimal transformations (4), we can establish the following identity:

$$\frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}G_{\mathrm{N}} = 0, \tag{5}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_s\frac{\partial}{\partial q_s} + \alpha_s\frac{\partial}{\partial \dot{q}_s}, \tag{6}$$

and $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is called gauge function.

If the infinitesimal generators ξ_s and the gauge function $G_{\mathrm{N}} = G_{\mathrm{N}}(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ satisfy Eq.(5), then we call it the special weak Noether symmetry. Eq.(5) is therefore the criterion of special weak Noether symmetry for Lagrange systems.

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3. Hojman conserved quantity deduced by special weak Noether symmetry

Similarly, under the infinitesimal transformations (4), the determining equations of Lie symmetry for the above system are

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s = \frac{\partial \alpha_s}{\partial q_k}\xi_k + \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{7}$$

For a given Lagrange system, if generators ξ_s of special weak Noether symmetry admit Eq.(7) and there exists a function $\mu = \mu(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ which satisfies

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{8}$$

then we have the Hojman conserved quantity caused by the special weak Noether symmetry which is expressed as

$$I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu\xi_s) + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}\left(\mu\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \text{const}. \tag{9}$$

We now prove the above conclusion. Consider

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{H}} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial \xi_s}{\partial q_s}\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left[\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + \frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right)\right]. \tag{10}$$

It is straightforward to prove that

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial}{\partial \dot{q}_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{\partial \alpha_k}{\partial \dot{q}_s}\left(\frac{\partial}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right), \tag{11a}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_s}{\partial q_s} = \frac{\partial}{\partial q_s}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial \alpha_k}{\partial q_s}\frac{\partial \xi_s}{\partial \dot{q}_k}, \tag{11b}$$

then we take partial derivative of Eq.(7) with respect to $\dot{q}_s$ and have

$$\frac{\partial}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial q_k}\xi_k\right) + \frac{\partial}{\partial \dot{q}_s}\left(\frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right). \tag{12}$$

Substituting Eqs.(11–12) into Eq.(10) we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{H}} = \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) + \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) + \frac{\partial^2 \alpha_s}{\partial q_k \partial \dot{q}_s}\xi_k + \frac{\partial^2 \alpha_s}{\partial \dot{q}_k \partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k. \tag{13}$$

Let us now turn our attention to Eq.(8). We can obtain the following formula through calculation:

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\xi_s\right) = -\frac{\partial^2 \alpha_s}{\partial q_k \partial \dot{q}_s}\xi_k + \frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s - \frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_k}\frac{\partial \alpha_k}{\partial q_s}\xi_s, \tag{14a}$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) = -\frac{\partial^2 \alpha_s}{\partial \dot{q}_k \partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k - \frac{1}{\mu}\frac{\partial \alpha_k}{\partial \dot{q}_s}\frac{\partial \mu}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s + \frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{1}{\mu}\frac{\partial \mu}{\partial q_s}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s, \tag{14b}$$

thus, substituting Eqs.(14a) and (14b) into Eq.(13) we have

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}I_{\mathrm{H}} = \frac{1}{\mu}\frac{\partial \mu}{\partial \dot{q}_s}\left\{\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s\right) - \frac{\partial \alpha_s}{\partial q_k}\xi_k - \frac{\partial \alpha_s}{\partial \dot{q}_k}\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_k\right\} = 0. \tag{15}$$

It is easy to see that the Hojman conserved quantity deduced from the weak Noether symmetry is not different from that of the Noether symmetry.

4. Weak Noether symmetry and Noether symmetry

For the system (1), the special Noether identity has a simple form as follows:

$$\frac{\partial L}{\partial q_s}\xi_s + \frac{\partial L}{\partial \dot{q}_s}\dot{\xi}_s + \dot{G}_{\mathrm{N}} = 0. \tag{16}$$

If the infinitesimal generators ξ_s is not dependent of generalized velocities, that is,

$$\frac{\partial \xi_k}{\partial \dot{q}_s} = 0, \quad (s, k = 1, \ldots, n). \tag{17}$$

Eq.(5) is the same as Eq.(16) formally; conversely, they are different.

Choosing

$$\xi_s = 0, \quad (s = 1, \ldots, n),$$

according to special Noether symmetry we have

$$G_{\rm N} = 0, \quad I_{\rm H} = 0,$$

which is trivial.

Likewise, according to the special weak Noether symmetry we have

$$\frac{\bar{\rm d}}{{\rm d}t} G_{\rm N} = 0,$$

so that we can not only take $G_{\rm N} = 0$ but choose arbitrary integrals.

From the above, we can say that the weak Noether symmetry is more prevalent than the Noether symmetry.

5. Illustrative example

The Lagrangian of a system with one degree of freedom is given as follows:

$$L = \frac{1}{2}\dot{q}^2 \exp(-\gamma t). \tag{18}$$

We can from Eqs.(5) and (18) deduce the weak Noether identity as

$$\dot{q}\exp(-\gamma t)\frac{\bar{\rm d}}{{\rm d}t}\xi + \frac{\bar{\rm d}}{{\rm d}t} G_{\rm N} = 0. \tag{19}$$

Similarly, it is well known that, Noether identity gives

$$\dot{q}\exp(-\gamma t)\dot{\xi} + \dot{G}_{\rm N} = 0. \tag{20}$$

If choosing the infinitesimal generator as

$$\xi = \gamma q - \dot{q}, \tag{21}$$

so that $\dfrac{\bar{\rm d}}{{\rm d}t}\xi = 0$, then substituting (21) into Eq.(19) we have

$$\frac{\bar{\rm d}}{{\rm d}t} G_{\rm N} = 0, \tag{22}$$

but if Noether identity is concerned, we will arrived at

$$\dot{q}\exp(-\gamma t)[\gamma\dot{q} - \ddot{q}] + \dot{G}_{\rm N} = 0. \tag{23}$$

It is clear to see that Eq.(22) have a solution

$$G_{\rm N} = \text{const},$$

from which we can verify that the special weak Noether symmetry exists; whereas mathematically there is no solution for equation (23). According to Ref.[4], the Killing equations have a solution which guarantees the existence of the Noether symmetry, however, this cannot be applied to Eq.(23). Up to now no one has discussed this problem. It is so complex that the problem needs to be further investigated. In a nutshell, the question is whether the existence of Noether symmetry is equivalent to having a solution for Killing equations or not.

The determining equation (7) of Lie symmetry gives

$$\frac{\bar{\rm d}}{{\rm d}t}\left(\frac{\bar{\rm d}}{{\rm d}t}\xi\right) = \gamma\frac{\bar{\rm d}}{{\rm d}t}\xi. \tag{24}$$

It is apparent that the generator of the special weak Noether symmetry admits Eq.(24).

Eq.(8) gives

$$\gamma + \frac{\bar{\rm d}}{{\rm d}t}\ln\mu = 0, \tag{25}$$

which has the following solutions:

$$\mu = \exp(-\gamma t), \tag{26}$$

$$\mu = \exp(-\gamma t + \gamma q - \dot{q}). \tag{27}$$

Thus, substituting Eqs.(26) and (27) into Eq.(9) we have

$$I_{\rm H} = \gamma,$$

which is trivial, and

$$I_{\rm H} = \gamma(1 + \gamma q - \dot{q}) = \text{const}.$$

In conclusion, applications of the weak Noether symmetry is more extensive than those of the Noether symmetry because the condition of the former is weaker. In this way, we can find more conserved quantities for a given system than before.

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