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The Lie symmetries and Noether conserved quantities of discrete non-conservative mechanical systems*

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This paper investigates the Lie symmetries and Noether conserved quantities of discrete non-conservative mechanical systems. The variational principle of discrete mechanics, from which discrete motion equations of systems are deduced, is generalized to the case of including the time variational. The requirement for an invariant group transformation is defined to be the Lie symmetry and the criterion when the Noether conserved quantities may be obtained from Lie symmetries is also presented. An example is discussed for applications of the results.

Keywords: discrete mechanics, total variational principle, Lie symmetry, discrete conserved quantity

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1. Introduction

The symmetries are fundamental issues in mathematical physics especially in analytical mechanics, and play an important role in identifying invariant quantities. In Refs.[1,2], the symmetries and conserved quantities have been thoroughly investigated in constrained mechanical systems by using group technique and this method has recently been extended to the discrete mechanical systems. The variational principles are essential for determining the formalisms of motion equations in mechanical systems and a number of authors have since proposed variational treatments of discrete mechanics: Lee used time as a dynamical variable,[3] Guo *et al* presented a discrete difference variational principle and introduced the concept of Euler–Lagrange cohomology,[4−6] Marsden *et al* developed a discrete integration algorithm that preserves the structure of a system when using the variational method,[7−9] and Zhang and Fu presented discrete variational principle in mechanical systems.[10−14]

Lie symmetries have long been used as a tool for solving and classifying differential equations ever since S. Lie extended group theory to his research on invariance in differential equations. Lutzky was the first to apply Lie symmetries to mechanical systems, by studying the differential equations of motion.[15,16] Investigations on Lie symmetries and their applications in constrained mechanical systems have many results.[17−23] Levi *et al* were the first to extend Lie symmetries to discrete systems,[24−26] but these works are highly mathematical.

A. E. Noether showed that there is a close relationship between the mathematical symmetries of a physical system and its conserved quantities. Specifically, a Noether symmetry is an infinitesimal transformation group in generalized coordinates and time under which the Hamilton action functional is invariant. Dorodnitsyn adapted Noether's theory to discrete Lagrangian systems,[27] which is generalized to non-conservative mechanical systems in Section 4 of this paper.

The purpose of this paper is to study the Lie symmetries and Noether conserved quantities of discrete non-conservative systems. The discrete equations of motion, whose invariance defines the Lie symmetries, are deduced by using a total variational principle. We also derive the criterion for obtaining the Noether conserved quantities from the Lie symmetries, and finally discuss an example.

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2. The total variational principle and discrete motion equations

For brevity of notation, we consider an one-dimensional mechanical system denoted by its Lagrangian $L(t, q, \dot{q})$. The Hamilton action functional is defined by

$$S = \int_{t_1}^{t_2} L(t, q, \dot{q}) \mathrm{d}t. \tag{1}$$

Suppose that the point transformations of time and generalized coordinates are

$$\begin{aligned} t^* &= t + \Delta t = t + \varepsilon\tau(t, q), \\ q^* &= q + \Delta q = q + \varepsilon\xi(t, q). \end{aligned} \tag{2}$$

In transformations (2), ε is a group parameter, Δt and Δq are the variational of time and generalized coordinates, then τ and ξ are generators of the infinitesimal transformations. The generalized quasi-symmetrical transformations of Eq.(2) should satisfy the condition[1]

$$\Delta S = -\int_{t_1}^{t_2} Q\delta q \mathrm{d}t, \tag{3}$$

where $Q(t, q, \dot{q})$ is generalized force and Δ indicates the total variational of S. Using the total variational relationship

$$\Delta q = \delta q + \dot{q}\Delta t, \tag{4}$$

We can rewrite the condition (3) as

$$\Delta S + \int_{t_1}^{t_2} Q\Delta q \mathrm{d}t - \int_{q_1}^{q_2} Q\Delta t \mathrm{d}q = 0. \tag{5}$$

In discrete mechanics, continuous curve in configuration space is replaced with the discrete sequence $(q_i(t_i))(i = 0, 1, \ldots, N)$. The discrete versions of L and Q are $L_\mathrm{d}(t_k, t_{k+1}, q_k, q_{k+1})$ and $Q_\mathrm{d}(t_k, t_{k+1}, q_k, q_{k+1})$ respectively $(k = 0, 1, \ldots, N-1)$. The total variational of corresponding discrete action functional (5) becomes

$$\begin{aligned} &\Delta S_\mathrm{d} + \sum_{k=0}^{N-1} Q_\mathrm{d}(t_{k+1} - t_k)\Delta q_k \\ &- \sum_{k=0}^{N-1} Q_\mathrm{d}(q_{k+1} - q_k)\Delta t_k = 0, \end{aligned} \tag{6}$$

where

$$S_\mathrm{d} = \sum_{k=0}^{N-1} L_\mathrm{d}(t_k, t_{k+1}, q_k, q_{k+1})(t_{k+1} - t_k). \tag{7}$$

The prolongation of Eq.(6) is

$$\begin{aligned} &\Delta \sum_{k=0}^{N-1} L_\mathrm{d}(\phi_k)(t_{k+1} - t_k) + \sum_{k=0}^{N-1} Q_\mathrm{d}(\phi_k)(t_{k+1} - t_k)\Delta q_k - \sum_{k=0}^{N-1} Q_\mathrm{d}(\phi_k)(q_{k+1} - q_k)\Delta t_k \\ &= \sum_{k=1}^{N-1} [D_3 L_\mathrm{d}(\phi_k)(t_{k+1} - t_k) + D_4 L_\mathrm{d}(\phi_{k-1})(t_k - t_{k-1}) + Q_\mathrm{d}(\phi_k)(t_{k+1} - t_k)]\Delta q_k \\ &+ \sum_{k=1}^{N-1} [D_1 L_\mathrm{d}(\phi_k)(t_{k+1} - t_k) + D_2 L_\mathrm{d}(\phi_{k-1})(t_k - t_{k-1}) + L_\mathrm{d}(\phi_{k-1}) - L_\mathrm{d}(\phi_k) - Q_\mathrm{d}(\phi_k)(q_{k+1} - q_k)]\Delta t_k \\ &+ [D_3 L_\mathrm{d}(\phi_0)(t_1 - t_0) + Q_\mathrm{d}(\phi_0)(t_1 - t_0)]\Delta q_0 + D_4 L_\mathrm{d}(\phi_{N-1})(t_N - t_{N-1})\Delta q_N \\ &+ [D_1 L_\mathrm{d}(\phi_0)(t_1 - t_0) - L_\mathrm{d}(\phi_0) - Q_\mathrm{d}(\phi_0)(q_1 - q_0)]\Delta t_0 + [D_2 L_\mathrm{d}(\phi_{N-1})(t_N - t_{N-1}) + L_\mathrm{d}(\phi_{N-1})]\Delta t_N = 0, \end{aligned} \tag{8}$$

where D_j is the partial derivative of the discrete functional with respect to the argument j and $\phi_k = (t_k, t_{k+1}, q_k, q_{k+1})$, $k = 0, 1, \ldots, N-1$.

In Eq.(8), we used the discrete version of integration by parts to rearrange the sequence. The terms Δt_k and Δq_k represent the variational in t_k and q_k respectively. Adopting the fixed boundary $\Delta t_0 = \Delta t_N = 0$ and $\Delta q_0 = \Delta q_N = 0$, we obtain the discrete Euler–Lagrange equation

$$\begin{aligned} &D_3 L_\mathrm{d}(\phi_k)(t_{k+1} - t_k) + D_4 L_\mathrm{d}(\phi_{k-1})(t_k - t_{k-1}) \\ &+ Q_\mathrm{d}(\phi_k)(t_{k+1} - t_k) = 0 \end{aligned} \tag{9}$$

from the variational Δq_k and the discrete energy evolution equation

$$D_1 L_\mathrm{d}(\phi_k)(t_{k+1} - t_k) + D_2 L_\mathrm{d}(\phi_{k-1})(t_k - t_{k-1})$$

$$+L_{\rm d}(\phi_{k-1})-L_{\rm d}(\phi_k)-Q_{\rm d}(\phi_k)(q_{k+1}-q_k)=0 \quad (10)$$

from the variational Δt_k.

3. The Lie symmetries of discrete non-conservative systems

The infinitesimal point transformations of discrete time and generalized coordinates are

$$\begin{aligned} t_k^* &= t_k+\Delta t_k = t_k+\varepsilon\tau_k(t_k,q_k),\\ q_k^* &= q_k+\Delta q_k = q_k+\varepsilon\xi_k(t_k,q_k), \end{aligned} \quad (11)$$

where τ_k,ξ_k are the discrete generators. The vector field of generators is

$$X_{\rm d}^{(0)}=\tau_k\frac{\partial}{\partial t_k}+\xi_k\frac{\partial}{\partial q_k}, \quad (12)$$

which can be prolonged to the two- and three-point schemes

$$X_{\rm d}^{(1)}=X_{\rm d}^{(0)}+\tau_{k+1}\frac{\partial}{\partial t_{k+1}}+\xi_{k+1}\frac{\partial}{\partial q_{k+1}}, \quad (13)$$

$$X_{\rm d}^{(2)}=X_{\rm d}^{(1)}+\tau_{k-1}\frac{\partial}{\partial t_{k-1}}+\xi_{k-1}\frac{\partial}{\partial q_{k-1}}. \quad (14)$$

We now rewrite the discrete motion equation (9) and the energy evolution equation (10) as

$$E(\varphi_k)=0, \quad (15)$$

$$F(\varphi_k)=0, \quad (16)$$

where $\varphi_k=(t_{k-1},t_k,t_{k+1},q_{k-1},q_k,q_{k+1})$, $k=1,\ldots,N-1$. Applying prolongation Eq.(14) to Eqs.(15)–(16), we obtain

$$X_{\rm d}^{(2)}[E(\varphi_k)]=0, \quad (17)$$

$$X_{\rm d}^{(2)}[F(\varphi_k)]=0. \quad (18)$$

Criterion 1 If the infinitesimal generators τ_k and ξ_k satisfy Eqs.(17)–(18), then invariance of the discrete equations of motion (9)–(10) is the Lie symmetry of the discrete non-conservative systems. Eqs.(17)–(18) are thus the determining equations of Lie symmetry.

By solving these equations, we can obtain generators for the infinitesimal Lie symmetry transformations.

4. The Noether conserved quantities of discrete non-conservative systems

The recursive and derivative operators of discrete systems for any discrete variable or function are represented as

$$R_\pm f(z_k)=f(z_{k\pm1}), \quad (19)$$

$$D_{\rm d}f(z_k)=\frac{R_+f(z_k)-f(z_k)}{t_{k+1}-t_k}. \quad (20)$$

Using Eqs.(19) and (20), we can rewrite Eqs.(9)–(10) as

$$D_3L_{\rm d}(\phi_k)+C_tD_4[R_-L_{\rm d}(\phi_k)]+Q_{\rm d}(\phi_k)=0, \quad (21)$$

$$\begin{aligned} &D_1L_{\rm d}(\phi_k)+C_tD_2[R_-L_{\rm d}(\phi_k)]-D_{\rm d}[R_-L_{\rm d}(\phi_k)]\\ &-Q_{\rm d}(\phi_k)D_{\rm d}(q_k)=0, \end{aligned} \quad (22)$$

where $C_t=(t_k-t_{k-1})/(t_{k+1}-t_k)$.

The requirement of Noether symmetry (invariance of the Hamilton action functional under the infinitesimal transformation groups) gives

$$\begin{aligned} &\tau_k\{D_1L_{\rm d}(\phi_k)+C_tD_2[R_-L_{\rm d}(\phi_k)]-D_{\rm d}[R_-L_{\rm d}(\phi_k)]\\ &-Q_{\rm d}(\phi_k)D_{\rm d}(q_k)\}+\xi_k\{D_3L_{\rm d}(\phi_k)\\ &+C_tD_4[R_-L_{\rm d}(\phi_k)]+Q_{\rm d}(\phi_k)\}=0. \end{aligned} \quad (23)$$

Criterion 2 If the infinitesimal group generators τ_k and ξ_k satisfy Eq.(23), and in addition there exists a discrete gauge function $G_k(t_k,q_k)$ such that the identity

$$\begin{aligned} &L_{\rm d}D_{\rm d}(\tau_k)+X_{\rm d}^{(1)}(L_{\rm d})+Q_{\rm d}[\xi_k-D_{\rm d}(q_k)\tau_k]\\ &+D_{\rm d}(G_k)=0 \end{aligned} \quad (24)$$

holds, then the discrete non-conservative systems have the discrete Noether conserved quantities

$$\begin{aligned} &\tau_kR_-L_{\rm d}(\phi_k)+\tau_k(t_k-t_{k-1})D_2[R_-L_{\rm d}(\phi_k)]\\ &+\xi_k(t_k-t_{k-1})D_4[R_-L_{\rm d}(\phi_k)]+G_k=\text{const}. \end{aligned} \quad (25)$$

and Eq.(24) is called the discrete Noether identity.

Proof Using representations Eqs.(13), (19) and (20), the prolongation of Eq.(24) gives

$$
\begin{aligned}
&L_{\rm d}D_{\rm d}(\tau_k)+X_{\rm d}^{(1)}(L_{\rm d})+Q_{\rm d}[\xi_k-D_{\rm d}(q_k)\tau_k]+D_{\rm d}(G_k)\\
=&\tau_k\left\{D_1L_{\rm d}(\phi_k)+C_tD_2[R_-L_{\rm d}(\phi_k)]-D_{\rm d}[R_-L_{\rm d}(\phi_k)]-Q_{\rm d}(\phi_k)D_{\rm d}(q_k)\right\}\\
&+\xi_k\left\{D_3L_{\rm d}(\phi_k)+C_tD_4[R_-L_{\rm d}(\phi_k)]+Q_{\rm d}(\phi_k)\right\}\\
&+D_{\rm d}\left\{\tau_kR_-L_{\rm d}(\phi_k)+\tau_k(t_k-t_{k-1})D_2[R_-L_{\rm d}(\phi_k)]+\xi_k(t_k-t_{k-1})D_4[R_-L_{\rm d}(\phi_k)]+G_k\right\}=0. \qquad (26)
\end{aligned}
$$

Using Eq.(23) or Eqs.(9)–(10), we find that Eq.(26) becomes

$$
\begin{aligned}
&D_{\rm d}\left\{\tau_kR_-L_{\rm d}(\phi_k)+\tau_k(t_k-t_{k-1})D_2[R_-L_{\rm d}(\phi_k)]\right.\\
&\left.+\xi_k(t_k-t_{k-1})D_4[R_-L_{\rm d}(\phi_k)]+G_k\right\}=0. \qquad (27)
\end{aligned}
$$

We therefore obtain the discrete Noether conserved quantities Eq.(25)

$$
\begin{aligned}
&\tau_kR_-L_{\rm d}(\phi_k)+\tau_k(t_k-t_{k-1})D_2[R_-L_{\rm d}(\phi_k)]\\
&+\xi_k(t_k-t_{k-1})D_4[R_-L_{\rm d}(\phi_k)]+G_k=\text{const.}
\end{aligned}
$$

It is now straightforward to acquire the criterion for obtaining the Noether conserved quantities from the Lie symmetries.

Criterion 3 If the Lie symmetry generators τ_k and ξ_k satisfy the discrete Noether identity Eq.(24), then the Lie symmetries of the discrete nonconservative systems lead to the discrete Noether conserved quantities Eq.(25).

Given Criteria 1 and 2, the proof of Criterion 3 is evident.

5. An example

Let us take an example to illustrate this procedure. The discrete Lagrangian of the system and its recursive operator are

$$
L_{\rm d}(\phi_k)=\frac{1}{2}\left(\frac{q_{k+1}-q_k}{t_{k+1}-t_k}\right)^2 \qquad (28)
$$

and

$$
R_-[L_{\rm d}(\phi_k)]=L_{\rm d}(\phi_{k-1})=\frac{1}{2}\left(\frac{q_k-q_{k-1}}{t_k-t_{k-1}}\right)^2. \qquad (29)
$$

The discrete generalized force is

$$
Q_{\rm d}(\phi_k)=-\frac{q_{k+1}-q_k}{t_{k+1}-t_k}. \qquad (30)
$$

Using Eqs.(28)–(30) and (15)–(16), we find that Eqs.(9)–(10) become

$$
E(\varphi_k)=-\frac{q_{k+1}-q_k}{t_{k+1}-t_k}+\frac{q_k-q_{k-1}}{t_k-t_{k-1}}-(q_{k+1}-q_k)=0, \qquad (31)
$$

$$
\begin{aligned}
F(\varphi_k)=&\frac{1}{2}\left(\frac{q_{k+1}-q_k}{t_{k+1}-t_k}\right)^2-\frac{1}{2}\left(\frac{q_k-q_{k-1}}{t_k-t_{k-1}}\right)^2\\
&+\frac{(q_{k+1}-q_k)^2}{t_{k+1}-t_k}=0. \qquad (32)
\end{aligned}
$$

Substituting Eqs.(31)–(32) into Eqs.(17)–(18), we get the determining equations

$$
\begin{aligned}
&\tau_{k-1}D_1E(\varphi_k)+\tau_kD_2E(\varphi_k)+\tau_{k+1}D_3E(\varphi_k)\\
&+\xi_{k-1}D_4E(\varphi_k)+\xi_kD_5E(\varphi_k)+\xi_{k+1}D_6E(\varphi_k)=0, \qquad (33)
\end{aligned}
$$

$$
\begin{aligned}
&\tau_{k-1}D_1F(\varphi_k)+\tau_kD_2F(\varphi_k)+\tau_{k+1}D_3F(\varphi_k)\\
&+\xi_{k-1}D_4F(\varphi_k)+\xi_kD_5F(\varphi_k)+\xi_{k+1}D_6F(\varphi_k)=0. \qquad (34)
\end{aligned}
$$

Now we will solve Eqs.(33)–(34) to obtain the Lie symmetry generators. First let us consider Eq.(33). The representations of $D_jE(\varphi_k)$ in Eq.(33) can be calculated from Eq.(31):

$$
D_1E=\frac{q_k-q_{k-1}}{(t_k-t_{k-1})^2}, \qquad (35)
$$

$$
D_2E=-\frac{q_k-q_{k-1}}{(t_k-t_{k-1})^2}-\frac{q_{k+1}-q_k}{(t_{k+1}-t_k)^2}, \qquad (36)
$$

$$
D_3E=\frac{q_{k+1}-q_k}{(t_{k+1}-t_k)^2}, \qquad (37)
$$

$$
D_4E=-\frac{1}{t_k-t_{k-1}}, \qquad (38)
$$

$$
D_5E=\frac{1}{t_{k+1}-t_k}+\frac{1}{t_k-t_{k-1}}+1, \qquad (39)
$$

$$
D_6E=-\frac{1}{t_{k+1}-t_k}-1. \qquad (40)
$$

Let us suppose that the generators are linear, so that

$$
\tau_k(t_k,q_k)=C_1t_k+C_2q_k+C_3, \qquad (41)
$$

$$
\xi_k(t_k,q_k)=C_4t_k+C_5q_k+C_6, \qquad (42)
$$

where C_1-C_6 are constants.

We then substitute Eqs.(41)–(42) and Eqs.(35)–(40) into Eq.(33), and compare the resulting coefficients to those of Eq.(31). This procedure yields

$$
C_1=C_2=C_4=0. \qquad (43)
$$

So we have

$$
\tau_k(t_k,q_k)=C_3, \qquad (44)
$$

$$\xi_k(t_k, q_k) = C_5 q_k + C_6. \tag{45}$$

As for Eq.(34), after substituting into Eqs.(44)–(45) it becomes

$$\left(\frac{q_{k+1}-q_k}{t_{k+1}-t_k}\right)^2 - \left(\frac{q_k-q_{k-1}}{t_k-t_{k-1}}\right)^2 + 2\frac{(q_{k+1}-q_k)^2}{t_{k+1}-t_k} = 2F(\varphi_k) = 0. \tag{46}$$

The representations of $D_j F(\varphi_k)$ in Eq.(34) can be calculated from Eq.(32), in the same manner as Eqs.(35)–(40). We have thus obtained one group of Lie symmetry generators Eqs.(44)–(45) that satisfies the determining Eqs.(33)–(34) of the discrete non-conservative system.

The next step is to calculate the Noether conserved quantities which correspond to this Lie symmetry. Using representations Eqs.(13), (20) and (28)–(30), substituting the Lie symmetry generators Eqs.(44)–(45) into the Noether identity Eq.(24), we have

$$-\frac{1}{2}\left(\frac{q_{k+1}-q_k}{t_{k+1}-t_k}\right)^2 D_{\rm d}(\tau_k) + \frac{q_{k+1}-q_k}{t_{k+1}-t_k} D_{\rm d}(\xi_k) - \frac{q_{k+1}-q_k}{t_{k+1}-t_k}\xi_k + D_{\rm d}^2(q_k)\tau_k + D_{\rm d}(G_k) = 0. \tag{47}$$

It follows that when

$$\tau_k(t_k, q_k) = 0, \tag{48}$$

$$\xi_k(t_k, q_k) = C_6, \tag{49}$$

so the function

$$G_k(t_k, q_k) = C_6 q_k \tag{50}$$

satisfies the Noether identity Eq.(24). According to Criterion 3, this discrete non-conservative system has the following Noether conserved quantity:

$$C_6\left(\frac{q_k-q_{k-1}}{t_k-t_{k-1}} + q_k\right) = \text{const.} \tag{51}$$

The representations of $D_2[R_- L_{\rm d}(\phi_k)], D_4[R_- L_{\rm d}(\phi_k)]$ in Eq.(24) can be calculated from Eq.(29).

6. Conclusions

There are four main results in this paper: (i) the discrete total variational representation Eq.(6) with non-conservative term, (ii) the discrete vector field and its prolongations Eqs.(12)–(14), (iii) the determining Eqs.(17)–(18) of discrete non-conservative systems, (iv) the discrete Noether identity Eq.(24) and Noether conserved quantities Eq.(25). Using Eq.(6), we can generalize the total variational approach to other discrete mechanical systems such as with non-holonomic constraints. Furthermore, we can study not only Lie symmetries through Criterion 1 but the Noether conserved quantities of these discrete systems using Criteria 2 and 3.

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