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The relativistic bound states for a new ring-shaped harmonic oscillator*

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In this paper a new ring-shaped harmonic oscillator for spin 1/2 particles is studied, and the corresponding eigenfunctions and eigenenergies are obtained by solving the Dirac equation with equal mixture of vector and scalar potentials. Several particular cases such as the ring-shaped non-spherical harmonic oscillator, the ring-shaped harmonic oscillator, non-spherical harmonic oscillator, and spherical harmonic oscillator are also discussed.

Keywords: new ring-shaped harmonic oscillator, bound state, Dirac equation
PACC: 0300, 0330, 0365

1. Introduction

It is well known that the harmonic oscillator potential is one of the exactly solvable potentials in quantum mechanics, and has been widely applied in many fields. In particular, it is a central potential of nuclear shell model, which has provided a good description of nuclear single-particle motion and shell structure for the spherical and axially deformed nuclei.[1,2] However, considering the fact that the realistic nuclei often deviate from the spherical and axial harmonic oscillator model, some authors have introduced several other oscillator models, such as the non-spherical harmonic oscillator (NHO),[3] the ring-shaped harmonic oscillator (RHO),[4−8] and the ring-shaped non-spherical harmonic oscillator (RNHO).[9] Their nonrelativistic energy spectra and wavefunctions have been obtained by solving the corresponding Schrödinger equation. The recurrence formulas of radial matrix elements[10] and the average values of radial operators[11] have been derived for the RNHO. The creation and annihilation operators have also been established from the eigenfunctions with the factorization method.[12]

In recent years, the relativistic harmonic oscillator has received considerable attention. The spherical relativistic harmonic oscillator with spin symmetry has been studied previously.[13−18] Using a Dirac Hamiltonian with scalar S and vector V potentials quadratic in space coordinates, Chen *et al*[19] found a harmonic-oscillator like second order equation which can be solved analytically for $\Delta = V - S = 0$, as considered before by Kukulin,[17] and also for $\Sigma = V + S = 0$. Ginocchio has solved the triaxial, axial, and spherical harmonic oscillators for the case $\Delta = 0$ and applied it to the study of antinucleons embedded in nuclei.[20,21] The case $\Sigma = 0$ is particularly relevant in nuclear physics, since it is usually pointed out as a necessary condition for occurrence of pseudospin symmetry in nuclei.[22,23] Lisboa *et al* have studied the generalized relativistic harmonic oscillator (GRHO) for spin 1/2 particles, obtained the analytical solutions for bound states of the corresponding Dirac equations by setting either Σ or Δ to zero.[24] Under the conditions of exact spin symmetry $\Delta = V - S =$ const and pseudospin symmetry $\Sigma = V + S =$ const, Qiang has solved the Klein–Gordon equation and Dirac equation with the spherical harmonic oscillator (SHO) potential, and presented the energy spectra of bound states and wavefunctions by setting $\Delta = 0$.[18] We have investigated the pseudospin symmetry in the relativistic harmonic oscillator by solving the Dirac equation with quadratic vector and scalar potentials, and found that it is a good approximation in realistic nuclei such as ^{208}Pb, shown the pseudospin breaking in correlation with nuclear mean field shaped by the harmonic oscil-

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lator frequency and the distance of well-bottom deviation from the centre.[25] Because non-spherical potentials are more practical for realistic nuclei, Zhang *et al* have solved the Klein–Gordon equation and Dirac equation with equal scalar and vector RNHO potentials, and obtained the energy spectrum of bound states and wavefunction.[26] We have discussed in details the Dirac-RNHO problem under the condition of pseudospin symmetry and given the spinor wavefunction and energy spectrum for bound states, where several particular cases such as the RHO, NHO and SHO models, are also checked.[27] Recently, Lu *et al* have proposed a double ring-shaped oscillator potential and presented the solution of the Klein–Gordon equation in an equal mixing of scalar and vector fields.[28] Especially, a generalized non-central potential is introduced in Ref.[29], where the exact bound state solutions are obtained for a generalized Hartmann potential in an equal mixing of vector and scalar potentials. In this paper, we add the non-central term proposed in Ref.[29] to harmonic oscillator potential and discuss a more generalized non-spherical harmonic oscillator and give its spinor wavefunction and energy spectrum for bound states under the condition that the scalar potential is equal to the vector potential, where the RNHO, RHO, and NHO models will be regarded as the particular examples.

In spherical coordinates, the new ring-shaped harmonic oscillator (NRHO) potential is defined as

$$U(r,\theta)=\frac{1}{2}M\omega^2r^2+\frac{h^2}{2M}\frac{f(\theta)}{r^2}, \tag{1}$$

where M, ω, and $f(\theta)$ are rest mass, frequency, and asymmetrical term of NRHO, respectively.

2. Solution of Dirac- NRHO problem

In the natural units ($\hbar=c=\omega=1$), the Dirac equation of a nucleon with mass M moving in an attractive scalar potential S and a repulsive vector potential V can be written as

$$\left[\boldsymbol{\alpha}\cdot\boldsymbol{p}+\beta\left(M+S\right)+V\right]\ \psi=E\psi, \tag{2}$$

where

$$\boldsymbol{p}=-\mathrm{i}\nabla,\quad \boldsymbol{\alpha}=\begin{pmatrix}0 & \boldsymbol{\sigma}\\ \boldsymbol{\sigma} & 0\end{pmatrix},\quad \beta=\begin{pmatrix}I & 0\\ 0 & -I\end{pmatrix}, \tag{3}$$

$\boldsymbol{\sigma}$ is the vector Pauli matrix. In the Pauli–Dirac representation, let

$$\psi=\begin{pmatrix}\varphi\\ \chi\end{pmatrix}, \tag{4}$$

there are

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi=\left(E-M-\Sigma\right)\varphi, \tag{5a}$$

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi=\left(E+M-\Delta\right)\chi, \tag{5b}$$

where $\Sigma=V+S$ and $\Delta=V-S$. Under the condition of spin symmetry ($\Delta=0$), Eq.(5) can be simplified into the following form:

$$\boldsymbol{\sigma}\cdot\boldsymbol{p}\chi=\left(E-M-\Sigma\right)\varphi, \tag{6a}$$

$$\chi=\frac{\boldsymbol{\sigma}\cdot\boldsymbol{p}\varphi}{E+M}. \tag{6b}$$

Substituting Eq.(6b) into Eq.(6a), we obtain an Schrödinger-like equation for the low component

$$\left[\boldsymbol{p}^2+\left(E+M\right)\Sigma\right]\ \varphi=\left(E^2-M^2\right)\ \varphi. \tag{7}$$

When Σ is taken to be the GRHO potential, Eq.(7) becomes

$$\left[-\nabla^2+\left(E+M\right)\left(\frac{1}{2}M\omega^2r^2+\frac{h^2}{2M}\frac{f(\theta)}{r^2}\right)\right]\varphi = \left(E^2-M^2\right)\varphi. \tag{8}$$

From Eq.(8), it can be seen that, as the decoupling of pseudospin and pseudo-orbital momentum, the low component spinor can be just a spin up or spin down spinor, i.e. $\begin{pmatrix}1\\0\end{pmatrix}$ or $\begin{pmatrix}0\\1\end{pmatrix}$, multiplied by a function of the spherical coordinates, i.e., χ can be written as

$$\varphi(r,\theta,\phi)=\frac{\mathrm{e}^{\mathrm{i}m\phi}}{\sqrt{2\pi}}\frac{R(r)}{r}\Theta(\theta)\varphi_{m_\mathrm{s}}, \tag{9}$$

in which m_s is $\pm1/2$ and χ_{m_s} a spin up or spin down two-component spinor. Substituting Eq.(9) into Eq.(8), we can separate the differential equations for the angular and radial wavefunctions as

$$\frac{\mathrm{d}^2R(r)}{\mathrm{d}r^2}+\left[\left(E^2-M^2\right)-(E+M)\,\frac{1}{2}Mr^2-\frac{\lambda}{r^2}\right]R(r)=0, \tag{10a}$$

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\sin\theta\frac{\mathrm{d}\Theta(\theta)}{\mathrm{d}\theta}+\left(\lambda-\frac{E+M}{2M}f(\theta)-\frac{m^2}{\sin^2\theta}\right)\Theta(\theta)=0, \tag{10b}$$

where λ is a constant arising from the separation of variables. In order to obtain the solution of angular part for the GRHO, $f(\theta)$ is adopted as

$$f(\theta)=\frac{\alpha\cos^2\theta+\beta\cos\theta+\gamma}{\sin^2\theta}\quad, \tag{11}$$

which arises from a reference in Ref.[29]. The other expressions on $f(\theta)$ in Ref.[29] can be investigated in the same footing without repeat. In order to solve Eq.(10b), a new variable $x=\cos^2(\theta/2)$ is introduced. By letting $\Theta=x^{\mu}(1-x)^{\nu}w(x)$, Eq.(10b) is transformed into the following form:

$$\begin{aligned}&x(1-x)\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+\left[1+2\mu-2\left(1+\mu+\nu\right)x\right]\frac{\mathrm{d}w}{\mathrm{d}x}\\&+\frac{E+M}{4M}(\alpha-\gamma)-\left(\mu+\nu+2\mu\nu+\frac{1}{2}m^2-\lambda\right)w\\=\;&0,\end{aligned} \tag{12}$$

where $\mu=\frac{1}{2}\sqrt{m^2+(\alpha-\beta+\gamma)(E+M)/2M}$, $\nu=\frac{1}{2}\sqrt{m^2+(\alpha+\beta+\gamma)(E+M)/2M}$. The solution of Eq.(12) is a hyper-geometric function $F(a,b,c;x)$ with

$$\begin{aligned}a=&\mu+\nu\\&+\frac{1}{2}\left[1-\sqrt{1+4\lambda+2\alpha(E+M)/M}\right],\end{aligned} \tag{13a}$$

$$\begin{aligned}b=&\mu+\nu\\&+\frac{1}{2}\left[1+\sqrt{1+4\lambda+2\alpha(E+M)/M}\right],\end{aligned} \tag{13b}$$

$$c=1+2\mu. \tag{13c}$$

To satisfy the condition of bound states, $a=-\kappa$ and the integer number $\kappa\geq 0$ are required. Then we obtain

$$\lambda+\frac{1}{4}=\frac{1}{4}(b+\kappa)^2-\frac{E+M}{2M}\alpha, \tag{14}$$

i.e.,

$$\begin{aligned}\lambda=&\frac{1}{2}\left[\sqrt{\frac{E+M}{2M}(\alpha-\beta+\gamma)+m^2}+\sqrt{\frac{E+M}{2M}(\alpha+\beta+\gamma)+m^2}\right.\\&\left.+\sqrt{\frac{E+M}{2M}(\alpha-\beta+\gamma)+m^2}\sqrt{\frac{E+M}{2M}(\alpha+\beta+\gamma)+m^2}\right]\\&+\kappa^2+\left(1+\sqrt{\frac{E+M}{2M}(\alpha-\beta+\gamma)+m^2}+\sqrt{\frac{E+M}{2M}(\alpha+\beta+\gamma)+m^2}\right)\kappa\\&+\frac{1}{2}\left(m^2-\frac{E+M}{2M}\alpha+\frac{E+M}{2M}\gamma\right).\end{aligned} \tag{15}$$

In order to solve the radial equation (10a), a new variable $\rho=k\,r$ is introduced. By setting

$$k=\sqrt[4]{(E+M)M/2}, \tag{16a}$$

$$s=(E-M)\sqrt{2(E+M)/M}, \tag{16b}$$

$$l'(l'+1)=\lambda, \tag{16c}$$

we turn the radial equation (10a) is into

$$\frac{\mathrm{d}^2R(\rho)}{\mathrm{d}\rho^2}+\left[s-\rho^2-\frac{l'(l'+1)}{\rho^2}\right]R(\rho)=0. \tag{17}$$

Considering the asymptotic behaviour of the radial wavefunction as $r\to 0$ and $r\to\infty$, we can express the physically acceptable solution as

$$R(\rho)=\rho^{l'+1}\mathrm{e}^{-\frac{1}{2}\rho^2}f(\rho). \tag{18}$$

Substituting Eq.(18) into Eq.(17), we obtain

$$\frac{\mathrm{d}^2 f(\rho)}{\mathrm{d}\rho^2} + 2\left(-\rho + \frac{l'+1}{\rho}\right)\frac{\mathrm{d}f(\rho)}{\mathrm{d}\rho} + (s - 2l' - 3)f(\rho) = 0. \tag{19}$$

Letting $\xi = \rho^2$, we can convert the radial equation into

$$\xi\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \left(l' + \frac{3}{2} - \xi\right)\frac{\mathrm{d}f}{\mathrm{d}\xi} - \frac{1}{4}(2l' + 3 - s)f = 0, \tag{20}$$

which is a confluent hyper geometric equation, and its solution is a confluent hyper-geometric function

$$f(\xi) = F\left(\frac{2l'+3-s}{4}, l' + \frac{3}{2}, \xi\right). \tag{21}$$

However, the only physically acceptable solutions are the ones that satisfy the boundary condition $R(\xi \to \infty) \to 0$. This fact requires that

$$\frac{2l'+3-s}{4} = -n_r, \quad n_r = 0,\ 1,\ 2,\ L, \tag{22}$$

where n_r is the radial quantum number. Eq.(22) gives the bound energy spectrum for the NRHO with pseudospin symmetry

$$(E - M)\sqrt{E+M} = \sqrt{2M}(2n_r + l' + 3/2), \tag{23}$$

where

$$l' = -1/2 + \sqrt{\lambda + 1/4}. \tag{24}$$

The corresponding unnormalized radial wavefunction is expressed as

$$R_{n_r l}(r) = (kr)^{l'+1}\exp\left(-\frac{1}{2}k^2 r^2\right) \times F(-n_r, l' + 3/2, k^2 r^2). \tag{25}$$

So the unnormalized spinor wavefunction is

$$\psi(r) = \begin{pmatrix} \varphi(\boldsymbol{r}) \\ \chi(\boldsymbol{r}) \end{pmatrix} = \frac{1}{\sqrt{2\pi}}\begin{pmatrix} \varphi_{m_s} \\ \dfrac{\boldsymbol{\sigma}\cdot\boldsymbol{p}}{E-M}\varphi_{m_s} \end{pmatrix}\frac{R_{n_r l'}(r)}{r}\Theta_{l'm}(\theta)\mathrm{e}^{\mathrm{i}m\phi}. \tag{26}$$

3. Discussion

From Eqs.(15) and (23), the bound energies $E(n_r,\ \kappa,\ m)$ with the quantum numbers $(n_r,\ \kappa,\ m)$ are found to be correlated with the parameters $(\alpha,\ \beta,\ \gamma)$ for GRHO. When the parameters $(\alpha,\ \beta,\ \gamma)$ are fixed to some special values, NRHO will reduce to some special cases. For instances, the examples appearing in the Ref.[27] can be reproduced with the proper choices of the parameters $(\alpha,\ \beta,\ \gamma)$. With the choice $\alpha = -A,\ \beta = 0,\ \gamma = A + B$, NRHO potential degenerates to RNHO potential. The corresponding parameter becomes

$$\lambda = k^2 + \left(1 + 2\sqrt{B\,(E+M)\,/2M + m^2}\right)k + \sqrt{B\,(E+M)\,/2M + m^2} + B\,(E+M)\,/2M + m^2 + A\,(E+M)\,/2M. \tag{27}$$

With the choice $\alpha = -A,\ \beta = 0,\ \gamma = A$, GRHO potential reduces to NHO potential. The corresponding parameter becomes

$$\lambda = k^2 + k + 2mk + m^2 + m + A\,(E+M)\,/2M. \tag{28}$$

While $\alpha = 0,\ \beta = 0,\ \gamma = B$, NRHO potential turns into RHO potential. The corresponding parameter becomes

$$\lambda = k^2 + \left(1 + 2\sqrt{B\,(E+M)\,/2M + m^2}\right)k + \sqrt{B\,(E+M)\,/2M + m^2} + B\,(E+M)\,/2M + m^2. \tag{29}$$

In particular, NRHO potential can be degenerated to a usual Harmonic oscillator potential with $\alpha = \beta = \gamma = 0$, which gives the energy spectrum

$$(E - M)\sqrt{E+M} = \sqrt{2M}(2n_r + l + 3/2). \tag{30}$$

4. Conclusions

Under the condition that the scalar potential is equal to the vector potential, we have solved exactly the Dirac equation for a new ring-shaped harmonic oscillator potential by using the ordinary method of separation of variables, and have given the bound-state energy spectra and spinor wavefunctions. Several particular cases, such as RNHO, RHO, NHO and usual HO are also discussed. Their bound-state energy spectra and spinor wavefunctions are also been obtained by setting the parameters in NRHO potential to appropriate values.

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