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Compact-like discrete breather and its stability in a discrete monatomic Klein–Gordon chain*

Xu Quan(徐 权)[a)b)†] and Tian Qiang(田 强)[b)]

[a)] *Department of Physics, Daqing Normal University, Daqing* 163712, *China*

[b)] *Department of Physics, Beijing Normal University, Beijing* 100875, *China*

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This paper studies a discrete one-dimensional monatomic Klein–Gordon chain with only quartic nearest-neighbour interactions, in which the compact-like discrete breathers can be explicitly constructed by an exact separation of their time and space dependence. Introducing the trying method, it proves that compact-like discrete breathers exist in this nonlinear system. It also discusses the linear stability of the compact-like discrete breathers, when the coefficient (β) of quartic on-site potential and the coupling constant (K_4) of quartic interactive potential satisfy the given conditions, they are linearly stable.

Keywords: compact-like discrete breather, Klein–Gordon chain, linear stability
PACC: 6320R, 6320P, 0547

1. Introduction

Energy localized due to nonlinearity in dynamical system has been observed for more than one century. Among the most important characteristics of a localized excitation there are its localization length and the spatial decay characteristics. Because the compact-like discrete breather (CDB) has not an exponential tail, they will not interact with each other until the point of contact, which lead to a much weaker mutual interaction between different CDB in the same lattice. The lifetime of the CDB is thus substantially increased and the CDB can be of interest for energy storage.

The compact-like localized model was first presented by Rosenau and Hyman,[1] afterwards Kivshar showed that the solitons in purely anharmonic lattices could be treated as compactions,[2] and Dusuel *et al* also proved that the generalized ϕ^4 or double-well model with nonlinear coupling could exhibit compact-like solution.[3] Recently, the discussions about CDB were found in Refs.[4–10], but many of them did not analyse their stability. This paper investigates the existence and linear stability of the CDB in a discrete one-dimensional monatomic Klein–Gordon chain with soft and hard quartic on-site potential and quartic interactive potential by using the trying method[8−10] and the linearly stable theory.[11−19]

This paper is organized as follows: in Section 2, we describe the model, a discrete one-dimensional monatomic Klein–Gordon chain with soft and hard quartic on-site potential and quartic interactive potential, and the evolution equation. In Section 3, we search the proof of existence of the CDB by using the trying method. In Section 4, we analyse the linear stability of the CDB. The paper concludes with a short summary of the main results in Section 5.

2. The model

We consider a discrete one-dimensional monatomic Klein–Gordon system. In such a chain, atoms with unit mass interact with their nearest-neighbours. The Hamiltonian of the system is given by

$$H = \sum \left[\frac{1}{2}\dot{u}_n^2 + V(u_n) + \frac{K_4}{4}(u_{n+1} - u_n)^4 \right], \quad (1)$$

where u_n is the displacement of the n-th atom, K_4 is parameters controlling the strength of the nonlinear coupling, and $V(u_n)$ is an on-site potential

$$V(u_n) = \frac{1}{2}\omega_0^2 u_n^2 + \frac{1}{4}\beta u_n^4. \quad (2)$$

The equation of motion for the displacement of

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†E-mail: xuquan@dqsy.net

the n-th atom from its equilibrium is

$$\ddot{u}_n = -\omega_0^2 u_n - \beta u_n^3 + K_4 \left[(u_{n+1} - u_n)^3 - (u_n - u_{n-1})^3 \right]. \quad (3)$$

We will discuss the system by using the trying method. These models appear in many realistic physical systems.

3. The proof of existence of CDB

In this section, we look for CDB solutions of Eq.(3). We choose dependent variable transformation

$$u_n(t) = (-1)^n \Phi_n G(t). \quad (4)$$

Substituting Eq.(4) into Eq.(3), we can rewrite Eq.(3) as:

$$\ddot{G} + \omega_0^2 G + C G^3 = 0, \quad (5a)$$

$$K_4 \left[(\Phi_{n+1} + \Phi_n)^3 + (\Phi_n + \Phi_{n-1})^3 \right] + \beta \Phi_n^3 - C \Phi_n = 0, \quad (5b)$$

where C is a determined constant.

When $C > 0$, equation (5a) has solution

$$G = G_0 \mathrm{cn}(\omega t, m), \quad (6)$$

where cn is a Jacobian elliptic function of modulus $m^2 = \dfrac{C G_0^2}{2(\omega_0^2 + C G_0^2)}$ and pulsation $\omega = \sqrt{\omega_0^2 + C G_0^2}$.

When $C < 0$, equation (5a) has solution

$$G = G_0 \mathrm{sn}(\omega t, m), \quad (7)$$

where sn is a Jacobian elliptic function of modulus $m^2 = -\dfrac{C G_0^2}{2(\omega_0^2 + C G_0^2)}$ and pulsation $\omega = \sqrt{\omega_0^2 + C G_0^2/2}$.

To seek out the CDB solutions, we assume the solution of Eq.(5b) to be in the following form:

$$\Phi_n = \cos[q(n - n_0)], \text{ when } |n - n_0| < \pi/2q, \quad (8)$$

and $\Phi_n = 0$, otherwise, where q and n_0 are constants. Substituting Eq.(8) into Eq.(5b), we find

$$6K_4(\cos q + 1)\sin^2 q + \Phi_n^2 \left[4K_4 \left(2\cos^3 q + 3\cos^2 q - 1 \right) + \beta \right] - C = 0. \quad (9)$$

Thus, we have

$$\left(2\cos^3 q + 3\cos^2 q - 1 \right) + \frac{\beta}{4K_4} = 0, \quad (10a)$$

$$C = 6K_4(\cos q + 1)\sin^2 q. \quad (10b)$$

When $K_4 > 0, C > 0$, equation (3) has the following solution:

$$u_n = (-1)^n G_0 \cos[q(n - n_0)] \mathrm{cn}(\omega t, m), \text{ when } |n - n_0| < \pi/2q, \quad (11)$$

and $u_n(t) = 0$, otherwise. The spatial width is $W_0 = \pi/q$.

When $K_4 < 0$, $C < 0$, equation (3) has the following solution:

$$u_n = (-1)^n G_0 \cos[q(n - n_0)] \mathrm{sn}(\omega t, m), \text{ when } |n - n_0| < \pi/2q, \quad (12)$$

and $u_n(t) = 0$, otherwise. The spatial width is $W_0 = \pi/q$.

When $\beta < 0$, Eq.(2) corresponds to soft quartic on-site potential. Equation (10a) can be written as

$$2\cos^3 q + 3\cos^2 q - 1 - |\beta|/4K_4 = 0, \quad (13)$$

where $K_4 > 0$. This equation has real q solution for $K_4 > |\beta|/16$; the q can be determined by the value of K_4.

When $\beta > 0$, equation (2) corresponds to hard quartic on-site potential. Equation (10a) can be written as

$$2\cos^3 q + 3\cos^2 q - 1 + |\beta|/4K_4 = 0, \quad (14)$$

here $K_4 > 0$. This equation has real q solution for $K_4 > |\beta|/4$; the q can be determined by the value of K_4, too. For $K_4 < 0$, the results are reverse.

The relations between K_4 and q of Eqs.(13) and (14) are shown in Fig.1.

The CDBs of Eqs.(11) and (12) are shown in Fig.2.

From the Figs.1 and 2 we know that if q, K_4 and β satisfy the conditions of Fig.1, equation (5b) has the solution Eq.(11) or Eq.(12). It indicates that the discrete one-dimensional monatomic Klein–Gordon chain has the CDB solution, indeed.

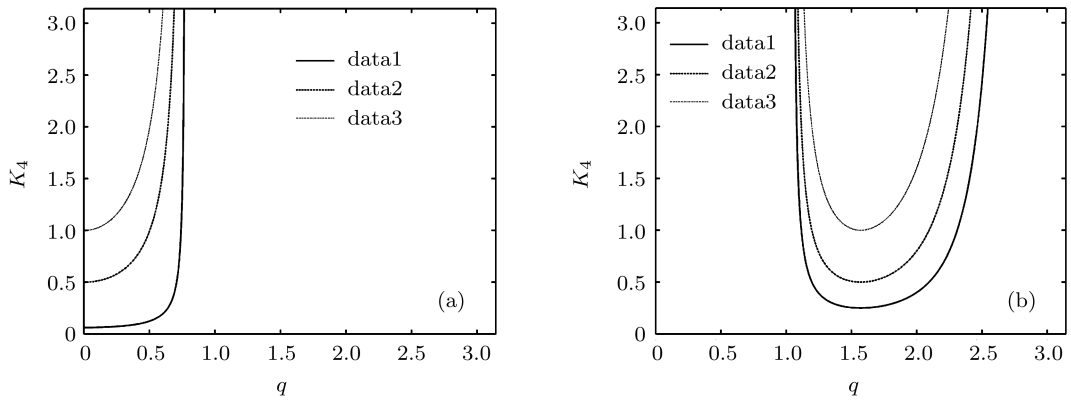


Fig.1. The relation of q with K_4. (a) The relation of q with K_4 in soft quartic on-site potential, the data1 denotes that $\beta = 1$, the data2 denotes that $\beta = 16$, the data3 denotes that $\beta = 32$ and the $K_4 > 0$. (b) The relation of q with K_4 in hard quartic on-site potential, the data1 denotes that $\beta = 1$, the data2 denotes that $\beta = 2$, the data3 denotes that $\beta = 4$ and the $K_4 > 0$.

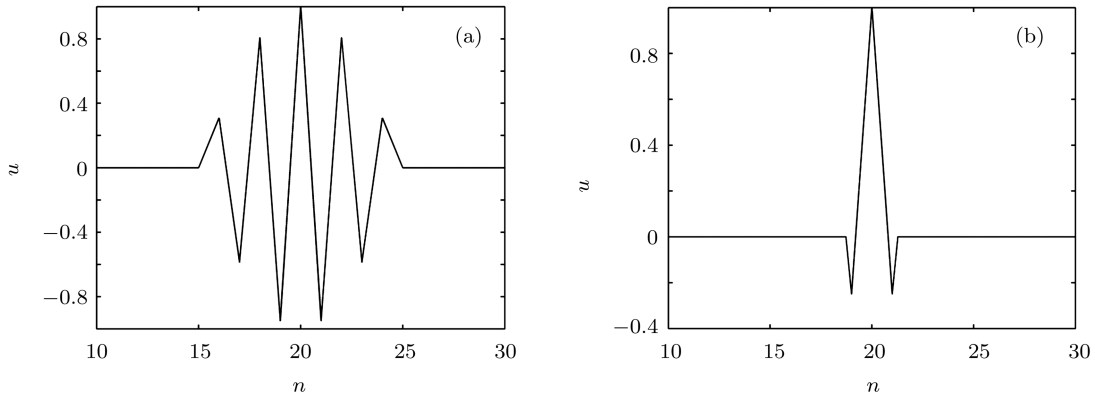


Fig.2. (a) CDB in soft quartic on-site potential $q = 0.1\pi$, (b) CDB in hard quartic on-site potential $q = 0.42\pi$.

4. The analyses of linear stability

Next, we discuss the stability of these solutions. The stability of discrete periodic solution of discrete nonlinear system was discussed in many papers.[11−19] According to Aubry's theory[14] we introduce the variable transformation $u_n(t) \to u_n(t) + \varepsilon_n(t)$, the linearized equations of Eq.(3) is:

$$\begin{aligned}\ddot{\varepsilon}_n = & \left(-\omega_0^2 - 3\beta u_n^2\right)\varepsilon_n \\ & +3K_4[(u_{n+1}-u_n)^2(\varepsilon_{n+1}-\varepsilon_n) \\ & -(u_n-u_{n-1})^2(\varepsilon_n-\varepsilon_{n-1})],\end{aligned} \tag{15}$$

where ε_n represents a small perturbation of the solution of the dynamical equations.

A solution $u_n(t)$ of Eq.(3) is said to be linearly stable when there is no solution $\varepsilon_n(t)$ of Eq.(15) which diverges exponentially at large time. This property does not imply that a perturbation on the initial conditions will remain small for all time for the real nonlinear dynamical system Eq.(1). However, this linear stability implies that this perturbation will not grow exponentially as a function of time. Such trajectories have physically a significantly longer lifetime than those which are linearly unstable.

In this case $u_n(t)$ is a real and time-independent solution of Eq.(3) the general solution of Eq.(15) can be written in the following form:

$$\varepsilon_n(t) = a_n e^{i\omega_\lambda t}, \tag{16}$$

where the eigenfrequencies ω_λ and eigenmodes a_n are determined by the following eigenequation:

$$\begin{aligned}&\left(\omega_0^2 + 3\beta u_n^2\right)a_n - 3K_4[(u_{n+1}-u_n)^2(a_{n+1}-a_n) \\ &-(u_n-u_{n-1})^2(a_n-a_{n-1})] = \omega_\lambda^2 a_n.\end{aligned} \tag{17}$$

Then, we note that if ω_λ is an eigenvalue for the eigenmodes a_n, then ω_λ^* is an eigenvalue for the eigenmodes a_n^*. As a result, linear stability is achieved if and only if all the eigenvalues of Eq.(17) are real. Thus, for a large but finite system with N sites, we

should have $2N$ real eigenvalues ω_λ for a linear stability solution. The eigenequation (17) is easily solved by introducing a wave vector k and can be written as

$$a_n = \mathrm{e}^{\mathrm{i}kn}. \tag{18}$$

Substituting Eqs.(18) and (11) into Eq.(17), we have

$$\begin{aligned}\omega_\lambda^2 = {} & \omega_0^2 + 3\beta\left[G_0\cos q\,(n-n_0)\right]^2 \\ & -6K_4G_0^2[\cos^2 q\,(n-n_0)\,(\cos q-1)^2\,(\cos k-1) \\ & +\sin^2(n-n_0)\sin^2 q\,(\cos k-1) \\ & -2\mathrm{i}\cos q\,(n-n_0)\sin q\,(n-n_0) \\ & \times(\cos q-1)\sin q\sin k].\end{aligned} \tag{19}$$

If ω_λ is real, $\cos q\,(n-n_0)\sin q\,(n-n_0)\,(\cos q-1)\sin q\sin k$ must be zero, and we obtain the following conclusions.

Case 1: $k = 2s\pi$, $s = 0, 1, 2\ldots$, equation (19) can be rewritten as:

$$\omega_\lambda^2 = \omega_0^2 + 3\beta\left[G_0\cos q\,(n-n_0)\right]^2. \tag{20}$$

When $\beta > 0$ (hard quartic on-site potential) ω_λ is always real, the CDB solution of Eq.(3) is said to be linearly stable.

When $\beta < 0$ (soft quartic on-site potential) ω_λ is real, the CDB solution of Eq.(3) is said to be linearly stable, if and only if β satisfies

$$|\beta| \le \frac{\omega_0^2}{3\left[G_0\cos q\,(n-n_0)\right]^2}. \tag{21}$$

Case 2: $k = (2s+1)\,\pi$, $s = 0, 1, 2\ldots$, equation (19) can be rewritten as:

$$\begin{aligned}\omega_\lambda^2 = {} & \omega_0^2 + 3\beta[G_0\cos q\,(n-n_0)]^2 \\ & +12K_4G_0^2[\cos^2 q\,(n-n_0)\,(\cos q-1)^2 \\ & +\sin^2(n-n_0)\sin^2 q].\end{aligned} \tag{22}$$

When $\beta > 0$ (hard quartic on-site potential) and $K_4 > 0$, ω_λ is always real, the CDB solution of Eq.(3) is said to be linearly stable.

When $\beta > 0$ (hard quartic on-site potential) and $K_4 < 0$, ω_λ is real, the CDB solution of Eq.(3) is said to be linearly stable if and only if β and K_4 satisfy

$$|K_4| \le \frac{\omega_0^2 + 3\beta\left[G_0\cos q\,(n-n_0)\right]^2}{12G_0^2\left[\cos^2 q\,(n-n_0)\,(\cos q-1)^2 + \sin^2(n-n_0)\sin^2 q\right]}. \tag{23}$$

When $\beta < 0$ (soft quartic on-site potential) and $K_4 > 0$, ω_λ is real, the CDB of Eq.(3) is said to be linearly stable, if and only if β and K_4 satisfy

$$|K_4| \ge \frac{3\beta\left[G_0\cos q\,(n-n_0)\right]^2 - \omega_0^2}{12G_0^2\left[\cos^2 q\,(n-n_0)\,(\cos q-1)^2 + \sin^2(n-n_0)\sin^2 q\right]}. \tag{24}$$

When $\beta < 0$ (soft quartic on-site potential) and $K_4 < 0$, ω_λ is real, the CDB solution of Eq.(3) is said to be linearly stable, if and only if β and K_4 satisfy

$$|K_4| \le \frac{\omega_0^2 - 3\beta\left[G_0\cos q\,(n-n_0)\right]^2}{12G_0^2\left[\cos^2 q\,(n-n_0)\,(\cos q-1)^2 + \sin^2(n-n_0)\sin^2 q\right]}. \tag{25}$$

Substituting Eqs.(18) and (12) into Eq.(17), we obtain the same results.

5. Conclusions

This paper has investigated the existence and the linear stability of the CDB in a discrete one-dimensional monatomic Klein–Gordon chain with soft and hard quartic on-site potential and quartic interactive potential, separately. Above discussion indicates that the discrete one-dimensional monatomic Klein–Gordon chain has the CDB solution and it is linearly stable when β and K_4 satisfy above conditions. It also indicates that the linear stability of the CDB solution of the discrete one-dimensional monatomic Klein–Gordon chain with hard quartic on-site potential is higher than that with soft quartic on-site potential.

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