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Agreement coordination for second-order multi-agent systems with disturbances

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This paper is devoted to the study of consensus problems for the second-order multi-agent systems with external disturbances, switching topology and communication time-delay. Firstly, we perform a model transformation and separate the agreement states from the disagreement states. Secondly, according to this system, we derive a sufficient condition for consensus problem without disturbances by using the Lyapunov-based approach. Furthermore, we give a design criterion in terms of bilinear matrix inequality for the control protocol in the presence of disturbances. Finally, numerical simulations are provided to show the effectiveness of our strategies.

Keywords: consensus, time-delay, switching topology, disturbance
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1. Introduction

Distributed coordinated control of multiple agents has attracted a great number of researchers in recent years. This is partly due to recent technological advances in communication and computation, and important applications, including unmanned vehicles, automated highway systems, formation control, minisatellites, etc.

In a multi-agent system, agents are usually coupled and interconnected with some simple neighbour rules. Generally, all agents need to agree on certain quantities of interest, such as the location and the speed. This problem is usually called consensus problem.

During the past decade, many researches have dealt with consensus problems.[1−12] Vicsek *et al*[1] introduced a simple model for phase transition of a group of self-driven particles and numerically demonstrated complex dynamics of the model. Jadbabaie *et al*[2] focused on attitude alignment for network of agents with an undirected graph in which each agent had a discrete-time integrator dynamics. Recently, one has investigated the stability properties for second-order multi-agent systems. For example, Tanner *et al*[3] introduced a set of control laws that enabled the second-order agents to generate stable flocking motion. Ren *et al*[4] proposed a second-order protocol and provided sufficient conditions in the case of fixed topology. Olfati-Saber[5] considered a group of mobile second-order agents moving in a plane and introduced a set of control laws that enabled the group to achieve a common velocity while avoiding collisions.

In engineering practice, multi-agent systems may not be able to be measured accurately due to various disturbances, such as time-delay, external disturbances and the variation of network topology.[3−16] Lin and Jia[11] considered consensus problems for first-order multi-agent systems with external disturbances and model uncertainty in fixed and switching topologies. By investigating recent results, little work has addressed the effects of external disturbances for second-order multi-agent systems. It is a challenging task to solve the agreement problems for the second-order systems, because of external disturbances of the position vector and the velocity vector which might cause the system to diverge or oscillate. Moreover, once time-delay and switching topology are involved, the analysis of second-order agent stability becomes much more complex. However, it is of great significance to improve the robustness of the second-order multi-agent systems in practical application.

With this background, we investigate consensus problems of second-order multi-agent systems in undirected graphs with external disturbances, switching topology and time-delay. In doing the analysis,

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we perform a model transformation for second-order multi-agent systems and separate the agreement states from the disagreement states first. A new model is obtained for the disagreement dynamics. Then, according to this system, we derive a sufficient condition for the state consensus of the system in the absence of disturbances by using the Lyapunov-based approach. Finally, we analyse the case where there exist the disturbances and present a design rule in terms of bilinear matrix inequality to make all agents reach consensus with a desired H_∞ performance.

The remainder of the present paper is organized as follows. We introduce some basic concepts and notation in algebraic graph theory in Section 2. In Section 3 we present the multi-agent system model and define the problem addressed. The main results of the second-order multi-agent systems are brought forth in Section 4. The numerical simulations are given in Section 5. Finally, some conclusions are drawn in Section 6.

2. Graph theory

Let $\boldsymbol{G}(V, \varepsilon, A)$ be an undirected graph of order n with a set of nodes $V = \{s_1, \ldots, s_n\}$, set of edges $\varepsilon \in V \times V$, and a weighted adjacency matrix $\boldsymbol{A} = [a_{ij}]$ with nonnegative elements a_{ij} for all $i, j \in \chi = \{1, 2, \ldots, n\}$. An edge of $\boldsymbol{G}$ is denoted by $e_{ij} = (s_i, s_j)$. The adjacency elements associated with the edges of the graph are positive, i.e. $e_{ij} \in \varepsilon \Leftrightarrow a_{ij} > 0$. Moreover, we assume $a_{ii} = 0$ for all $i \in \chi$. The set of neighbours of node s_i is denoted by $N_i = \{s_j \in V : (s_i, s_j) \in \varepsilon\}$. A diagonal matrix $\boldsymbol{D} = \mathrm{diag}\{d_1, \ldots d_n\}$ is a degree matrix of $\boldsymbol{G}$, whose diagonal elements $d_i = \sum_{j=1}^{n} a_{ij}$ for $i \in \chi$. Then the Laplacian of the weighted graph $\boldsymbol{G}$ is defined as

$$\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A} \in R^{n \times n}. \tag{1}$$

Correspondingly, the graph Laplacian with the undirected graph is defined as $\boldsymbol{L} = [l_{ij}]$, where $l_{ii} = d_i$ and $l_{ij} = -a_{ij}, \quad i \neq j$. An important fact of $\boldsymbol{L}$ is that all the row sums of $\boldsymbol{L}$ are zero and thus $1_n = [1, 1, \ldots, 1]^{\mathrm{T}} \in R^n$ is an eigenvector of $\boldsymbol{L}$ associated with the eigenvalue $\lambda = 0$, and for an undirected graph $\boldsymbol{G}$, $\boldsymbol{L}$ is symmetric and positive semi-definite. A path is a sequence of ordered edges of the form (s_{i_1}, s_{i_2}), $(s_{i_2}, s_{i_3}), \ldots$, where $s_{i_j} \in V$. An undirected graph is called being connected if there is a path between any distinct pair of nodes.

Lemma 1 If the undirected graph $\boldsymbol{G}$ is connected, then its Laplacian $\boldsymbol{L}$ satisfies[12]

1) $\mathrm{rank}(\boldsymbol{L}) = n - 1$, zero is an eigenvalue of $\boldsymbol{L}$, and 1_n is the corresponding eigenvector, i.e. $\boldsymbol{L}1_n = 0$.

2) The remaining $n-1$ eigenvalues are all positive and real.

3. Problem description

We consider a network system composed of n agents. Each agent is regarded as a node in an undirected graph $\boldsymbol{G}$. Each edge $(s_i, s_j) \in \varepsilon(\boldsymbol{G}(t))$ corresponds to an available information link between agent i and agent j at time t. Moreover, each agent updates its current state based on the information received from its physical or logical neighbours.

In addition, we assume that the dynamics of each agent is

$$\begin{aligned} \dot{x}_i(t) &= v_i(t), \\ m_i \dot{v}_i(t) &= u_i(t) + w_i(t), i = 1, \ldots, n \end{aligned} \tag{2}$$

with initial condition $x_i(t) = x_i(0)$ and $v_i(t) = v_i(0)$, $t \in (-\infty, 0]$, where $x_i(t) \in R^n$ is the position vector of agent i, $m_i > 0$ is its mass, $v_i(t) \in R^n$ is its velocity vector, $u_i(t) \in R^n$ is the control input (or protocol) and $w_i(t) \in R^n$ is the external disturbance input that belongs to $L_2[0, \infty)$ at time t. Without loss of generality, we assume $m_1 = m_2 = \ldots = m_n = 1$. We refer to $\boldsymbol{G} = (\boldsymbol{G}, x, v)$ with $x = [x_1, \ldots, x_n]^{\mathrm{T}}$, and $v = [v_1, \ldots, v_n]^{\mathrm{T}}$ as a network (or algebraic graph) $\boldsymbol{G}$.

Our objective is to find appropriate protocols which make all agents reach agreement with a common speed in the presence of external disturbances. Considering communication time-delay, we use the following control protocol for agent i:

$$\begin{aligned} u_i(t) = {} & k_1 \sum\nolimits_{s_j \in N_i} a_{ij}(t)(x_j(t-\tau) - x_i(t-\tau)) \\ & + k_2 \sum\nolimits_{s_j \in N_i} a_{ij}(t)(v_j(t-\tau) \\ & - v_i(t-\tau)) \end{aligned} \tag{3}$$

where τ denotes the time-delay, and $a_{ij}(t)$ is the weight of each edge.

Given the dynamical system in (2), the protocol u_i is said to asymptotically solve the consensus problem, if the states of agents satisfy

$$\begin{aligned} \lim_{t \to +\infty} ||x_i(t) - x_j(t)|| = 0, \\ \lim_{t \to +\infty} ||v_i(t) - v_j(t)|| = 0 \end{aligned} \tag{4}$$

for all $i, j \in \chi$.

A natural way to combine the relative information is to define output functions $z_i(t)$ computed from an average of the relative displacements of all agents as follows:

let

$$\xi(t) = [x_1(t), v_1(t), x_2(t), v_2(t), \ldots, x_n(t), v_n(t)]^{\mathrm{T}},$$

and

$$z_i(t) = \xi_i(t) - \frac{1}{n}\sum_{i=1}^{n}\xi_i(t), \quad i = 1, \ldots, n$$

then expression (4) will be equivalent to

$$\lim_{t\to+\infty} ||z_i(t)|| = 0, \quad i = 1, \ldots, 2n. \tag{5}$$

Denote

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ \boldsymbol{B}_1 = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}, \ \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and

$$\boldsymbol{C} = \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & -\frac{1}{n} & \cdots \\ -\frac{1}{n} & \frac{n-1}{n} & -\frac{1}{n} & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{n} & \cdots & -\frac{1}{n} & \frac{n-1}{n} \end{bmatrix},$$

and use control protocol (3), the network dynamics will be

$$\begin{aligned} \dot{\xi}(t) &= (I_n \otimes A)\xi(t) - (L_\sigma \otimes B_1)\xi(t-\tau) \\ &\quad +(I_n \otimes B_2)w(t) \\ z(t) &= (C \otimes I_2)\xi(t) \end{aligned} \tag{6}$$

with $w(t) = [w_1(t), \ldots, w_n(t)]$, and $z(t) = [z_1(t), \ldots, z_{2n}(t)]$.

Note that $\boldsymbol{L}_\sigma$ is the Laplacian of the graph $\boldsymbol{G}_\sigma$, σ denotes the switching signal that determines the topology, and $\otimes$ is the Kronecker product.

In this paper, we design the nonzero entries a_{ij} of $\boldsymbol{L}_\sigma$ for the state consensus of the multi-agent systems with a given H_∞ performance index. To do this, we define the following performance index:

$$J(w) = \int_0^\infty [z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t)]\mathrm{d}t < 0, \tag{7}$$

where γ is a given positive constant.

4. Stability analysis of network dynamics

In this section, we provide the convergence and performance analysis of the consensus problem in undirected second-order networks with external disturbances, switching topology and time-delay. Before presenting the main theorems, we first introduce some lemmas.

Lemma 2 Consider the matrix

$$\boldsymbol{M} = \begin{bmatrix} n-1 & -1 & -1 & \ldots \\ -1 & n-1 & -1 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ -1 & \ldots & -1 & n-1 \end{bmatrix}.$$

The following statements hold:[11]

1) The eigenvalues of $\boldsymbol{M}$ are n with a multiplicity of $n-1$, and 0 with a multiplicity of 1. The vectors 1_n^{T} and 1_n are the left and the right eigenvectors of $\boldsymbol{M}$ associated with the zero eigenvalue, respectively.

2) There exists an orthogonal matrix $\boldsymbol{U}$ such that $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{M}\boldsymbol{U} = \begin{bmatrix} nI_{n-1} & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 0 \end{bmatrix}$ and $\boldsymbol{U}$ is the matrix of eigenvectors of $\boldsymbol{M}$. For the Laplacian $\boldsymbol{N} \in R^{n\times n}$ of any undirected graph, $\boldsymbol{U}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{U} = \begin{bmatrix} * & 0_{(n-1)\times 1} \\ 0_{1\times(n-1)} & 0 \end{bmatrix}$.

Lemma 3 For any $x, y \in R^n$ and any positive matrix $\boldsymbol{Q} \in R^{n\times n}$, $2x^{\mathrm{T}}y \le x^{\mathrm{T}}\boldsymbol{Q}x + y^{\mathrm{T}}\boldsymbol{Q}^{-1}y$.

Lemma 4 (Schur Complement)[17] For a given symmetric matrix $\boldsymbol{S}$ with the form $\boldsymbol{S} = [\boldsymbol{S}_{ij}]$, $\boldsymbol{S}_{11} \in R^{r\times r}$, $\boldsymbol{S}_{12} \in R^{r\times(n-r)}$, and $\boldsymbol{S}_{22} \in R^{(n-r)\times(n-r)}$, then, $\boldsymbol{S} < 0$ if and only if either $\boldsymbol{S}_{11} < 0$ and $\boldsymbol{S}_{22} - \boldsymbol{S}_{21}\boldsymbol{S}_{11}^{-1}\boldsymbol{S}_{12} < 0$ or $\boldsymbol{S}_{22} < 0$ and $\boldsymbol{S}_{11} - \boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}_{21} < 0$.

In the following theorems, we are interested in finding design rules for the weight of each edge to guarantee that all agents asymptotically achieve consensus while satisfying the H_∞ performance index (7).

We denote $\boldsymbol{U} = [\,\boldsymbol{U}_1 \ \bar{\boldsymbol{U}}_1\,]$, where $\bar{\boldsymbol{U}}_1 = \frac{1_n}{\sqrt{n}}$ is the last column of $\boldsymbol{U}$ and $\boldsymbol{U}_1$ is the rest part. It is easy to check that $\boldsymbol{U}_1^{\mathrm{T}}\bar{\boldsymbol{U}}_1 = 0$.

Let

$$W_1(t) = \frac{1}{n}\sum_{i=1}^{n}\int_0^{\mathrm{T}}\int_0^{\mathrm{T}} w_i(s)\mathrm{d}s\mathrm{d}s,$$

$$W_2(t) = \frac{1}{n}\sum_{i=1}^{n}\int_0^{\mathrm{T}} w_i(s)\mathrm{d}s,$$

$$\hat{\xi}(t) = \xi(t) - 1_n \otimes [W_1(t)\ W_2(t)]^{\mathrm{T}},$$

$$\delta(t) = (\boldsymbol{U}_1 \otimes \boldsymbol{I}_2)^{\mathrm{T}} \hat{\xi}(t),$$

and

$$\bar{\delta}(t) = (\bar{\boldsymbol{U}}_1 \otimes \boldsymbol{I}_2)^{\mathrm{T}} \hat{\xi}(t),$$

where $W_1(t)$ is an average disturbance of the position vector, and $W_2(t)$ is an average disturbance of the velocity vector of all agents. $\hat{\xi}(t)$ denotes the states of all agents, which takes out the average of external disturbances. $\bar{\delta}(t)$ describes the average state (or agreement state) of all agents, while $\delta(t)$ depicts the disagreement states of all agents.

In view of Lemma 2, we have

$$\begin{aligned} z(t) &= (\boldsymbol{C} \otimes \boldsymbol{I}_2)\xi(t) \\ &= (\boldsymbol{C} \otimes \boldsymbol{I}_2)\hat{\xi}(t) \\ &= (\boldsymbol{C} \otimes \boldsymbol{I}_2)(\boldsymbol{U} \otimes \boldsymbol{I}_2) \begin{bmatrix} \delta(t) \\ \bar{\delta}(t) \end{bmatrix} \\ &= (\boldsymbol{U}_1 \otimes \boldsymbol{I}_2)\delta(t). \end{aligned} \tag{8}$$

We can see that $z(t)$ does not comprise $\bar{\delta}(t)$, which means that the consensus problem for systems (6) is irrespective of the average position and velocity.

In doing the analysis, we discuss the dynamics as follows:

$$\begin{aligned} &(\boldsymbol{U} \otimes \boldsymbol{I}_2)^{\mathrm{T}} \dot{\xi}(t) \\ &= \begin{bmatrix} \dot{\delta}(t) \\ \dot{\bar{\delta}}(t) \end{bmatrix} + (\boldsymbol{U} \otimes \boldsymbol{I}_2)^{\mathrm{T}} \left(1_n \otimes \begin{bmatrix} \dot{W}_1(t) \\ \dot{W}_2(t) \end{bmatrix} \right) \\ &= \begin{bmatrix} \dot{\delta}(t) \\ \dot{\bar{\delta}}(t) \end{bmatrix} + \begin{bmatrix} 0_{2n-2} \\ \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^{n} \int_0^t w_i(s)\mathrm{d}s \\ \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^{n} w_i(t) \end{bmatrix}. \end{aligned} \tag{9}$$

From the assumption, we have $\boldsymbol{L}_\sigma 1_n = 0$, then, from Eq.(6), it is evident that

$$\begin{aligned} (\boldsymbol{U} \otimes \boldsymbol{I}_2)^{\mathrm{T}} \dot{\xi}(t) &= (\boldsymbol{U} \otimes \boldsymbol{I}_2)^{\mathrm{T}} (\boldsymbol{I}_n \otimes \boldsymbol{A}) \left(\hat{\xi}(t) + 1_n \otimes \begin{bmatrix} W_1(t) & W_2(t) \end{bmatrix}^{\mathrm{T}} \right) \\ &\quad -(\boldsymbol{U} \otimes \boldsymbol{I}_2)^{\mathrm{T}} (L_\sigma \otimes \boldsymbol{B}_1)(\hat{\xi}(t-\tau) + 1_n \otimes \begin{bmatrix} W_1(t-\tau) & W_2(t-\tau) \end{bmatrix}^{\mathrm{T}}) + (\boldsymbol{U}^{\mathrm{T}} \otimes \boldsymbol{B}_2) w(t) \\ &= (\boldsymbol{U} \otimes \boldsymbol{I}_2)^{\mathrm{T}} (\boldsymbol{I}_n \otimes \boldsymbol{A})(\boldsymbol{U} \otimes \boldsymbol{I}_2)[\, \delta(t) \ \bar{\delta}(t) \,]^{\mathrm{T}} - (\boldsymbol{U} \otimes \boldsymbol{I}_2)^{\mathrm{T}} (L_\sigma \otimes \boldsymbol{B}_1) \\ &\quad \times (\boldsymbol{U} \otimes \boldsymbol{I}_2)[\, \delta(t-\tau) \ \bar{\delta}(t-\tau) \,]^{\mathrm{T}} + (\boldsymbol{U} \otimes \boldsymbol{I}_2)^{\mathrm{T}} (\boldsymbol{I}_n \otimes \boldsymbol{A})(1_n \otimes [\, W_1(t) \ W_2(t) \,]^{\mathrm{T}}) \\ &\quad +(\boldsymbol{U}^{\mathrm{T}} \otimes \boldsymbol{B}_2) w(t) \\ &= \begin{bmatrix} \boldsymbol{I}_{n-1} \otimes \boldsymbol{A} & 0 \\ 0 & \boldsymbol{A} \end{bmatrix} \begin{bmatrix} \delta(t) \\ \bar{\delta}(t) \end{bmatrix} - \begin{bmatrix} (\boldsymbol{U}_1^{\mathrm{T}} L_\sigma \boldsymbol{U}_1) \otimes \boldsymbol{B}_1 & 0_{2(n-1)\times 2} \\ 0_{2\times 2(n-1)} & 0_{2\times 2} \end{bmatrix} \\ &\quad \times \begin{bmatrix} \delta(t-\tau) \\ \bar{\delta}(t-\tau) \end{bmatrix} + \begin{bmatrix} (\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2) w(t) \\ \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^{n} \int_0^t w_i(s)\mathrm{d}s \\ \dfrac{1}{\sqrt{n}} \displaystyle\sum_{i=1}^{n} w_i(t) \end{bmatrix}. \end{aligned} \tag{10}$$

From Eqs.(9) and (10), it is easy to see that

$$\begin{aligned} \dot{\delta}(t) &= (\boldsymbol{I}_{n-1} \otimes \boldsymbol{A})\delta(t) - (\bar{\boldsymbol{L}}_\sigma \otimes \boldsymbol{B}_1)\delta(t-\tau) \\ &\quad +(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2) w(t), \\ \dot{\bar{\delta}}(t) &= \boldsymbol{A}\bar{\delta}(t) \end{aligned} \tag{11}$$

where $\bar{\boldsymbol{L}}_\sigma = \boldsymbol{U}_1^{\mathrm{T}} \boldsymbol{L}_\sigma \boldsymbol{U}_1$.

We separate the average state $\bar{\delta}(t)$ from the disagreement states $\delta(t)$, and the component $\bar{\delta}(t)$ is completely dependent on the initial condition but not $\delta(t)$. $\bar{\delta}(t)$ and $\delta(t)$ are independent of each other in dynamics, and the external disturbance $w(t)$ affects only the disagreement states $\delta(t)$. At the same time, we can see that $z(t) = (\boldsymbol{U}_1 \otimes \boldsymbol{I}_2)\delta(t)$ from expression (8). Notice that if $\lim_{t\to+\infty} \delta(t) = 0$, then $\lim_{t\to+\infty} z(t) = 0$; that is, consensus is achieved. So we know the stability of systems (6) is irrelevant to the component $\bar{\delta}(t)$. Then, we can only analyse the effects of external disturbances for the disagreement states to improve the consensus performance of the network system. Hence,

we obtain a new second-order model as follows:

$$\begin{aligned}\dot{\delta}(t) &= (\boldsymbol{I}_{n-1}\otimes A)\delta(t)-(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\delta(t-\tau)\\ &\quad+(\boldsymbol{U}_1^{\mathrm{T}}\otimes\boldsymbol{B}_2)w(t)\\ z(t) &= (\boldsymbol{U}_1\otimes\boldsymbol{I}_2)\delta(t)\end{aligned}\tag{12}$$

And system (12) is equivalent to system (6). In this case, we can address the robust H_∞ problem for the second-order network system (12) instead of system (6).

First, by using the Lyapunov-based approach, we consider the consensus problem in networks of second-order agents in the absence of disturbances.

Theorem 1 Consider an undirected network of agents with switching topology and constant time-delay τ. Assume that the communication topology $\boldsymbol{G}_\sigma$ is kept connected. System (12) is globally asymptotically stable with $w(t)\equiv 0$, i.e. consensus can be achieved in the absence of disturbances, if there exist positive definite matrices $\boldsymbol{P}$, $\boldsymbol{Q}$, $\boldsymbol{R}\in\boldsymbol{R}^{2(n-1)\times 2(n-1)}$ satisfying

$$\boldsymbol{H}=\begin{bmatrix}\boldsymbol{H}_{11} & \boldsymbol{H}_{12} & \boldsymbol{H}_{13}\\ \boldsymbol{H}_{12}^{\mathrm{T}} & \boldsymbol{H}_{22} & 0_{2(n-1)\times 2(n-1)}\\ \boldsymbol{H}_{13}^{\mathrm{T}} & 0_{2(n-1)\times 2(n-1)} & -\dfrac{\boldsymbol{R}}{\tau}\end{bmatrix}<0,\tag{13}$$

where

$$\begin{aligned}\boldsymbol{H}_{11} &= \boldsymbol{P}(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})+(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})^{\mathrm{T}}\boldsymbol{P}\\ &\quad-\boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)-(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{P}\\ &\quad+\tau(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})+\boldsymbol{Q},\\ \boldsymbol{H}_{12} &= -\tau(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})^{\mathrm{T}}\boldsymbol{R}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1),\\ \boldsymbol{H}_{13} &= \boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1),\end{aligned}$$

and

$$\boldsymbol{H}_{22}=-\boldsymbol{Q}+\tau(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{R}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1).$$

Proof We define a common Lyapunov–Krasovskii function for system (12) as follows:

$$\begin{aligned}V &= \delta^{\mathrm{T}}(t)\boldsymbol{P}\delta(t)+\int_{t-\tau}^{t}\delta^{\mathrm{T}}(s)\boldsymbol{Q}\delta(s)\mathrm{d}s\\ &\quad+\int_{-\tau}^{0}\int_{t+\theta}^{t}\dot{\delta}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\delta}(s)\mathrm{d}s\mathrm{d}\theta,\end{aligned}\tag{14}$$

where $\boldsymbol{P},\boldsymbol{Q},\boldsymbol{R}\in\boldsymbol{R}^{2(n-1)\times 2(n-1)}$ are symmetric positive definite matrices.

Calculating $\dot{V}$ along the solution of the system (12), it follows that

$$\begin{aligned}\dot{V} &= 2\delta^{\mathrm{T}}(t)\boldsymbol{P}\dot{\delta}(t)+\delta^{\mathrm{T}}(t)\boldsymbol{Q}\delta(t)-\delta^{\mathrm{T}}(t-\tau)\boldsymbol{Q}\delta(t-\tau)\\ &\quad+\tau\dot{\delta}^{\mathrm{T}}(t)\boldsymbol{R}\dot{\delta}(t)-\int_{t-\tau}^{t}\dot{\delta}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\delta}(s)\mathrm{d}s.\end{aligned}\tag{15}$$

From the dynamics of system (12), we have

$$\begin{aligned}2\delta^{\mathrm{T}}(t)\boldsymbol{P}\dot{\delta}(t) &= 2\delta^{\mathrm{T}}(t)\boldsymbol{P}(I_{n-1}\otimes\boldsymbol{A})\delta(t)\\ &\quad-2\delta^{\mathrm{T}}(t)\boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\delta(t-\tau)\\ &\quad+2\delta^{\mathrm{T}}(t)\boldsymbol{P}(U_1^{\mathrm{T}}\otimes\boldsymbol{B}_2)w(t).\end{aligned}\tag{16}$$

Owing to the fact that $\delta(t-\tau)=\delta(t)-\int_{t-\tau}^{t}\dot{\delta}(s)\mathrm{d}s$, and from lemma 3, we obtain

$$\begin{aligned}&-2\delta^{\mathrm{T}}(t)\boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\delta(t-\tau)\\ &=-2\delta^{\mathrm{T}}(t)\boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\delta(t)\\ &\quad+\int_{t-\tau}^{t}2((\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{P}\delta(t))^{\mathrm{T}}\dot{\delta}(s)\mathrm{d}s\\ &\le-2\delta^{\mathrm{T}}(t)\boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\delta(t)\\ &\quad+\tau\delta^{\mathrm{T}}(t)\boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\boldsymbol{R}^{-1}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)^{\mathrm{T}}\\ &\quad\times\boldsymbol{P}\delta(t)+\int_{t-\tau}^{t}\dot{\delta}^{\mathrm{T}}(s)\boldsymbol{R}\dot{\delta}(s)\mathrm{d}s.\end{aligned}\tag{17}$$

And we find that

$$\begin{aligned}&\dot{\delta^{\mathrm{T}}}(t)\boldsymbol{R}\dot{\delta}(t)\\ &=\delta^{\mathrm{T}}(t)(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})\delta(t)\\ &\quad+\delta^{\mathrm{T}}(t-\tau)(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{R}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\delta(t-\tau)\\ &\quad+w^{\mathrm{T}}(t)(\boldsymbol{U}_1^{\mathrm{T}}\otimes\boldsymbol{B}_2)^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}}\otimes\boldsymbol{B}_2)w(t)\\ &\quad-2\delta^{\mathrm{T}}(t)(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})^{\mathrm{T}}\boldsymbol{R}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\delta(t-\tau)\\ &\quad-2\delta^{\mathrm{T}}(t-\tau)(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}}\otimes\boldsymbol{B}_2)w(t)\\ &\quad+2\delta^{\mathrm{T}}(t)(\boldsymbol{I}_{n-1}\otimes\boldsymbol{A})^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}}\otimes\boldsymbol{B}_2)w(t).\end{aligned}\tag{18}$$

Now, we consider the system with $w(t)\equiv 0$, and system (12) is exponentially stable. Consequently, we have that

$$\begin{bmatrix}\delta(t)\\ \delta(t-\tau)\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\tilde{\boldsymbol{H}}_{11} & \tilde{\boldsymbol{H}}_{12}\\ \tilde{\boldsymbol{H}}_{12}^{\mathrm{T}} & \tilde{\boldsymbol{H}}_{22}\end{bmatrix}\begin{bmatrix}\delta(t)\\ \delta(t-\tau)\end{bmatrix}<0,\tag{19}$$

where

$$\begin{aligned}\tilde{\boldsymbol{H}}_{11} &= \boldsymbol{P}(I_{n-1}\otimes\boldsymbol{A})+(I_{n-1}\otimes\boldsymbol{A})^{\mathrm{T}}\boldsymbol{P}-\boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\\ &\quad-(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{P}+\tau(I_{n-1}\otimes\boldsymbol{A})^{\mathrm{T}}\boldsymbol{R}(I_{n-1}\otimes\boldsymbol{A})\\ &\quad+\boldsymbol{Q}+\tau\boldsymbol{P}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)\boldsymbol{R}^{-1}(\bar{\boldsymbol{L}}_\sigma\otimes\boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{P},\end{aligned}$$

and

$$\tilde{\boldsymbol{H}}_{12}=\boldsymbol{H}_{12},\quad\tilde{\boldsymbol{H}}_{22}=\boldsymbol{H}_{22}.$$

Then, a sufficient condition for $\dot{V}<0$ holds if and only if $\tilde{\boldsymbol{H}}<0$. On the second thoughts, from Schur Complement Formula 4, $\tilde{\boldsymbol{H}}<0$ holds if and only if $\boldsymbol{H}<0$. This completes the proof.

In Ref.[6], Lin *et al* tried to analyse the states all together for the stability of the second-order agent systems without disturbances, and the analysis was complex. In this paper, we first perform a model transformation and separate the agreement states from the disagreement states. Then, we study the states related to the consensus problem instead of all the states, so our theoretical results are much more comprehensible and obvious.

Next, according to the results obtained in Theorem 1, we discuss the performance of the second-order network systems with disturbance $w(t)$. The following theorem provides a sufficient condition in terms of bilinear matrix inequality, under which all agents asymptotically achieve consensus with the desired H_∞ performance.

Theorem 2 Assume that the switching topology $\boldsymbol{G}_\sigma$ is kept connected. For system (12) with $w(t)$, consensus can be achieved with $||T_{wz}||_\infty < \gamma$, if there exist positive matrices $\boldsymbol{P}$,$\boldsymbol{Q}$, $\boldsymbol{R} \in R^{2(n-1)\times 2(n-1)}$ satisfying

$$\phi = \begin{bmatrix} \boldsymbol{\varphi}_{11} + (\boldsymbol{U}_1 \otimes \boldsymbol{I}_2)^{\mathrm{T}}(\boldsymbol{U}_1 \otimes \boldsymbol{I}_2) & \boldsymbol{\varphi}_{12} & \boldsymbol{\varphi}_{13} & \boldsymbol{\varphi}_{14} \\ \boldsymbol{\varphi}_{12}^{\mathrm{T}} & \boldsymbol{\varphi}_{22} & \boldsymbol{\varphi}_{23} & 0_{2(n-1)\times 2(n-1)} \\ \boldsymbol{\varphi}_{13}^{\mathrm{T}} & \boldsymbol{\varphi}_{23}^{\mathrm{T}} & \boldsymbol{\varphi}_{33} & 0_{n\times 2(n-1)} \\ \boldsymbol{\varphi}_{14}^{\mathrm{T}} & 0_{2(n-1)\times 2(n-1)} & 0_{2(n-1)\times n} & -\frac{\boldsymbol{R}}{\tau} \end{bmatrix} < 0, \tag{20}$$

where

$$\begin{aligned} \boldsymbol{\varphi}_{13} &= \boldsymbol{P}(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2) + \tau(\boldsymbol{I}_{n-1} \otimes \boldsymbol{A})^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2), \\ \boldsymbol{\varphi}_{23} &= -\tau(\bar{\boldsymbol{L}}_\sigma \otimes \boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2), \\ \boldsymbol{\varphi}_{33} &= -\gamma^2\boldsymbol{I} + \tau(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2)^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2), \\ \boldsymbol{\varphi}_{11} &= \boldsymbol{H}_{11}, \boldsymbol{\varphi}_{12} = \boldsymbol{H}_{12}, \boldsymbol{\varphi}_{14} = \boldsymbol{H}_{13}, \boldsymbol{\varphi}_{22} = \boldsymbol{H}_{22}, \end{aligned}$$

$\boldsymbol{H}_{11}$, $\boldsymbol{H}_{12}$, $\boldsymbol{H}_{13}$, $\boldsymbol{H}_{22}$ are as defined in Theorem 1.

Proof Similar to the proof of Theorem 1, calculating $\dot{V}$, we have

$$\begin{aligned} \dot{V}(t) \le\ & \delta^{\mathrm{T}}(t)\tilde{\boldsymbol{H}}_{11}\delta(t) + 2\delta^{\mathrm{T}}(t)\tilde{\boldsymbol{H}}_{12}\delta(t-\tau) \\ & +\delta^{\mathrm{T}}(t-\tau)\tilde{\boldsymbol{H}}_{22}\delta(t-\tau) \\ & +2\delta^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2)w(t) \\ & +2\tau\delta^{\mathrm{T}}(t)(I_{n-1} \otimes A)^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2)w(t) \\ & -2\tau\delta^{\mathrm{T}}(t-\tau)(\bar{L}_\sigma \otimes \boldsymbol{B}_1)^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2)w(t) \\ & +\tau w^{\mathrm{T}}(t)(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2)^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{U}_1^{\mathrm{T}} \otimes \boldsymbol{B}_2)w(t) \end{aligned}$$

Consider the following cost performance for $T > 0$, $J_T = \int_0^{\mathrm{T}} [z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t)]\mathrm{d}t$ for nonzero $w(t) \in \boldsymbol{L}_2[0,\infty), t \in [0,\infty)$, and zero-valued initial condition $\delta(t) = 0, t \in (-\infty, 0]$. Because $\delta(t) = 0, t \in (-\infty, 0]$, we can obtain $V(0) = 0$. And

$$\begin{aligned} J_{\mathrm{T}} &= \int_0^{\mathrm{T}} [z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t)]\mathrm{d}t \\ &= \int_0^{\mathrm{T}} [z^{\mathrm{T}}(t)z(t) - \gamma^2 w^{\mathrm{T}}(t)w(t) \\ &\quad +\dot{V}(t)]\mathrm{d}t - (V(T) - V(0)) \\ &\le \int_0^{\mathrm{T}} \eta^{\mathrm{T}}(t)\boldsymbol{M}\eta(t)\mathrm{d}t - V(T), \end{aligned} \tag{21}$$

where

$$\boldsymbol{M} = \begin{bmatrix} \tilde{\boldsymbol{H}}_{11} + (\boldsymbol{U}_1 \otimes \boldsymbol{I}_2)^{\mathrm{T}}(\boldsymbol{U}_1 \otimes \boldsymbol{I}_2) & \tilde{\boldsymbol{H}}_{12} & \tilde{\boldsymbol{H}}_{13} \\ \tilde{\boldsymbol{H}}_{12}^{\mathrm{T}} & \tilde{\boldsymbol{H}}_{22} & \tilde{\boldsymbol{H}}_{23} \\ \tilde{\boldsymbol{H}}_{13}^{\mathrm{T}} & \tilde{\boldsymbol{H}}_{23}^{\mathrm{T}} & \tilde{\boldsymbol{H}}_{33} \end{bmatrix},$$

$$\tilde{\boldsymbol{H}}_{13} = \varphi_{13}, \tilde{\boldsymbol{H}}_{23} = \varphi_{23}, \tilde{\boldsymbol{H}}_{33} = \varphi_{33},$$

$$\eta(t) = \begin{bmatrix} \delta(t) & \delta(t-\tau) & w(t) \end{bmatrix}^{\mathrm{T}},$$

$\tilde{\boldsymbol{H}}_{11}$, $\tilde{\boldsymbol{H}}_{12}$, $\tilde{\boldsymbol{H}}_{22}$ are defined as in Theorem 1.

From Lemma 4, $\boldsymbol{M} < 0$ is equivalent to $\phi < 0$. Furthermore, we know $V(T) > 0$. So, if $\phi < 0$ holds, then $J(T) < 0$, i.e. $\int_0^{\mathrm{T}} z^{\mathrm{T}}(t)z(t)\mathrm{d}t < \gamma^2 \int_0^{\mathrm{T}} w^{\mathrm{T}}(t)w(t)\mathrm{d}t$. Let $T \to +\infty$, then we will have $||z(t)||_2^2 < \gamma^2||w(t)||_2^2$, i.e. $||T_{wz}(s)||_\infty < \gamma$. Thus, this completes the proof.

Next, we investigate whether all agents converge to a limited value in the presence of disturbance $w(t)$. From Eq.(11), it is easy to see that the external disturbance $w(t)$ does not affect the agreement state $\bar{\delta}(t)$. Denote $\bar{\delta}(t) = \begin{bmatrix} \bar{\delta}_1(t) & \bar{\delta}_2(t) \end{bmatrix}^{\mathrm{T}}$, we have

$$\begin{aligned} \dot{\bar{\delta}}_1(t) &= \bar{\delta}_1(0) + \bar{\delta}_2(0)t, \\ \dot{\bar{\delta}}_2(t) &= \bar{\delta}_2(0). \end{aligned} \tag{22}$$

As we have defined,

$$\hat{\xi}(t) = \xi(t) - 1_n \otimes \left[W_1(t)\ W_2(t) \right]^{\mathrm{T}},$$

$\delta(t) = (\bar{\boldsymbol{U}}_1 \otimes \boldsymbol{I}_2)^{\mathrm{T}} \hat{\xi}(t)$, and $\bar{\boldsymbol{U}}_1 = \dfrac{1_n}{\sqrt{n}}$. Then, we have

$$\begin{aligned} \frac{\sum_i x_i(t)}{n} &= \frac{\bar{\delta}_1(t)}{\sqrt{n}} + \frac{1}{n}\sum_{i=1}^{n}\int_0^t \int_0^t w_i(s)\mathrm{d}s\mathrm{d}s, \\ \frac{\sum_i v_i(t)}{n} &= \frac{\bar{\delta}_2(t)}{\sqrt{n}} + \frac{1}{n}\sum_{i=1}^{n}\int_0^t w_i(s)\mathrm{d}s. \end{aligned} \tag{23}$$

Observing that $W_1(0) = 0$, and $W_2(0) = 0$, then $\bar{\delta}_1(0) = \dfrac{\sum_i x_i(0)}{n}$, and $\bar{\delta}_2(0) = \dfrac{\sum_i v_i(0)}{n}$. Notice that the component $\bar{\delta}(t)$ describes the average state of all agents at time t in the new system after the model transformation, and the component $\bar{\delta}(t)$ is independent of disturbance. Thus, the agreement state $\bar{\delta}(t)$ is related only to the initial condition $\bar{\delta}(0)$. However, the real average state of all agents in the original system is the state $\left[\dfrac{\sum_i x_i(t)}{n}\ \dfrac{\sum_i v_i(t)}{n} \right]^{\mathrm{T}}$. From expression (23), we can see that whether all agents converge to a limited value is related to external disturbance, which fits well to the engineering practice.

5. Simulation results

Numerical simulations will be given to illustrate the theoretical results obtained in the previous sections. Figure 1 shows three different graphs each with $n = 4$ nodes. Moreover, all graphs in this figure are undirected and connected. In Fig.2, a finite state machine is shown with three states $\{\boldsymbol{G}_{\mathrm{a}}, \boldsymbol{G}_{\mathrm{b}}, \boldsymbol{G}_{\mathrm{c}}\}$ which represent a second-agent network with switching topology and time-delay as a hybrid system. The hybrid system starts at the state $\boldsymbol{G}_{\mathrm{a}}$, and switches every 0.01 s to the next state according to the state machine as shown in Fig.2. Suppose that all initial conditions are set to be zero.

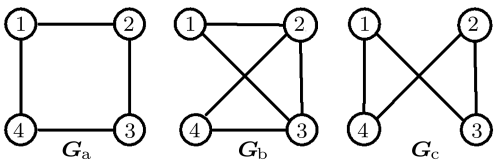


Fig.1. Three examples of connected undirected graphs.

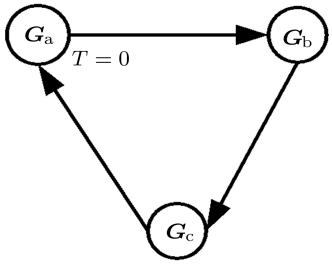


Fig.2. Finite automation with three states representing the states of a second-agent network with switching topology.

Now, we present simulation results for the consensus problems of second-order multi-agent systems in the presence of external disturbances in undirected graphs with switching topology and time- delay, and these simulations are performed with three states $\{\boldsymbol{G}_{\mathrm{a}}, \boldsymbol{G}_{\mathrm{b}}, \boldsymbol{G}_{\mathrm{c}}\}$. By choosing the performance index $\gamma = 1$ and applying Theorem 2, it is obtained that for the communication time-delay $\tau = 0.1s$ and switching topology, a feasible solution is $\boldsymbol{P} = \boldsymbol{Q} = \boldsymbol{R} = \boldsymbol{I}$ and the weights of the edges (1,2), (2,3), (3,4), (1,4), (1,3) and (2,4) are all 0.6. So we get the Laplacians of the three graphs in Figure 1, which are respectively

$$\boldsymbol{L}_{\mathrm{a}} = \begin{bmatrix} 1.2 & -0.6 & 0 & -0.6 \\ -0.6 & 1.2 & -0.6 & 0 \\ 0 & -0.6 & 1.2 & -0.6 \\ -0.6 & 0 & -0.6 & 1.2 \end{bmatrix},$$

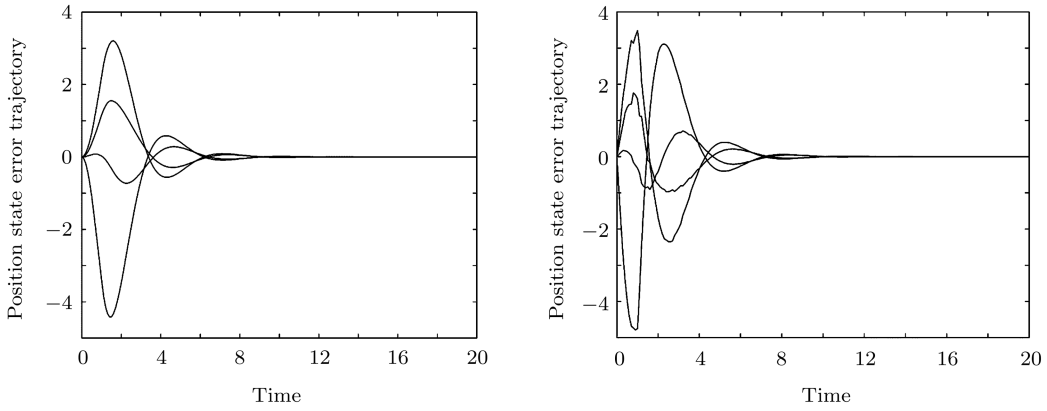
$$\boldsymbol{L}_{\mathrm{b}} = \begin{bmatrix} 1.2 & -0.6 & -0.6 & 0 \\ -0.6 & 1.2 & 0 & -0.6 \\ -0.6 & 0 & 1.2 & -0.6 \\ 0 & -0.6 & -0.6 & 1.2 \end{bmatrix},$$

$$\boldsymbol{L}_{\mathrm{c}} = \begin{bmatrix} 1.2 & 0 & -0.6 & -0.6 \\ 0 & 1.2 & -0.6 & -0.6 \\ -0.6 & -0.6 & 1.2 & 0 \\ -0.6 & -0.6 & 0 & 1.2 \end{bmatrix}.$$

Figure 3 shows the state error trajectories with disturbance $w(t) = \left[1\ 1.5\ 2\ -1 \right] \bar{\omega}(t)$, while Fig.4 shows the corresponding energy trajectories for the second-order multi-agent systems, where

$$\bar{\omega}(t) = \begin{cases} 1, & 0 \le t \le 1; \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, from Fig.3, consensus is asymptotically achieved, and from Fig.4 it follows that the energy of the output signal $z(t)$ is smaller than that of the external disturbance $w(t)$. In other words, consensus is achieved with the performance $||T_{wz}(s)||_\infty < \gamma$.



(a) The trajectories of position error (b) The trajectories of velocity error

Fig.3. State trajectories of the second-order agent network with switching topology and constant time-delay.

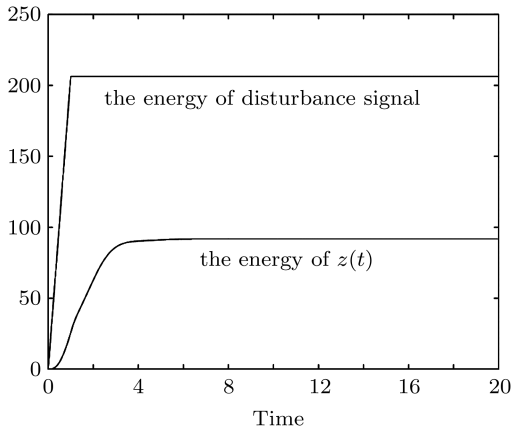


Fig.4. The energy trajectories of $z(t)$.

6. Conclusions

In this paper, we investigate the consensus problems in undirected networks of second-order agents with switching topology and time-delay in the presence of external disturbances. The convergence analysis is carried out based on the Lyapunov Theory. Moreover, a stability criterion in terms of bilinear matrix inequalities is established under which all agents reach consensus while satisfying a desired H_∞ performance. Finally, we demonstrate its practicability through computer simulations.

References

[1] Vicsek T, Cziroo′k A, Ben-Jacob E, Cohen O and Shochet I 1995 *Phys. Rev. Lett.* **75** 1226
[2] Jadbabaie A, Lin J and Morse A S 2003 *IEEE Trans. Autom. Control* **48** 988
[3] Tanner H, Jadbabaie A and Pappas G 2003 *Proceedings of the 42nd IEEE Conference on Decision and Control* P2016
[4] Ren W and Beard R W 2005 *IEEE Trans. Automat. Control* **50** 655
[5] Olfati–Saber R 2006 *IEEE Trans. Automat. Control* **51** 410
[6] Lin P, Jia Y, Du J and Yuan S 2008 *Asian J. Control* **10** 254
[7] Moreau L 2005 *IEEE Trans. Automat. Control* **50** 169
[8] Hu J and Hong Y 2007 *Physica* A **374** 853
[9] Li Z 2008 *Physica* A **387** 1369
[10] Lin P and Jia Y 2008 *Physica* A **387** 303
[11] Lin P and Jia Y 2008 *Syst. Control Lett.* **57** 643
[12] Olfati–Saber R and Murray R M 2004 *IEEE Trans. Autom. Control* **49** 1520
[13] Chen F, Chen Z Q and Yuan Z Z 2007 *Chin. Phys.* **16** 287
[14] Pei W D, Chen Z Q and Yuan Z Z 2008 *Chin. Phys.* B **17** 373
[15] Chen D L and Zhang W D 2008 *Chin. Phys.* B **17** 1506
[16] Liu T L and Huang H J 2007 *Acta Phys. Sin.* **56** 6321 (in Chinese)
[17] Jia Y 2007 Robust H_∞ *Control* (China: Science Press) p145