Vol 17 No 12, December 2008
1674-1056/2008/17(12)/4407-11

Chinese Physics B

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Exponential synchronization of stochastic impulsive perturbed chaotic Lur'e systems with time-varying delay and parametric uncertainty*

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(Received 2 June 2008; revised manuscript received 7 July 2008)

This paper is devoted to investigating the scheme of exponential synchronization for uncertain stochastic impulsive perturbed chaotic Lur'e systems. The parametric uncertainty is assumed to be norm bounded. Based on the Lyapunov function method, time-varying delay feedback control technique and a modified Halanay inequality for stochastic differential equations, several sufficient conditions are presented to guarantee the exponential synchronization in mean square between two identical uncertain chaotic Lur'e systems with stochastic and impulsive perturbations. These conditions are expressed in terms of linear matrix inequalities (LMIs), which can easily be checked by utilizing the numerically efficient Matlab LMI toolbox. It is worth pointing out that the approach developed in this paper can provide a more general framework for the synchronization of multi–perturbation chaotic Lur'e systems, which reflects a more realistic dynamics. Finally, a numerical example is provided to demonstrate the effectiveness of the proposed method.

Keywords: chaotic Lur'e systems, exponential synchronization, time-varying delay, parametric uncertainty

PACC: 0540, 0545

1. Introduction

During the last two decades, synchronization of the chaotic systems has attracted considerable attention due to its great potential applications in secure communication, chemical reactions and biological systems.[1] The first idea of synchronization of two chaotic systems with different initial conditions was introduced in Refs.[2, 3]. Many different methods are applied theoretically and experimentally to synchronizing the chaotic systems, such as linear and nonlinear feedback control,[4−6] backstepping control,[7,8] variable structure control,[9,10] adaptive control,[11−13] impulsive control,[14−17] fuzzy model–based control,[18−20] active control,[21] etc. As is well known there are some nonlinear systems, such as Chua's circuit, n-scroll attractors and hyperchaotic attractors, which can be represented as Lur'e systems. Therefore, master–slave synchronization for Lur'e systems has been well studied in recent years.[6,22−25]

Owing to the propagation delay that is frequently encountered in remote master-slave synchronization schemes and often results in instabilities, recently, there have been made some research efforts to investigate the delay effect on master-slave synchronization.[22−25] In order to handle the case where the master-slave systems cannot be synchronized by a pure time-delay static error output feedback control, in Refs.[6, 22, 23] considered is the feedback control including both the current error state feedback and the delayed static error output feedback. Moreover, for some special applications, the parameters of some systems cannot be exactly known beforehand, and the effect of these uncertainties will destroy the synchronization and even break it. In the past few years, to deal with the difficulties brought by parametric uncertainty, the problem of master-slave synchronization for all kinds of uncertain chaotic systems has gained much research attention.[6,9,14,18]

When chaos synchronization is used for engineering application, such as secure communication, these chaos–based synchronization schemes are rather sensitive to noise and disturbance in channels, which

*Project supported by the National Natural Science Foundation of China (Grant Nos 60534010, 60572070, 60774048 and 60728307), the Program for Changjiang Scholars and Innovative Research Team in University (Grant No 60521003) and the National High Technology Development Program of China (Grant No 2006AA04Z183).

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can destroy many good properties of the synchronized systems. Channel noise may be presented in many forms among which the impulsive noise is a common one.[25−27] Moreover, besides impulsive perturbation, a real system is usually affected by external perturbations which in many cases are of great uncertainties and hence may be treated as random, fluctuations from the release of neurotransmitters and other probabilistic causes. By virtue of the stability theory for stochastic differential equations, there has been an increasing interest in the study of synchronization of chaotic systems with stochastic perturbations.[28−30] In the studies mentioned above considered is only one kind of perturbation, i.e. impulsive or stochastic one. However, in a real system, it is more reasonable that the two perturbations coexist simultaneously. Therefore, the synchronization scheme between chaotic systems with impulsive and stochastic perturbations is significant from theoretical and practical points of view. To the best of our knowledge, so far, the problem of exponential synchronization of chaotic systems with impulsive and stochastic perturbations remains a challenging open problem.

Motivated by the aforementioned discussion, in this paper we address the scheme of exponential synchronization for uncertain Lur'e systems with impulsive and stochastic perturbations via time-varying delay feedback control. By virtue of the modified Halanay inequality for stochastic differential equations and mathematical induction, a time-varying delay feedback controller is designed to ensure the exponential synchronization of uncertain Lur'e systems with impulsive and stochastic perturbations. The derived criteria are robust for all admissible uncertainties and expressed in terms of linear matrix inequalities (LMIs), which can be solved numerically by using the methods in Ref.[31].

Throughout the paper, R and R^n denote the real number and the n–dimensional Euclidean space, respectively. $R^{n\times m}$ is the set of all real matrices. $R^+ \triangleq [0,\infty)$. $N \triangleq \{1,2,\ldots\}$. For real symmetric matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, the notation $\boldsymbol{X} > \boldsymbol{Y}$ ($\boldsymbol{X} \geq \boldsymbol{Y}$, $\boldsymbol{X} < \boldsymbol{Y}$, $\boldsymbol{X} \leq \boldsymbol{Y}$) means that $\boldsymbol{X} - \boldsymbol{Y}$ is positive definite (positive semi-definite, negative definite, negative semi-definite). $\boldsymbol{I}$ denotes the identity matrix with appropriate dimensions. $\|\cdot\|$ refers to Euclidean vector norm or the induced matrix 2-norm. $\lambda_{\max}(\boldsymbol{S})$ ($\lambda_{\min}(\boldsymbol{S})$) denotes the maximal (minimal) eigenvalue of matrix $\boldsymbol{S}$. $C(T_1,T_2)$ denotes the space of continuous mappings from the topological space T_1 to the topological space T_2. For $\tau > 0$, $\mathscr{C}([t_0-\tau,t_0];R^n)$ denotes the family of continuous function φ from $[t_0-\tau,t_0]$ to R^n. Let $(\Omega,\ \mathscr{F},\ \mathscr{F}_{t\geq 0},\ \mathscr{P})$ be a complete probability space with a filtration $\mathscr{F}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathscr{F}_0$ contains all $\mathscr{P}$-null sets); $L^p_{\mathscr{F}_0}([t_0-\tau,t_0];R^n)$ denotes the family of all $\mathscr{F}_0$-measurable $\mathscr{C}([t_0-\tau,t_0];R^n)$-valued random variables $\phi = \phi(s) : t_0-\tau \leq s \leq t_0$ such that $\sup_{t_0-\tau\leq s\leq t_0} E\left(\|\phi(s)\|^p\right) < \infty$, where $p > 0$ and $E\{\cdot\}$ represents the mathematical expectation operator with respect to the given probability measure $\mathscr{P}$; $\mathscr{C}^{1,2}(R^+\times R^n;R^+)$ denotes the family of all nonnegative functions $V(t,\boldsymbol{x})$ on $R^+\times R^n$ which are continuously twice differentiable with respect to $\boldsymbol{x}$ and differentiable with respect to t. The symbol "$*$" within a matrix represents the symmetric term of the matrix.

2. Problem formulation and some preliminaries

Consider the following master-slave synchronization scheme with time-varying delay feedback for an uncertain Lur'e system:

$$
\begin{aligned}
M_{\rm u}:\ &\begin{cases}\dot{\boldsymbol{x}}(t) = (\boldsymbol{A}+\Delta\boldsymbol{A}(t))\boldsymbol{x}(t)\\ \qquad\quad +(\boldsymbol{B}+\Delta\boldsymbol{B}(t))\sigma(\boldsymbol{C}\boldsymbol{x}(t)),\\ \boldsymbol{p}(t) = \boldsymbol{H}\boldsymbol{x}(t),\end{cases}\\
S_{\rm u}:\ &\begin{cases}\dot{\boldsymbol{y}}(t) = (\boldsymbol{A}+\Delta\boldsymbol{A}(t))\boldsymbol{y}(t)\\ \qquad\quad +(\boldsymbol{B}+\Delta\boldsymbol{B}(t))\sigma(\boldsymbol{C}\boldsymbol{y}(t))+\boldsymbol{u}(t),\\ \boldsymbol{q}(t) = \boldsymbol{H}\boldsymbol{y}(t),\end{cases}\\
C_{\rm u}:\ &\boldsymbol{u}(t) = -\boldsymbol{K}(\boldsymbol{x}(t)-\boldsymbol{y}(t)) + \boldsymbol{M}(\boldsymbol{p}_\tau(t)-q_\tau(t)),
\end{aligned}
\tag{1}
$$

with master system $M_{\rm u}$, slave system $S_{\rm u}$ and time-varying delay feedback controller $C_{\rm u}$, which is disturbed by stochastic and impulsive perturbations (see Fig.1). The subscript u refers to the uncertain case of Lur'e system. State vectors are $\boldsymbol{x}(t)$, $\boldsymbol{y}(t) \in R^n$, outputs of subsystems $\boldsymbol{p}(t)$ and $\boldsymbol{q}(t) \in R^l$, $\boldsymbol{p}_\tau(t) \triangleq \boldsymbol{p}(t-\tau(t))$, $q_\tau(t) \triangleq q(t-\tau(t))$, $\boldsymbol{H} \in R^{l\times n}$, $\boldsymbol{A},\Delta\boldsymbol{A}(t) \in R^{n\times n}$, $\boldsymbol{B},\Delta\boldsymbol{B}(t) \in R^{n\times n_{\rm h}}$ and $\boldsymbol{C} \in R^{n_{\rm h}\times n}$. The diagonal nonlinearity $\sigma \in C^1(R^{n_{\rm h}},R^{n_{\rm h}})$ satisfies a sector condition[6] with $\sigma_i(\cdot)$ $(i=1,2,\ldots,n_h)$ belonging to sector $[0,\kappa]$, i.e. $\sigma_i(\zeta)(\sigma_i(\zeta)-\kappa\zeta) \leq 0$, $\forall\zeta \in R$ for $i = 1,2,\ldots,n_{\rm h}$, $\boldsymbol{u}(t) \in R^n$, $\boldsymbol{K} \in R^{n\times n}$, and $\boldsymbol{M} \in R^{n\times l}$.

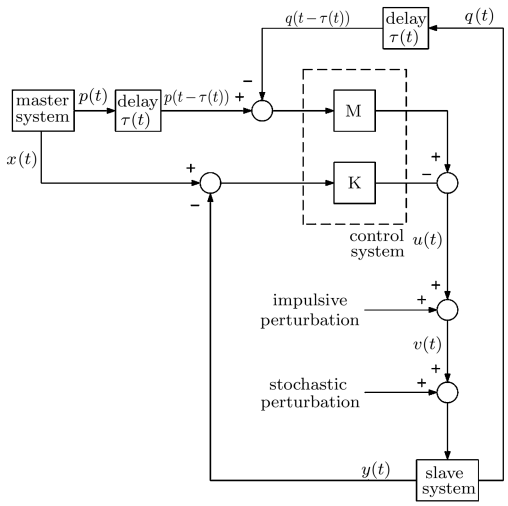


Fig.1. Master–slave synchronization framework with stochastic and impulsive perturbations.

The controller C_{u} is assumed to be disturbed by an impulsive noise sequence with constant coefficients, which can be described by

$$\sum_{k=1}^{\infty}\delta(t-t_k)\boldsymbol{G}_k\boldsymbol{u}(t), \quad \boldsymbol{G}_k\in R^{n\times n},$$
$$\|\boldsymbol{G}_k\|\le l_k, \quad k\in N,$$

where the fixed impulsive moments t_k satisfy $t_{k-1}<t_k$ and $\lim_{k\to\infty}t_k=\infty$, $\delta(t)$ is the Dirac function, thus, the real control signal transmitted to the slave system at time t is

$$\boldsymbol{v}(t)=\boldsymbol{u}(t)+\sum_{k=1}^{\infty}\delta(t-t_k)\boldsymbol{G}_k\boldsymbol{u}(t). \tag{2}$$

In addition to stochastic perturbation, thus, we rewrite the master-slave synchronization scheme (1) as follows:

$$M'_{\mathrm{u}}:\begin{cases}\mathrm{d}\boldsymbol{x}(t)=[(\boldsymbol{A}+\Delta\boldsymbol{A}(t))\boldsymbol{x}(t)\\ \qquad\qquad +(\boldsymbol{B}+\Delta\boldsymbol{B}(t))\sigma(\boldsymbol{C}\boldsymbol{x}(t))]\mathrm{d}(t),\\ \boldsymbol{p}(t)=\boldsymbol{H}\boldsymbol{x}(t),\end{cases}$$

$$S'_{\mathrm{u}}:\begin{cases}\mathrm{d}\boldsymbol{y}(t)=[(\boldsymbol{A}+\Delta\boldsymbol{A}(t))\boldsymbol{y}(t)\\ \qquad\qquad +(\boldsymbol{B}+\Delta\boldsymbol{B}(t))\sigma(\boldsymbol{C}\boldsymbol{y}(t))+\boldsymbol{v}(t)]\mathrm{d}(t)\\ \qquad\qquad +\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\mathrm{d}\omega(t),\\ \boldsymbol{q}(t)=\boldsymbol{H}\boldsymbol{y}(t),\end{cases}$$

and

$$\begin{aligned}C'_{\mathrm{u}}:\boldsymbol{v}(t)=&-\boldsymbol{K}(\boldsymbol{x}(t)-\boldsymbol{y}(t))+\boldsymbol{M}(\boldsymbol{p}_\tau(t)-q_\tau(t))\\ &+\sum_{k=1}^{\infty}\delta(t-t_k)\boldsymbol{G}_k\boldsymbol{u}(t),\end{aligned}\tag{3}$$

where $\boldsymbol{e}(t)=\boldsymbol{x}(t)-\boldsymbol{y}(t)=[x_1(t)-y_1(t),x_2(t)-y_2(t),\ldots,x_n(t)-y_n(t)]^{\mathrm{T}}$ is the error vector, $\boldsymbol{e}_\tau(t)\triangleq\boldsymbol{e}(t-\tau(t))$, $\omega(t)=[\omega_1(t),\omega_2(t),\ldots,\omega_n(t)]^{\mathrm{T}}$ is an n–dimensional Brown motion defined on $(\Omega,\mathscr{F},\mathscr{F}_{t\ge0},\mathscr{P})$ and $\vartheta:R^+\times R^n\times R^n\to R^{n\times n}$ is the diffusion coefficient matrix.

From Eq.(3), we can obtain the following error system:

$$\begin{aligned}\mathrm{d}\boldsymbol{e}(t)=&\Big[(\boldsymbol{A}+\Delta\boldsymbol{A}(t)+\boldsymbol{K})\boldsymbol{e}(t)-\boldsymbol{M}\boldsymbol{H}\boldsymbol{e}_\tau(t)\\ &+(\boldsymbol{B}+\Delta\boldsymbol{B}(t))\eta(\boldsymbol{C}\boldsymbol{e}(t),\boldsymbol{y}(t))\\ &+\sum_{k=1}^{\infty}\delta(t-t_k)(\boldsymbol{D}_k\boldsymbol{e}(t_k^-)+\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k))\Big]\mathrm{d}t\\ &-\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\mathrm{d}\omega(t),\quad t\in[t_{k-1},t_k),\end{aligned}\tag{4}$$

where $\eta(\boldsymbol{C}\boldsymbol{e}(t),\boldsymbol{y}(t))=\sigma(\boldsymbol{C}\boldsymbol{e}(t)+\boldsymbol{C}\boldsymbol{y}(t))-\sigma(\boldsymbol{C}\boldsymbol{y}(t))$, $\boldsymbol{D}_k=\boldsymbol{G}_kK$, $\tilde{\boldsymbol{D}}_k=-\boldsymbol{G}_k\boldsymbol{M}\boldsymbol{H}$, $\boldsymbol{e}_\tau(t_k)=\boldsymbol{e}_\tau(t_k^-)=\boldsymbol{e}(t_k-\tau(t_k))$, and $k\in N$.

Alternatively, by using the properties of Dirac measure, error system (4) can also be expressed as

$$\begin{aligned}\mathrm{d}\boldsymbol{e}(t)=&[(\boldsymbol{A}+\Delta\boldsymbol{A}(t)+\boldsymbol{K})\boldsymbol{e}(t)-\boldsymbol{M}\boldsymbol{H}\boldsymbol{e}_\tau(t)\\ &+(\boldsymbol{B}+\Delta\boldsymbol{B}(t))\eta(\boldsymbol{C}\boldsymbol{e}(t),\boldsymbol{y}(t))]\mathrm{d}t\\ &-\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\mathrm{d}\omega(t),\ t\ne t_k,\\ \Delta\boldsymbol{e}(t_k)=&\boldsymbol{e}(t_k)-\boldsymbol{e}(t_k^-)\\ =&\boldsymbol{D}_k\boldsymbol{e}(t_k^-)+\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k),\ k\in N,\end{aligned}\tag{5}$$

where $\lim\limits_{t\to t_k^+}\boldsymbol{e}(t)=\boldsymbol{e}(t_k)$, i.e. $\boldsymbol{e}(t)$ is right–continuous at $t=t_k$, $\lim\limits_{t\to t_k^-}\boldsymbol{e}(t)=\boldsymbol{e}(t_k^-)$ exists, $k\in N$.

For the proof of our main results, some necessary preliminaries are needed.

Assumption 1 The nonlinearity $\eta(\boldsymbol{C}\boldsymbol{e},\boldsymbol{y})$ belongs to the sector $[0,\kappa]$ such that

$$0\le\frac{\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y})}{\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e}}=\frac{\sigma_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e}+\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{y})-\sigma_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{y})}{\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e}}\le\kappa,$$
$$\forall\boldsymbol{e},\boldsymbol{y};\ i=1,2,\ldots,n_h\ (\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e}\ne0),\tag{6}$$

where $\boldsymbol{c}_i^{\mathrm{T}}$ denotes the ith row vector of $\boldsymbol{C}$, then inequality (6) implies the following inequality:

$$\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y})(\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y})-\kappa\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e})\le0.\tag{7}$$

Assumption 2 $\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))$ is locally Lipschitz continuous and satisfies the linear growth condition. $\vartheta(t,0,0)=0$. Moreover, there exist constant matrices with appropriate dimensions Σ_1 and Σ_2 such that

$$\begin{aligned}&\mathrm{trace}[\vartheta^{\mathrm{T}}(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))]\\ &\le\|\Sigma_1\boldsymbol{e}(t)\|^2+\|\Sigma_2\boldsymbol{e}_\tau(t)\|^2.\end{aligned}$$

Assumption 3 The time-varying delay $\tau(t)$ is a bounded function on R with $0 \leq \tau(t) \leq \tau$, where τ is a positive constant.

Assumption 4 The parametric uncertainties $\Delta\boldsymbol{A}(t)$ and $\Delta\boldsymbol{B}(t)$ are of the forms

$$\Delta\boldsymbol{A}(t) = \boldsymbol{E}_1\boldsymbol{F}_1(t)\boldsymbol{H}_1, \text{ and } \Delta\boldsymbol{B}(t) = \boldsymbol{E}_2\boldsymbol{F}_2(t)\boldsymbol{H}_2, \quad (8)$$

in which $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, $\boldsymbol{H}_1$ and $\boldsymbol{H}_2$ are known real constant matrices with appropriate dimensions, and the uncertain matrices $\boldsymbol{F}_1(t)$ and $\boldsymbol{F}_2(t)$ satisfy

$$\boldsymbol{F}_1^{\mathrm{T}}(t)\boldsymbol{F}_1(t) \leq \boldsymbol{I}, \quad \boldsymbol{F}_2^{\mathrm{T}}(t)\boldsymbol{F}_2(t) \leq \boldsymbol{I}. \quad (9)$$

Remark 1 As shown in expression (8), the parametric uncertainties $\Delta\boldsymbol{A}(t)$ and $\Delta\boldsymbol{B}(t)$ are time-varying perturbations, which are used in other papers to deal with the scheme of robust stabilization for uncertain systems,[6,32] and the parametric uncertainties $\Delta\boldsymbol{A}(t)$ and $\Delta\boldsymbol{B}(t)$ are said to be admissible if expressions (8) and (9) hold.[6]

Remark 2 As seen in Refs.[6,9], the parametric uncertainties of two chaotic systems are assumed to be the same. Similarly, for simplicity, the parametric uncertainties in the master system M'_{u} and the slave system S'_{u} of Eq.(3) are also assumed to be the same in this paper.

Definition 1 The master system M'_{u} and the slave system S'_{u} are said to be exponentially synchronized in mean square if, for a suitably designed feedback controller C'_{u}, the trivial solution $\boldsymbol{e}(t) = 0$ of error system (5) is exponentially stable in mean square, i.e. there exist constants $\mu \geq 1$ and $\lambda > 0$, such that

$$E(\|\boldsymbol{e}(t)\|^2) \leq \mu \sup_{s\in[t_0-\tau,t_0]} E(\|\phi(s)\|^2)e^{-\lambda(t-t_0)} \quad (10)$$

for any initial conditions $\boldsymbol{e}(s) = \phi(s)$ on $t_0-\tau \leq s \leq t_0$ in $L^2_{\mathscr{F}_0}([t_0-\tau, t_0]; R^n)$, and the constant $\lambda > 0$ is defined as the exponential synchronization rate.

Lemma 1[22] Given any real matrices $\boldsymbol{W}_1$, $\boldsymbol{W}_2$ and $\boldsymbol{Q}$ with appropriate dimensions and a positive constant $\varsigma > 0$ such that $0 < \boldsymbol{Q} = \boldsymbol{Q}^{\mathrm{T}}$, then the following inequality holds:

$$\boldsymbol{W}_1^{\mathrm{T}}\boldsymbol{W}_2 + \boldsymbol{W}_2^{\mathrm{T}}\boldsymbol{W}_1 \leq \varsigma\boldsymbol{W}_1^{\mathrm{T}}\boldsymbol{Q}\boldsymbol{W}_1 + \varsigma^{-1}\boldsymbol{W}_2^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{W}_2. \quad (11)$$

Lemma 2[31] For a given matrix

$$\boldsymbol{S} = \begin{bmatrix} \boldsymbol{S}_{11} & \boldsymbol{S}_{12} \\ \boldsymbol{S}_{12}^{\mathrm{T}} & \boldsymbol{S}_{22} \end{bmatrix} < 0,$$

where $\boldsymbol{S}_{11} = \boldsymbol{S}_{11}^{\mathrm{T}}$ and $\boldsymbol{S}_{22} = \boldsymbol{S}_{22}^{\mathrm{T}}$, it is equivalent to

$$\boldsymbol{S}_{22} < 0, \ \boldsymbol{S}_{11} - \boldsymbol{S}_{12}\boldsymbol{S}_{22}^{-1}\boldsymbol{S}_{12}^{\mathrm{T}} < 0.$$

Lemma 3[33] Let $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{W}$ and $\boldsymbol{Z} = \boldsymbol{Z}^{\mathrm{T}}$ be real matrices with appropriate dimensions, then

$$\boldsymbol{Z} + \boldsymbol{U}\boldsymbol{V}\boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}}\boldsymbol{V}^{\mathrm{T}}\boldsymbol{U}^{\mathrm{T}} < 0 \text{ for all } \boldsymbol{V}^{\mathrm{T}}\boldsymbol{V} \leq \boldsymbol{I},$$

if and only if there exists a scalar $\theta > 0$ such that

$$\boldsymbol{Z} + \theta^{-1}\boldsymbol{U}\boldsymbol{U}^{\mathrm{T}} + \theta\boldsymbol{W}^{\mathrm{T}}\boldsymbol{W} < 0.$$

Lemma 4 Under Assumptions 1 and 2, let $\boldsymbol{e}(t)$ be a solution of system (5) in the case of $\boldsymbol{G}_k = 0$ and assume that there exist a positive, continuous function $V(t, \boldsymbol{e}(t))$ and positive constants c_1 and c_2 such that

$$c_1\|\boldsymbol{e}(t)\|^2 \leq V(t, \boldsymbol{e}(t)) \leq c_2\|\boldsymbol{e}(t)\|^2, \quad t \geq t_0 - \tau. \quad (12)$$

When $t \geq t_0$ and for some values $0 \leq \beta < \alpha$,

$$\begin{aligned} &D^+E(V(t, \boldsymbol{e}(t))) \\ &\leq -\alpha E(V(t, \boldsymbol{e}(t))) + \beta E(\widetilde{V}(t)), \end{aligned} \quad (13)$$

where $E(\widetilde{V}(t)) \triangleq \sup\limits_{s\in[t-\tau,t]} E(V(s, \boldsymbol{e}(s)))$. Then

$$E(V(t, \boldsymbol{e}(t))) \leq E(\widetilde{V}(t_0))\exp\left(-\nu^+(t-t_0)\right) \quad (14)$$

and

$$\begin{aligned} &E(\|\boldsymbol{e}(t, t_0, \phi)\|^2) \\ &\leq \frac{c_2}{c_1} \sup_{s\in[t_0-\tau,t_0]} E(\|\phi(s)\|^2)\exp\left(-\nu^+(t-t_0)\right), \end{aligned} \quad (15)$$

where $\nu^+ \in (0, \alpha-\beta]$ is the unique positive solution of the equation $\nu = \alpha - \beta e^{\nu\tau} \cdot \phi(s) \triangleq \boldsymbol{e}(s)$ for $s \in [t_0-\tau, t_0]$.

Proof See Appendix A.

Remark 3 Theorem 8 in Ref.[34] shows the global exponential mean square stability (with $p = 2$) of a class of stochastic delay differential equations (SDDEs) with constant time delay. In this paper, Lemma 4 extends the results in Ref.[34] to the time-varying case, which is less conservative and can be applied to a wider class of practical purposes.

3. Master-slave synchronization for uncertain Lur'e systems

To facilitate the readers, before starting our main results, some related results are cited here. For a general stochastic system $\mathrm{d}\boldsymbol{x}(t) = g_1(t, \boldsymbol{x})\mathrm{d}t + g_2(t, \boldsymbol{x})\mathrm{d}\omega(t)$ at $t \geq 0$ with initial value $\boldsymbol{x}_0 \in R^n$, $g_1 : R^+ \times R^n \to R^n$ and $g_2 : R^+ \times R^n \to R^{n\times n}$, for

each $V(t,\boldsymbol{x}) \in \mathscr{C}^{1,2}(R^+ \times R^n; R^+)$, an operator $\mathscr{L}$ is defined from $R^+ \times R^n$ to R by

$$\begin{aligned}\mathscr{L}V(t,\boldsymbol{x}) &= V_t(t,\boldsymbol{x}) + V_x(t,\boldsymbol{x})g_1(t,\boldsymbol{x}) \\ &\quad + \frac{1}{2}\mathrm{tr}[g_2^{\mathrm{T}}(t,\boldsymbol{x})V_{xx}(t,\boldsymbol{x})g_2(t,\boldsymbol{x})],\end{aligned}$$

where

$$\begin{aligned}V_t(t,\boldsymbol{x}) &= \frac{\partial V(t,\boldsymbol{x})}{\partial t}, \\ V_x(t,\boldsymbol{x}) &= \left(\frac{\partial V(t,\boldsymbol{x})}{\partial x_1}, \ldots, \frac{\partial V(t,\boldsymbol{x})}{\partial x_n}\right), \\ V_{xx}(t,\boldsymbol{x}) &= \left(\frac{\partial^2 V(t,\boldsymbol{x})}{\partial x_i \partial x_j}\right)_{n\times n}.\end{aligned}$$

In this section, instead of directly dealing with the uncertain case, we consider the master-slave synchronization scheme for nominal Lur'e systems without parametric uncertainties via time-varying delay feedback control, and the corresponding master-slave synchronization scheme can be formulated as

$$M_{\mathrm{u}}: \begin{cases} \mathrm{d}\boldsymbol{x}(t) = (\boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\sigma(\boldsymbol{C}\boldsymbol{x}(t)))\mathrm{d}(t), \\ \boldsymbol{p}(t) = \boldsymbol{H}\boldsymbol{x}(t), \end{cases}$$

$$S_{\mathrm{u}}: \begin{cases} \dot{\boldsymbol{y}}(t) = (\boldsymbol{A}\boldsymbol{y}(t) + \boldsymbol{B}\sigma(\boldsymbol{C}\boldsymbol{y}(t)) + \boldsymbol{v}(t))\mathrm{d}(t) \\ \qquad\quad + \vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\mathrm{d}\omega(t), \\ \boldsymbol{q}(t) = \boldsymbol{H}\boldsymbol{y}(t), \end{cases}$$

$$\begin{aligned}C_{\mathrm{u}}: \boldsymbol{v}(t) &= -\boldsymbol{K}(\boldsymbol{x}(t) - \boldsymbol{y}(t)) + \boldsymbol{M}(\boldsymbol{p}_\tau(t) - q_\tau(t)) \\ &\quad + \sum_{k=1}^{\infty}\delta(t-t_k)\boldsymbol{G}_k u(t) \qquad (16)\end{aligned}$$

with master system M_{u}, slave system S_{u} and time-varying delay feedback controller C_{u}. The subscript n refers to the nominal case of Lur'e systems. Other symbols possess the same meanings as those in Eq.(3). Similar to the derivation for error systems (4) and (5), by defining the error vector $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t)$, one can obtain the following error system:

$$\begin{aligned}\mathrm{d}\boldsymbol{e}(t) &= \Big[(\boldsymbol{A}+\boldsymbol{K})\boldsymbol{e}(t) - \boldsymbol{M}\boldsymbol{H}\boldsymbol{e}_\tau(t) + \boldsymbol{B}\eta(\boldsymbol{C}\boldsymbol{e}(t),\boldsymbol{y}(t)) \\ &\quad + \sum_{k=1}^{\infty}\delta(t-t_k)(\boldsymbol{D}_k\boldsymbol{e}(t_k^-) + \tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k))\Big]\mathrm{d}t \\ &\quad + \vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\mathrm{d}\omega(t),\ t \in [t_{k-1},t_k), \\ &\qquad k \in N, \qquad (17)\end{aligned}$$

or

$$\begin{aligned}\mathrm{d}\boldsymbol{e}(t) &= [(\boldsymbol{A}+\boldsymbol{K})\boldsymbol{e}(t) - \boldsymbol{M}\boldsymbol{H}\boldsymbol{e}_\tau(t) \\ &\quad + \boldsymbol{B}\eta(\boldsymbol{C}\boldsymbol{e}(t),\boldsymbol{y}(t))]\mathrm{d}t \\ &\quad + \vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\mathrm{d}\omega(t),\ t \neq t_k, \\ \Delta\boldsymbol{e}(t_k) &= \boldsymbol{e}(t_k) - \boldsymbol{e}(t_k^-) \\ &= \boldsymbol{D}_k\boldsymbol{e}(t_k^-) + \tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k),\ k \in N. \qquad (18)\end{aligned}$$

Theorem 1 Assume that Assumptions 1–3 and expression (2) hold. If there exist matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, positive definite matrices $\boldsymbol{P} > 0$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_{\mathrm{h}}}) > 0$, some constants $0 \le \beta < \alpha$, $\rho \ge 1$, $\varepsilon_k > 0$ and $\omega_k > 0$ such that the following conditions hold:

$$\Pi = \begin{bmatrix} \Psi & \boldsymbol{P}\boldsymbol{B} + \kappa\boldsymbol{C}^{\mathrm{T}}\Lambda & -\boldsymbol{P}\boldsymbol{M}\boldsymbol{H} \\ * & -2\Lambda & 0 \\ * & * & \gamma\Sigma_2^{\mathrm{T}}\Sigma_2 - \beta\boldsymbol{P} \end{bmatrix} \le 0, \qquad (19)$$

$$\boldsymbol{P} \le \gamma\boldsymbol{I}, \qquad (20)$$

and

$$\max\{e^{\nu\tau},\ a_k + b_k e^{\nu\tau}\} \le \rho < e^{\nu\tau_m}, \qquad (21)$$

then the synchronization scheme (16) will achieve global exponential synchronization in mean square, where $\Psi = \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}} + \gamma\Sigma_1^{\mathrm{T}}\Sigma_1 + \alpha\boldsymbol{P}$, $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{X}$, $a_k = (1+\varepsilon_k)\left((1+\omega_k) + (1+\omega_k^{-1})L_k^2\dfrac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\right)$, $b_k = (1+\varepsilon_k^{-1})\tilde{L}_k^2\dfrac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}$, $\boldsymbol{L}_k = l_k\|K\|$, $\tilde{\boldsymbol{L}}_k = l_k\|\boldsymbol{M}\boldsymbol{H}\|$, $\tau_m = \inf\{t_k - t_{k-1}\}$, $k \in N$, and $\nu \in (0, \alpha-\beta]$ satisfying

$$\nu = \alpha - \beta e^{\nu\tau}. \qquad (22)$$

Proof See Appendix B.

Remark 4 Theorem 1 gives one criterion to guarantee global exponential synchronization of nominal Lur'e systems with time-varying delay. The criterion only depends on the magnitude of time delay, i.e. $0 \le \tau(t) \le \tau$, which is not severally required to depend on the magnitude of the time-derivative of time delay.

Next, we discuss the master-slave synchronization scheme (3) for Lur'e systems with parametric uncertainties. Base on the results obtained in Theorem 1, the following synchronization criterion can be derived easily, which is robust for all admissible uncertainties.

Theorem 2 Assume that Assumptions 1–4 and expression (2) hold. If there exist matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, positive definite matrices $\boldsymbol{P} > 0$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_{\mathrm{h}}}) > 0$, positive constants $0 \le \beta < \alpha$, $\rho \ge 1$, $\mu_1 > 0$, $\mu_2 > 0$, $\varepsilon_k > 0$ and $\omega_k > 0$ such that

the following conditions hold:

$$\widetilde{\Pi} = \begin{bmatrix} \widetilde{\Psi}_{11} & \widetilde{\Psi}_{12} & -\boldsymbol{PMH} & \boldsymbol{PE}_1 & \boldsymbol{PE}_2 \\ * & \widetilde{\Psi}_{22} & 0 & 0 & 0 \\ * & * & \widetilde{\Psi}_{33} & 0 & 0 \\ * & * & * & -\mu_1\boldsymbol{I} & 0 \\ * & * & * & * & -\mu_2\boldsymbol{I} \end{bmatrix} \leq 0, \quad (23)$$

$$\boldsymbol{P} \leq \gamma \boldsymbol{I}, \quad (24)$$

$$\max\{e^{\nu\tau},\ a_k + b_k e^{\nu\tau}\} \leq \rho < e^{\nu\tau_m}, \quad (25)$$

then the synchronization scheme (3) will achieve global exponential synchronization in mean square, where $\widetilde{\Psi}_{11} = \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}} + \gamma\Sigma_1^{\mathrm{T}}\Sigma_1 + \mu_1\boldsymbol{H}_1^{\mathrm{T}}\boldsymbol{H}_1 + \alpha\boldsymbol{P}$, $\widetilde{\Psi}_{12} = \boldsymbol{PB} + \kappa\boldsymbol{C}^{\mathrm{T}}\Lambda$, $\widetilde{\Psi}_{22} = -2\Lambda + \mu_2\boldsymbol{H}_2^{\mathrm{T}}\boldsymbol{H}_2$, $\widetilde{\Psi}_{33} = \gamma\Sigma_2^{\mathrm{T}}\Sigma_2 - \beta\boldsymbol{P}$, $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{X}$, $a_k = (1+\varepsilon_k)\left((1+\omega_k) + (1+\omega_k^{-1})L_k^2\dfrac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\right)$, $b_k = (1+\varepsilon_k^{-1})\tilde{L}_k^2\dfrac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}$, $L_k = l_k\|\boldsymbol{K}\|$, $\tilde{L}_k = l_k\|\boldsymbol{MH}\|$, $\tau_m = \inf\{t_k - t_{k-1}\}$, $k \in N$, and $\nu \in (0, \alpha-\beta]$ satisfying

$$\nu = \alpha - \beta e^{\nu\tau}. \quad (26)$$

Proof See Appendix C.

If there is no stochastic perturbation in slave system S'_{u} of expression (3), then we can obtain the following corollary.

Corollary 1 Assume that Assumptions 1, 3, 4 and expression (2) hold. If there exist matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, positive definite matrices $\boldsymbol{P} > 0$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_{\mathrm{h}}}) > 0$, positive constants $0 \leq \beta < \alpha$, $\rho \geq 1$, $\mu_1 > 0$, $\mu_2 > 0$, $\varepsilon_k > 0$ and $\omega_k > 0$ such that the following conditions hold:

$$\begin{bmatrix} \Theta_{11} & \Theta_{12} & -\boldsymbol{PMH} & \boldsymbol{PE}_1 & \boldsymbol{PE}_2 \\ * & \Theta_{22} & 0 & 0 & 0 \\ * & * & -\beta\boldsymbol{P} & 0 & 0 \\ * & * & * & -\mu_1\boldsymbol{I} & 0 \\ * & * & * & * & -\mu_2\boldsymbol{I} \end{bmatrix} \leq 0, \quad (27)$$

$$\boldsymbol{P} \leq \gamma \boldsymbol{I}, \quad (28)$$

and

$$\max\{e^{\nu\tau},\ a_k + b_k e^{\nu\tau}\} \leq \rho < e^{\nu\tau_m}, \quad (29)$$

then the synchronization scheme (3) will achieve global exponential synchronization, where $\Theta_{11} = \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}} + \mu_1\boldsymbol{H}_1^{\mathrm{T}}\boldsymbol{H}_1 + \alpha\boldsymbol{P}$, $\Theta_{12} = \boldsymbol{PB} + \kappa\boldsymbol{C}^{\mathrm{T}}\Lambda$, $\Theta_{22} = -2\Lambda + \mu_2\boldsymbol{H}_2^{\mathrm{T}}\boldsymbol{H}_2$, $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{X}$, $a_k = (1+\varepsilon_k)\left((1+\omega_k) + (1+\omega_k^{-1})L_k^2\dfrac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\right)$, $b_k = (1+\varepsilon_k^{-1})\tilde{L}_k^2\dfrac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}$, $L_k = l_k\|\boldsymbol{K}\|$, $\tilde{L}_k = l_k\|\boldsymbol{MH}\|$, $\tau_m = \inf\{t_k - t_{k-1}\}$, $k \in N$, and $\nu \in (0, \alpha-\beta]$ satisfying

$$\nu = \alpha - \beta e^{\nu\tau}. \quad (30)$$

Proof Compared with Theorem 2, the only difference is that we have adopted the well-known Halanay inequality described in Lemma 5 instead of Lemma 4, and the remaining proof is similar to that of Theorem 2, which is omitted here. This completes the proof.

Remark 5 If we do not consider stochastic perturbation and the current error state feedback term in controller C'_{u} simultaneously, Corollary 1 can reduce to the results of Theorem 1 in Ref.[25] (the criterion expressed by LMIs). Furthermore, in Ref.[25], if only a pure time-delay static error feedback control is used, a necessary condition is that matrix A is negative semidefinite. By comparison, in this paper considered is the feedback control including both the current error state feedback and the delayed static error output feedback, and the restriction on the positive definiteness of A is removed. Therefore, our results have wider application fields than that of in Ref.[25].

If there is no impulsive perturbation in slave system S'_{u} of expression (3), then we can obtain the following corollary.

Corollary 2 Assume that Assumptions 1–4 hold. If there exist matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, positive definite matrices $\boldsymbol{P} > 0$ and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{n_{\mathrm{h}}}) > 0$, positive constants $0 \leq \beta < \alpha$, $\mu_1 > 0$ and $\mu_2 > 0$ such that the following conditions hold:

$$\begin{bmatrix} \widetilde{\Theta}_{11} & \widetilde{\Theta}_{12} & -\boldsymbol{PMH} & \boldsymbol{PE}_1 & \boldsymbol{PE}_2 \\ * & \widetilde{\Theta}_{22} & 0 & 0 & 0 \\ * & * & \widetilde{\Theta}_{33} & 0 & 0 \\ * & * & * & -\mu_1\boldsymbol{I} & 0 \\ * & * & * & * & -\mu_2\boldsymbol{I} \end{bmatrix} \leq 0, \quad (31)$$

$$\boldsymbol{P} \leq \gamma \boldsymbol{I}, \quad (32)$$

then the synchronization scheme (3) will achieve global exponential synchronization in mean square, where $\widetilde{\Theta}_{11} = \boldsymbol{PA} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}} + \gamma\Sigma_1^{\mathrm{T}}\Sigma_1 + \mu_1\boldsymbol{H}_1^{\mathrm{T}}\boldsymbol{H}_1 + \alpha\boldsymbol{P}$, $\widetilde{\Theta}_{12} = \boldsymbol{PB} + \kappa\boldsymbol{C}^{\mathrm{T}}\Lambda$, $\widetilde{\Theta}_{22} = -2\Lambda +$

$\mu_2 \boldsymbol{H}_2^{\mathrm{T}} \boldsymbol{H}_2$, $\widetilde{\Theta}_{33} = \gamma \Sigma_2^{\mathrm{T}} \Sigma_2 - \beta \boldsymbol{P}$, $\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{X}$, and $\nu \in (0, \alpha - \beta]$ satisfying

$$\nu = \alpha - \beta e^{\nu\tau}. \tag{33}$$

Proof This is a particular case of Theorem 2 with $\boldsymbol{G}_k = 0$, and the proof is similar to the first half of the proof of Theorem 2. Here we omit the procedure. This completes the proof.

4. Illustrated example

Consider the following Chua circuit:[35]

$$\begin{aligned} \dot{x}_1 &= \theta_1(x_2 - h(x_1)), \\ \dot{x}_2 &= x_1 - x_2 + x_3, \\ \dot{x}_3 &= -\theta_2 x_2, \end{aligned} \tag{34}$$

with nonlinear characteristic $h(x_1) = m_1 x_1 + (1/2)(m_0 - m_1)(|x_1 + \theta_3| - |x_1 - \theta_3|)$ and parameters $\theta_1 = 9$, $\theta_2 = 14.28$, $\theta_3 = 1$, $m_0 = -(1/7)$, and $m_1 = 2/7$. The system can be represented in the Lur'e form by

$$\boldsymbol{A} = \begin{bmatrix} -\theta_1 m_1 & \theta_1 & 0 \\ 1 & -1 & 1 \\ 0 & -\theta_2 & 0 \end{bmatrix}, \ \boldsymbol{B} = \begin{bmatrix} -\theta_1(m_0 - m_1) \\ 0 \\ 0 \end{bmatrix},$$

and

$$\boldsymbol{C} = \boldsymbol{H} = [1 \ \ 0 \ \ 0],$$

which implies $l = n_{\mathrm{h}} = 1$. $\boldsymbol{E}_1 = [0.1, 0.1, -0.1]^{\mathrm{T}}$, $\boldsymbol{E}_2 = [0.1, -0.1, -0.2]^{\mathrm{T}}$, $\boldsymbol{H}_1 = [1, 1, 0]$, $\boldsymbol{H}_2 = 0.5$, $\boldsymbol{F}_1(t) = \sin(t)$, $\boldsymbol{F}_2(t) = \cos(t)$. $\sigma(\zeta) = (1/2)(|\zeta + \theta_3| - |\zeta - \theta_3|)$ belongs to the sector $[0, \kappa]$ with $\kappa = 1$. Figure 2 shows the chaotic attractor of system (34) with parametric uncertainties $\Delta\boldsymbol{A}(t)$ and $\Delta\boldsymbol{B}(t)$, and the initial condition is taken as $\boldsymbol{x}(s) = [0.2, 0.13, 0.17]^{\mathrm{T}}$ for $-0.3 \le s \le 0$. Letting $\vartheta(t, \boldsymbol{e}(t), \boldsymbol{e}_\tau(t)) = \mathrm{diag}\Big\{\frac{1}{\sqrt{2}}(e_1(t) + e_{1\tau}(t)), \frac{1}{\sqrt{2}}(e_2(t) + e_{2\tau}(t)), \frac{1}{\sqrt{2}}(e_3(t) + e_{3\tau}(t))\Big\}$, we can easily have $\Sigma_1 = \Sigma_2 = \boldsymbol{I}$. Let $\alpha = 4$, $\beta = 2$, and $\boldsymbol{M} = [4, 1, -2]^{\mathrm{T}}$ and solve LMIs (23) and (24) in Theorem 2, a set of feasible solution will be obtained as follows:

$$\boldsymbol{P} = \begin{bmatrix} 6.9416 & -0.2105 & 0.4211 \\ -0.2105 & 6.4614 & 0.1638 \\ 0.4211 & 0.1638 & 6.2144 \end{bmatrix},$$

$$\boldsymbol{X} = \begin{bmatrix} -128.0016 & -28.7830 & 25.5531 \\ -28.7830 & -42.4071 & 46.1320 \\ 25.5531 & 46.1320 & -43.0684 \end{bmatrix},$$

$$\boldsymbol{K} = \boldsymbol{P}^{-1}\boldsymbol{X} = \begin{bmatrix} -18.9335 & -4.8376 & 4.3580 \\ -5.2116 & -6.9219 & 7.4698 \\ 5.5323 & 7.9337 & -7.4227 \end{bmatrix},$$

$\Lambda = 25.1538$, $\gamma = 8.5315$, $\mu_1 = 31.3908$, and $\mu_2 = 24.9069$. The time-varying delay is given as $\tau(t) = 0.3|\sin(t)|$ with $\tau = 0.3$. Obviously, $\tau(t)$ is not differentiable at $t = k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$. Let $l_k = 0.05$, $\varepsilon_k = 0.1$, and $\omega_k = 1.3$, and carry out the computation, then we will have $\nu = 1.1641$, $a_k = 5.9928$, $b_k = 0.6987$, and $\max\{e^{\nu\tau}, a_k + b_k e^{\nu\tau}\} = 6.9834$. Hence, from inequality (25), we can choose $\rho = 6.99$ and $\tau_m = 1.8$. Therefore, the synchronization scheme (3) can achieve global exponential synchronization in mean square, and the corresponding exponential synchronization rate $\nu - \dfrac{\ln\rho}{\tau_m} = 0.0838$. Figure 3 depicts the state response of error system (5). The initial conditions are taken as $\boldsymbol{x}(s) = [0.2, 0.13, 0.17]^{\mathrm{T}}$ and $\boldsymbol{y}(s) = [-0.1, -0.14, 0.45]^{\mathrm{T}}$ for $-0.3 \le s \le 0$, respectively. Throughout the simulations, we have used the fourth Runge-Kutta approach in time steps of 0.0001.

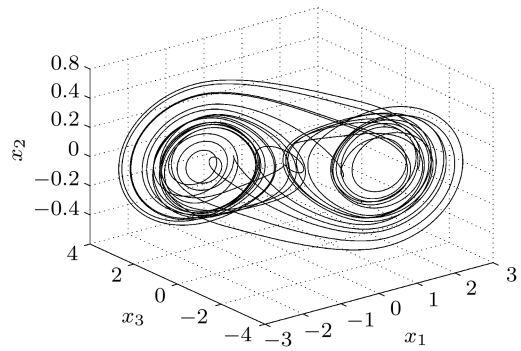


Fig.2. The chaotic attractor of Chua's circuit (34) with parametric uncertainties.

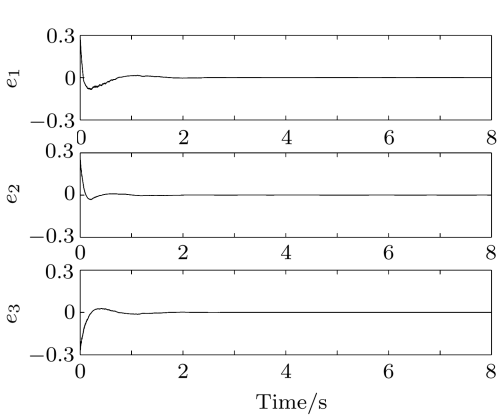


Fig.3. The state response of error system (5).

5. Conclusions

In this paper, we have investigated the problem of the exponential synchronization of stochastic impulsive perturbed chaotic Lur'e systems with parametric uncertainty. Based on the theory of stochastic impulsive differential equations, time-varying delay feedback control techniques and the modified Halanay inequality of stochastic version, some new sufficient conditions are established. It should be pointed out that the topic of synchronization with both stochastic and impulsive noise perturbations reflects a more realistic dynamics and deserves a great deal of attention. The theoretical results derived in this paper can strongly support other experimental observations with noise perturbation and arouse increasing interest in the synchronization of chaotic systems.

Appendix A

Before the proof of Lemma 4, we first introduce the well-known Halanay inequality.

Lemma 5[34] Let α, β and $\tau \geq 0$ be given constants in R, suppose that the positive-valued function, $v : [t_0 - \tau, \infty) \to R^+$, is continuous on $[t_0 - \tau, \infty)$ for given $\tau \geq 0$, and assume that the following conditions hold:

$$0 < \beta < \alpha, \tag{A1}$$

and

$$D^+v(t) \leq -\alpha v(t) + \beta \sup_{s\in[t-\tau,t]} v(s),\ t \in [t_0, \infty), \tag{A2}$$

then the following inequality will holds:

$$v(t) \leq \left(\sup_{s\in[t_0-\tau,t_0]} v(s)\right) \exp\left(-\nu^+(t - t_0)\right) \tag{A3}$$

for $t \in [t_0, \infty)$, where $\nu^+ \in (0, \alpha - \beta]$ satisfies

$$\nu = \alpha - \beta e^{\nu\tau}. \tag{A4}$$

Proof of Lemma 4

Denote

$$v(t) \triangleq E(V(t, \boldsymbol{e}(t))) = E(V(t, \boldsymbol{e}(t, t_0, \phi))).$$

Then from inequality (13), we have

$$D^+v(t) \leq -\alpha v(t) + \beta \sup_{s\in[t-\tau,t]} v(s).$$

Hence, from Lemma 5, we can obtain the exponentially decreasing bound on $v(t)$

$$v(t) \leq \left(\sup_{s\in[t_0-\tau,t_0]} v(s)\right) \exp\left(-\nu^+(t - t_0)\right) \tag{A5}$$

for $t \in [t_0, \infty)$. On the other hand, from condition (12), we obtain

$$\begin{aligned}\sup_{s\in[t_0-\tau,t_0]} v(s) &= \sup_{s\in[t_0-\tau,t_0]} E(V(s, \phi(s))) \\ &\leq c_2 \sup_{s\in[t_0-\tau,t_0]} E(\|\phi(s)\|^2)\end{aligned} \tag{A6}$$

and

$$E(\|\boldsymbol{e}(t)\|^2) \leq \frac{1}{c_1} v(t). \tag{A7}$$

From inequalities (A5)–(A7), we can obtain the desired result (15) easily. This completes the proof.

Appendix B

Proof of Theorem 1

Construct the following Lyapunov functional:

$$V(t, \boldsymbol{e}(t)) = \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t). \tag{B1}$$

It can be easily verified that $V(t, \boldsymbol{e}(t))$ is nonnegative over $[t_0 - \tau, \infty)$. By using the Itô's differential formula, the stochastic derivative of $V(t, \boldsymbol{e}(t))$ along the trajectory of system (18) is obtained as

$$\begin{aligned}\mathrm{d}V(t, \boldsymbol{e}(t)) =& \mathscr{L}V(t, \boldsymbol{e}(t))\mathrm{d}(t) \\ &+ V_e(t, \boldsymbol{e}(t))\vartheta(t, \boldsymbol{e}(t), \boldsymbol{e}(t - \tau(t)))\mathrm{d}\omega(t).\end{aligned} \tag{B2}$$

For $t \in [t_{k-1}, t_k)$, $k \in N$, taking the mathematical expectation on both sides of Eq.(B2), we can obtain the following Dini derivative:

$$D^+E(V(t, \boldsymbol{e}(t))) = E(\mathscr{L}V(t, \boldsymbol{e}(t))), \tag{B3}$$

where

$$\begin{aligned}&\mathscr{L}V(t, \boldsymbol{e}(t)) \\ &= V_t(t, \boldsymbol{e}(t)) + V_e(t, \boldsymbol{e}(t))[(\boldsymbol{A} + \boldsymbol{K})\boldsymbol{e}(t) - \boldsymbol{M}\boldsymbol{H}\boldsymbol{e}_\tau(t) + \boldsymbol{B}\eta(\boldsymbol{C}\boldsymbol{e}(t), \boldsymbol{y}(t))]\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\mathrm{trace}[\vartheta^{\mathrm{T}}(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))V_{ee}(t,\boldsymbol{e}(t))\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))]\\
&=2\boldsymbol{e}^{\mathrm{T}}\boldsymbol{P}[(\boldsymbol{A}+\boldsymbol{K})\boldsymbol{e}(t)-\boldsymbol{MH}\boldsymbol{e}_\tau(t)+\boldsymbol{B}\eta(\boldsymbol{Ce}(t),\boldsymbol{y}(t))]\\
&\quad+\mathrm{trace}[\vartheta^{\mathrm{T}}(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\boldsymbol{P}\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))]. \qquad \text{(B4)}
\end{aligned}
$$

It follows from Assumption 2 and inequality (20) that

$$
\begin{aligned}
&\mathrm{trace}[\vartheta^{\mathrm{T}}(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\boldsymbol{P}\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))]\\
&\le\gamma\mathrm{trace}[\vartheta^{\mathrm{T}}(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))]\\
&\le\gamma\left[\boldsymbol{e}^{\mathrm{T}}(t)\Sigma_1^{\mathrm{T}}\Sigma_1\boldsymbol{e}(t)+\boldsymbol{e}_\tau^{\mathrm{T}}(t)\Sigma_2^{\mathrm{T}}\Sigma_2\boldsymbol{e}_\tau(t)\right]. \qquad \text{(B5)}
\end{aligned}
$$

From inequality (7), for positive constants λ_i, $i=1,2,\ldots,n_{\mathrm{h}}$, we have

$$
-2\sum_{i=1}^{n_{\mathrm{h}}}\lambda_i\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y})(\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y})-\kappa\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e})\ge 0. \qquad \text{(B6)}
$$

Substituting inequalities (B5) and (B6) into expression (B4), we have

$$
\begin{aligned}
&\mathscr{L}V(t,\boldsymbol{e}(t))\\
&\le\boldsymbol{e}^{\mathrm{T}}(t)((\boldsymbol{A}+\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{P}(\boldsymbol{A}+\boldsymbol{K})+\gamma\Sigma_1^{\mathrm{T}}\Sigma_1)\boldsymbol{e}(t)\\
&\quad-2\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{PMH}\boldsymbol{e}_\tau(t)+2\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{PB}\eta(\boldsymbol{Ce}(t),\boldsymbol{y}(t))\\
&\quad-2\sum_{i=1}^{n_{\mathrm{h}}}\lambda_i\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y})(\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y})-\kappa\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e})\\
&\quad+\gamma\boldsymbol{e}_\tau^{\mathrm{T}}(t)\Sigma_2^{\mathrm{T}}\Sigma_2\boldsymbol{e}_\tau(t)\\
&=\xi^{\mathrm{T}}(t)\Pi\xi(t)-\alpha\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{Pe}(t)+\beta\boldsymbol{e}_\tau^{\mathrm{T}}(t)\boldsymbol{Pe}_\tau(t)\\
&\le-\alpha\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{Pe}(t)+\beta\boldsymbol{e}_\tau^{\mathrm{T}}(t)\boldsymbol{Pe}_\tau(t), \qquad \text{(B7)}
\end{aligned}
$$

where $\xi(t)=[\boldsymbol{e}^{\mathrm{T}}(t),\eta^{\mathrm{T}}(\boldsymbol{Ce}(t),\boldsymbol{y}(t)),\boldsymbol{e}_\tau^{\mathrm{T}}(t)]^{\mathrm{T}}$.

From expression (B3) and inequality (B7), we have

$$
\begin{aligned}
&D^+E(V(t,\boldsymbol{e}(t)))\\
&\le-\alpha E(V(t,\boldsymbol{e}(t)))+\beta\sup_{s\in[t-\tau,t]}E(V(s,\boldsymbol{e}(s))).
\end{aligned} \qquad \text{(B8)}
$$

From Lemma 4 and inequality (B8), there exists a constant $\nu>0$ such that for $t\in[t_{k-1},t_k)$ and $k\in N$,

$$
E(V(t,\boldsymbol{e}(t)))\le E(\widetilde{V}(t_{k-1}))\exp\left(-\nu(t-t_{k-1})\right). \qquad \text{(B9)}
$$

When $t=t_k$, from Lemma 1, we have

$$
\begin{aligned}
&V(t_k,\boldsymbol{e}(t_k))\\
&=[(\boldsymbol{D}_k+\boldsymbol{I})\boldsymbol{e}(t_k^-)+\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k)]^{\mathrm{T}}\boldsymbol{P}\\
&\quad\times[(\boldsymbol{D}_k+\boldsymbol{I})\boldsymbol{e}(t_k^-)+\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k)]\\
&=\boldsymbol{e}^{\mathrm{T}}(t_k^-)(\boldsymbol{D}_k+\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{D}_k+\boldsymbol{I})\boldsymbol{e}(t_k^-)\\
&\quad+2\boldsymbol{e}^{\mathrm{T}}(t_k^-)(\boldsymbol{D}_k+\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k)\\
&\quad+\boldsymbol{e}_\tau^{\mathrm{T}}(t_k)\tilde{\boldsymbol{D}}_k^{\mathrm{T}}\boldsymbol{P}\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k)\\
&\le\boldsymbol{e}^{\mathrm{T}}(t_k^-)(\boldsymbol{D}_k+\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{D}_k+\boldsymbol{I})\boldsymbol{e}(t_k^-)\\
&\quad+\varepsilon_k\boldsymbol{e}^{\mathrm{T}}(t_k^-)(\boldsymbol{D}_k+I)^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{D}_k+I)\boldsymbol{e}(t_k^-)\\
&\quad+\varepsilon_k^{-1}\boldsymbol{e}_\tau^{\mathrm{T}}(t_k)\tilde{\boldsymbol{D}}_k^{\mathrm{T}}\boldsymbol{P}\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k)\\
&\quad+\boldsymbol{e}_\tau^{\mathrm{T}}(t_k)\tilde{\boldsymbol{D}}_k^{\mathrm{T}}\boldsymbol{P}\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k)\\
&=(1+\varepsilon_k)\boldsymbol{e}^{\mathrm{T}}(t_k^-)(\boldsymbol{D}_k+\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{D}_k+\boldsymbol{I})\boldsymbol{e}(t_k^-)\\
&\quad+(1+\varepsilon_k^{-1})\boldsymbol{e}_\tau^{\mathrm{T}}(t_k)\tilde{\boldsymbol{D}}_k^{\mathrm{T}}\boldsymbol{P}\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k). \qquad \text{(B10)}
\end{aligned}
$$

For the first term on the right-hand side of inequality (B10), using Lemma 1 again, yields

$$
\begin{aligned}
&(1+\varepsilon_k)\boldsymbol{e}^{\mathrm{T}}(t_k^-)(\boldsymbol{D}_k+\boldsymbol{I})^{\mathrm{T}}\boldsymbol{P}(\boldsymbol{D}_k+\boldsymbol{I})\boldsymbol{e}(t_k^-)\\
&=(1+\varepsilon_k)\big(\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{D}_k^{\mathrm{T}}PD_k\boldsymbol{e}(t_k^-)\\
&\quad+2\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{D}_k^{\mathrm{T}}\boldsymbol{Pe}(t_k^-)+\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{Pe}(t_k^-)\big)\\
&\le(1+\varepsilon_k)\big(\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{D}_k^{\mathrm{T}}PD_k\boldsymbol{e}(t_k^-)\\
&\quad+\omega_k^{-1}\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{D}_k^{\mathrm{T}}PD_k\boldsymbol{e}(t_k^-)\\
&\quad+\omega_k\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{Pe}(t_k^-)+\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{Pe}(t_k^-)\big)\\
&=(1+\varepsilon_k)((1+\omega_k)\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{Pe}(t_k^-)\\
&\quad+(1+\omega_k^{-1})\boldsymbol{e}^{\mathrm{T}}(t_k^-)\boldsymbol{D}_k^{\mathrm{T}}PD_k\boldsymbol{e}(t_k^-))\\
&\le(1+\varepsilon_k)\left(1+\omega_k+(1+\omega_k^{-1})L_k^2\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\right)\\
&\quad\times V(t_k^-,\boldsymbol{e}(t_k^-)). \qquad \text{(B11)}
\end{aligned}
$$

For the second term on the right-hand side of inequality (B10), we have

$$
\begin{aligned}
&(1+\varepsilon_k^{-1})\boldsymbol{e}_\tau^{\mathrm{T}}(t_k)\tilde{\boldsymbol{D}}_k^{\mathrm{T}}\boldsymbol{P}\tilde{\boldsymbol{D}}_k\boldsymbol{e}_\tau(t_k)\\
&\le(1+\varepsilon_k^{-1})\tilde{L}_k^2\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}V(t_k,\boldsymbol{e}_\tau(t_k)). \qquad \text{(B12)}
\end{aligned}
$$

Substituting inequalities (B11) and (B12) into inequality (B10), it follows that

$$
\begin{aligned}
&V(t_k,\boldsymbol{e}(t_k))\\
&\le(1+\varepsilon_k)\left(1+\omega_k+(1+\omega_k^{-1})L_k^2\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}\right)\\
&\quad\times V(t_k^-,\boldsymbol{e}(t_k^-))+(1+\varepsilon_k^{-1})\tilde{L}_k^2\frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})}V(t_k,\boldsymbol{e}_\tau(t_k))\\
&=a_kV(t_k^-,\boldsymbol{e}(t_k^-))+b_kV(t_k,\boldsymbol{e}_\tau(t_k)). \qquad \text{(B13)}
\end{aligned}
$$

Then, take the expectation on both sides of inequality (B13), and the following inequality will hold:

$$\begin{aligned} &E(V(t_k, \boldsymbol{e}(t_k))) \\ &\leq a_k E(V(t_k^-, \boldsymbol{e}(t_k^-))) + b_k E(V(t_k, \boldsymbol{e}_\tau(t_k))). \end{aligned} \tag{B14}$$

In the following, by mathematical induction, for $t \in [t_{k-1}, t_k)$ and $k \in N$, we will prove the following inequality:

$$E(V(t, e(t))) \leq \rho^{k-1} E(\widetilde{V}(t_0)) \exp\left(-\nu(t - t_0)\right). \tag{B15}$$

Indeed, when $k = 1$, i.e. $t \in [t_0, t_1)$, it follows from inequality (B9) that

$$\begin{aligned} &E(V(t, \boldsymbol{e}(t))) \\ &\leq E(\widetilde{V}(t_0)) \exp\left(-\nu(t - t_0)\right) \\ &= \rho^0 E(\widetilde{V}(t_0)) \exp\left(-\nu(t - t_0)\right). \end{aligned} \tag{B16}$$

Thus, inequality (B15) holds for $k = 1$.

Next, assume that inequality (B15) holds for $k \leq m$, $m \in N$ and $m > 1$. Then, one needs to show inequality (B15) still holds when $k = m + 1$.

From inequalities (21), (B14) and the above induction assumption, inequality (B15), one has

$$\begin{aligned} &E(V(t_m, \boldsymbol{e}(t_m))) \\ &\leq a_m E(V(t_m^-, \boldsymbol{e}(t_m^-))) + b_m E(V(t_m, \boldsymbol{e}_\tau(t_m))) \\ &\leq \rho^{m-1} a_m E(\widetilde{V}(t_0)) \exp\left(-\nu(t_m - t_0)\right) \\ &\quad + \rho^{m-1} b_m E(\widetilde{V}(t_0)) \exp\left(-\nu(t_m - \tau(t_m) - t_0)\right) \\ &\leq \rho^{m-1}(a_m + b_m e^{\nu\tau}) E(\widetilde{V}(t_0)) \exp\left(-\nu(t_m - t_0)\right) \\ &\leq \rho^m E(\widetilde{V}(t_0)) \exp\left(-\nu(t_m - t_0)\right). \end{aligned} \tag{B17}$$

Hence, for $k = m+1$, it follows from inequalities (21), (B9), (B15) and (B17) that,

$$\begin{aligned} &E(V(t, \boldsymbol{e}(t))) \\ &\leq E(\widetilde{V}(t_m)) \exp\left(-\nu(t - t_m)\right) \\ &= \sup_{s \in [t_m - \tau, t_m]} E(V(s, \boldsymbol{e}(s))) \exp\left(-\nu(t - t_m)\right) \\ &= \max\left\{ \sup_{s \in [t_m - \tau, t_m)} E(V(s, \boldsymbol{e}(s))),\ E(V(t_m, \boldsymbol{e}(t_m))) \right\} \\ &\quad \times \exp\left(-\nu(t - t_m)\right) \\ &\leq \max\Bigg\{ \rho^{m-1} E(\widetilde{V}(t_0)) \exp\left(-\nu(t_m - \tau - t_0)\right), \\ &\quad \rho^m E(\widetilde{V}(t_0)) \exp\left(-\nu(t_m - t_0)\right) \Bigg\} \exp\left(-\nu(t - t_m)\right) \\ &= \max\left\{\rho^{m-1} e^{\nu\tau},\ \rho^m\right\} E(\widetilde{V}(t_0)) \exp(-\nu(t_m - t_0)) \\ &\quad \times \exp(-\nu(t - t_m)) \\ &\leq \rho^m E(\widetilde{V}(t_0)) \exp\left(-\nu(t - t_0)\right). \end{aligned}$$

Therefore, by induction, inequalities (B15) holds for all $k \in N$.

Since $\tau_m = \inf\{t_k - t_{k-1}\}$, one has $k - 1 \leq \dfrac{t_{k-1} - t_0}{\tau_m}$, which implies

$$\rho^{k-1} \leq \exp\left(\frac{\ln\rho}{\tau_m}(t_{k-1} - t_0)\right). \tag{B18}$$

Thus, for $t \in [t_{k-1}, t_k)$, from inequalities (B15) and (B18), we have

$$\begin{aligned} &E(\|\boldsymbol{e}(t)\|^2) \\ &\leq \frac{\rho^{k-1}}{\lambda_{\min}(\boldsymbol{P})} E(\widetilde{V}(t_0)) \exp\left(-\nu(t - t_0)\right) \\ &\leq \frac{\lambda_{\max}(\boldsymbol{P})}{\lambda_{\min}(\boldsymbol{P})} \sup_{s \in [t_0 - \tau, t_0]} E(\|\phi(s)\|^2) \\ &\quad \times \exp\left(-(\nu - \frac{\ln\rho}{\tau_m})(t - t_0)\right). \end{aligned}$$

Thus, the synchronization scheme (16) achieves global exponential synchronization in mean square, and the exponential synchronization rate is $\left(\nu - \dfrac{\ln\rho}{\tau_m}\right)$. This completes the proof.

Appendix C

Proof of Theorem 2

From Assumption 4 with $\Delta\boldsymbol{A}(t)$ and $\Delta\boldsymbol{B}(t)$, replace $\boldsymbol{A}$ and $\boldsymbol{B}$ in condition (19) with $\boldsymbol{A} + \boldsymbol{E}_1\boldsymbol{F}_1(t)\boldsymbol{H}_1$ and $\boldsymbol{B} + \boldsymbol{E}_2\boldsymbol{F}_2(t)\boldsymbol{H}_2$, respectively, then one has

$$\begin{aligned} \mathit{\Pi}_1 = \mathit{\Pi} &+ \boldsymbol{X}_1\boldsymbol{F}_1(t)\boldsymbol{Y}_1 + \boldsymbol{Y}_1^{\mathrm{T}}\boldsymbol{F}_1^{\mathrm{T}}(t)\boldsymbol{X}_1^{\mathrm{T}} \\ &+ \boldsymbol{X}_2\boldsymbol{F}_2(t)\boldsymbol{Y}_2 + \boldsymbol{Y}_2^{\mathrm{T}}\boldsymbol{F}_2^{\mathrm{T}}(t)\boldsymbol{X}_2^{\mathrm{T}}, \end{aligned} \tag{C1}$$

where

$$\boldsymbol{X}_1 = [\,\boldsymbol{E}_1^{\mathrm{T}}\boldsymbol{P} \quad 0 \quad 0\,]^{\mathrm{T}}, \quad \boldsymbol{Y}_1 = [\,\boldsymbol{H}_1 \quad 0 \quad 0\,],$$

$$\boldsymbol{X}_2 = [\,\boldsymbol{E}_2^{\mathrm{T}}\boldsymbol{P} \quad 0 \quad 0\,]^{\mathrm{T}}, \quad \text{and} \quad \boldsymbol{Y}_2 = [\,\boldsymbol{H}_2 \quad 0 \quad 0\,],$$

By using Lemma 3 with expression (C1), we have

$$\begin{aligned} \mathit{\Pi}_2 &= \mathit{\Pi} + \mu_1^{-1}\boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}} + \mu_1\boldsymbol{Y}_1^{\mathrm{T}}\boldsymbol{Y}_1 \\ &\quad + \mu_2^{-1}\boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}} + \mu_2\boldsymbol{Y}_2^{\mathrm{T}}\boldsymbol{Y}_2 \\ &= \begin{bmatrix} \widetilde{\Psi}_{11} & \widetilde{\Psi}_{12} & -\boldsymbol{P}\boldsymbol{M}\boldsymbol{H} \\ * & \widetilde{\Psi}_{22} & 0 \\ * & * & \widetilde{\Psi}_{33} \end{bmatrix} \\ &\quad + \mu_1^{-1}\boldsymbol{X}_1\boldsymbol{X}_1^{\mathrm{T}} + \mu_2^{-1}\boldsymbol{X}_2\boldsymbol{X}_2^{\mathrm{T}}. \end{aligned} \tag{C2}$$

From Lemma 2, it is easy to see that the condition $\widetilde{\mathit{\Pi}} < 0$ in condition (23) is equivalent to $\mathit{\Pi}_2 < 0$ in

expression (C2). Moreover, from Lemma 3, $\Pi_2 < 0$ is also equivalent to $\Pi_1 < 0$. Hence, we can obtain

$$\begin{aligned}
&\mathscr{L}V(t,\boldsymbol{e}(t))\\
&= V_t(t,\boldsymbol{e}(t)) + V_e(t,\boldsymbol{e}(t))[(\boldsymbol{A}+\Delta\boldsymbol{A}(t)+\boldsymbol{K})\boldsymbol{e}(t)\\
&\quad - \boldsymbol{M}\boldsymbol{H}\boldsymbol{e}_\tau(t) + (\boldsymbol{B}+\Delta\boldsymbol{B}(t))\eta(\boldsymbol{C}\boldsymbol{e}(t),\boldsymbol{y}(t))]\\
&\quad + \frac{1}{2}\mathrm{trace}[\vartheta^{\mathrm{T}}(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))V_{ee}(t,\boldsymbol{e}(t))\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))]\\
&= 2\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}[(\boldsymbol{A}+\Delta\boldsymbol{A}(t)+\boldsymbol{K})\boldsymbol{e}(t) - \boldsymbol{M}\boldsymbol{H}\boldsymbol{e}_\tau(t)\\
&\quad + (\boldsymbol{B}+\Delta\boldsymbol{B}(t))\eta(\boldsymbol{C}\boldsymbol{e}(t),\boldsymbol{y}(t))]\\
&\quad + \mathrm{trace}[\vartheta^{\mathrm{T}}(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))\boldsymbol{P}\vartheta(t,\boldsymbol{e}(t),\boldsymbol{e}_\tau(t))]\\
&\le \boldsymbol{e}^{\mathrm{T}}(t)\Bigg[(\boldsymbol{A}+\Delta\boldsymbol{A}(t)+\boldsymbol{K})^{\mathrm{T}}\boldsymbol{P} + \boldsymbol{P}(\boldsymbol{A}+\Delta\boldsymbol{A}(t)+\boldsymbol{K})\\
&\quad + \gamma\Sigma_1^{\mathrm{T}}\Sigma_1\Bigg]\boldsymbol{e}(t) - 2\boldsymbol{e}^{\mathrm{T}}(t)PMHe_\tau(t)\\
&\quad + 2\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{B}+\Delta\boldsymbol{B}(t))\eta(\boldsymbol{C}\boldsymbol{e}(t),\boldsymbol{y}(t))\\
&\quad - 2\sum_{i=1}^{n_h}\lambda_i\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y})(\eta_i(\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e},\boldsymbol{y}) - \kappa\boldsymbol{c}_i^{\mathrm{T}}\boldsymbol{e})\\
&\quad + \gamma\boldsymbol{e}_\tau^{\mathrm{T}}(t)\Sigma_2^{\mathrm{T}}\Sigma_2\boldsymbol{e}_\tau(t)\\
&= \xi^{\mathrm{T}}(t)\Pi_1\xi(t) - \alpha\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) + \beta\boldsymbol{e}_\tau^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_\tau(t)\\
&\le -\alpha\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t) + \beta\boldsymbol{e}_\tau^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_\tau(t),
\end{aligned}$$

where $\xi(t)$ and $V(t,\boldsymbol{e}(t))$ are defined in Theorem 1. Therefore, we have

$$\begin{aligned}
&D^+E(V(t,\boldsymbol{e}(t)))\\
&= E(\mathscr{L}V(t,\boldsymbol{e}(t)))\\
&\le -\alpha E(V(t,\boldsymbol{e}(t)))\\
&\quad + \beta\sup_{s\in[t-\tau,t]}E(V(s,\boldsymbol{e}(s))).
\end{aligned}$$

The remaining proof is similar to the proof of Theorem 1, which is omitted here. This completes the proof.

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