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Dispersion relation of excitation mode in strongly interacting fermions matter*

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This paper analyses the dispersion relation of the excitation mode in non-relativistic interacting fermion matter. The polarization tensor is calculated with the random phase approximation in terms of finite temperature field theory. With the polarization tensor, the influences of temperature, particle number density and interaction strength on the dispersion relation are discussed in detail. It finds that the collective effects are qualitatively more important in the unitary fermions than those in the finite contact interaction matter.

Keywords: dispersion relation, unitary fermions, polarization tensor
PACC: 0530F, 6740D, 7145G

1. Introduction

With low temperature refrigeration technology breakthrough, the quantum phenomenon of Bose–Einstein condensation (BEC) about weak interaction Bose gas was observed by several experimental groups.[1,2] The prediction made by Bose and Einstein was confirmed.

Gradually, the international physics raises the unprecedented research upsurge to ultra-cold atomic fermions superfluidity. In recent years, the remarkable achievements in the physics of cold Fermi gases with tunable interactions have opened up intriguing possibilities to study the crossover from bosonic to fermionic behaviour in strongly interacting many-body quantum systems.[3]

With the Feshbach resonance technology, the ultra-cold fermionic atoms undergoing BEC have indeed been observed for ^{40}K[4] as well as for ^{6}Li.[5−7] By sweeping an external magnetic field, the scattering length-interaction strength is very large in the vicinity of the resonance. The scattering length between the atoms can be negative on one side of the resonance and positive on the other side. Therefore, a Bardeen–Cooper–Schrieffer (BCS) superfluidity of paired fermions is smoothly connected with a Bose–Einstein condensate of dimer-molecules. In this so-called BCS–BEC crossover resonance physics, the divergent S-wave scattering length implies the thermodynamic universality; i.e., the thermodynamic property does not reply on the system details.[8−15] Much attention has been attracted to this challenging BCS–BEC crossover quantum many-body physics.

The statistical and thermodynamical properties of strongly interacting fermions have become an extensively investigated topic in the literature. Some progresses have been achieved in the theoretical description of the zero temperature ground state, and fundamental properties have been experimentally tested with considerable accuracy. In principle, finite temperature properties of the strongly interacting fermions physics are more intriguing than the theoretical ground state energy.

From the many-body viewpoint of interacting system, the basic topic is about the dispersion relation of the excitation mode, which relates the energy ω to the momentum $\boldsymbol{k}$. The information about the excitation mode propagation and experimental measurement of the equation of state can be reflected by the dispersion relation.[8] Most of the important parameters, such as resonance point position, energy spectrum width or particle lifetime, can be characterized by the dispersion equation. In this paper, we discuss the collective mode dispersion relations excited in the ultra-cold fermions system with the imaginary-time formalism finite temperature field theory.

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The paper is organized as follows. In Section 2, the polarization tensor is calculated with the non-relativistic random phase approximation (RPA). The analytical properties of the polarization tensor and the static infrared limit will be discussed in detail. In Section 3, the particle number density n is calculated by the fermion propagator. In Section 4, the dispersion relation of excitation mode is discussed by the pole of in-medium two-particle T^*-matrix. The influences of temperature, scattering length and particle number on the dispersion relation will be analysed and the self-consistent numerical results are presented. The conclusion remarks are made in Section 5.

Throughout the paper, the natural units $\hbar = k_{\rm B} = 1$ are employed.

2. Polarization tensor with random phase approximation

In discussing the dispersion relation of the collective excitation mode, one of the central task is addressing the polarization tensor $\Pi^*(\omega_n, \boldsymbol{p})$.[16,17]

With finite temperature field theory, the polarization tensor $\Pi^*(\omega_n, \boldsymbol{p})$ is calculated through the non-relativistic RPA

$$\Pi^*(\omega_n, \boldsymbol{p}) = T\sum_{\omega_1}\int_k \mathcal{G}(\omega_1, \boldsymbol{k})\mathcal{G}(\omega_1+\omega_n, \boldsymbol{k}+\boldsymbol{p}). \quad (1)$$

The $\mathcal{G}$ appearing in Eq.(1) is the fermion Green function

$$\mathcal{G}(\omega_1, \boldsymbol{k}) = \frac{1}{{\rm i}\omega_1 - (\varepsilon_{\boldsymbol{k}} - \mu)}, \quad (2)$$

which is given by the self-consistent mean-field theory. The μ is the effective chemical potential.

The calculation formalism is standard. With Eq.(2), the polarization tensor $\Pi^*(\omega_n, \boldsymbol{p})$ can be rewritten as

$$\begin{aligned}\Pi^*(\omega_n, \boldsymbol{p}) =& T\sum_{\omega_1}\int_k \frac{1}{{\rm i}\omega_1 - (\varepsilon_{\boldsymbol{k}} - \mu)} \\ & \frac{1}{{\rm i}(\omega_1+\omega_n) - (\varepsilon_{\boldsymbol{k}+\boldsymbol{p}} - \mu)}. \end{aligned} \quad (3)$$

Equation (3) is the typical formalism of summation to discrete frequency, which has $|\omega_1|^{-2}$ magnitude along with $|\omega_1| \longrightarrow \infty$. Usually, this summation is not absolutely convergent. For the sake of performing the summation, one can insert a redundant convergent factor ${\rm e}^{{\rm i}\omega_1}$. This technique divides the expression into two sub-terms and each term can be calculated with the residual theorem.

Let us firstly recall Matsubara discrete frequency summing technique. The analytical expression including such as the discrete frequency summation $\sum_n \frac{{\rm e}^{{\rm i}\omega_n\eta}}{{\rm i}\omega_n - x}$ is not absolutely convergent. For fermions in this work, the discrete frequency ω_n is $\omega_n = (2n+1)\pi T$. Note that the $-\beta({\rm e}^{\beta z}+1)^{-1}$ has a pole in the odd integer $z = {\rm i}\omega_n$ position, and the residual is 1. According to the residual theorem

$$\sum_n \frac{{\rm e}^{{\rm i}\omega_n\eta}}{{\rm i}\omega_n - x} = \frac{-\beta}{{\rm i}2\pi}\int_c \frac{{\rm d}z}{{\rm e}^{z\beta}+1}\frac{{\rm e}^{z\eta}}{z-x}, \quad (4)$$

the loop integral with a pole ($z = x$) is

$$\lim_{\eta\to 0}\sum_n \frac{{\rm e}^{{\rm i}\omega_n\eta}}{{\rm i}\omega_n - x} = \frac{\beta}{{\rm e}^{\beta x}+1}. \quad (5)$$

With Eq.(4), the polarization tensor $\Pi^*(\omega_n, \boldsymbol{p})$ is reduced to

$$\begin{aligned}\Pi^*(\omega_n, \boldsymbol{p}) =& T\int_k \frac{1}{{\rm i}\omega_n - (\varepsilon_{\boldsymbol{k}+\boldsymbol{p}} - \varepsilon_{\boldsymbol{k}})}\sum_{\omega_1}{\rm e}^{{\rm i}\omega_1\eta} \\ & \times\left[\frac{1}{{\rm i}\omega_1 - (\varepsilon_{\boldsymbol{k}} - \mu)} - \frac{1}{{\rm i}(\omega_1+\omega_n) - (\omega_{\boldsymbol{k}+\boldsymbol{p}} - \mu)}\right] \\ =& -\int_k \frac{f_{\boldsymbol{k}+\boldsymbol{p}} - f_{\boldsymbol{k}}}{{\rm i}\omega_n - (\varepsilon_{\boldsymbol{k}+\boldsymbol{p}} - \varepsilon_{\boldsymbol{k}})}. \end{aligned} \quad (6)$$

Here,

$$f_{\boldsymbol{k}} = \frac{1}{{\rm e}^{\beta(\frac{k^2}{2m} - \mu)}+1} \quad (7)$$

is the Fermi-Dirac function and $\beta = 1/T$ is the inverse temperature in unit Boltzmann constant. In this work, the shorthand notation $\int_k = \int {\rm d}^3\boldsymbol{k}/(2\pi)^3$ represents the momentum integral. The m is the bare fermion mass.

Substituting $\int {\rm d}^3\boldsymbol{k} = \int_0^\infty k^2{\rm d}k\int_0^\pi \sin\theta{\rm d}\theta\int_0^{2\pi}{\rm d}\phi$ into Eq.(6), one can have

$$\Pi^*(\omega_n,\boldsymbol{p}) = \frac{m}{2\pi^2}\int_0^\infty \frac{kf_k\mathrm{d}k}{2p}\left[\ln\frac{\mathrm{i}2m\omega_n - p^2 + 2pk}{\mathrm{i}2m\omega_n - p^2 - 2pk} - \ln\frac{\mathrm{i}2m\omega_n + p^2 + 2pk}{\mathrm{i}2m\omega_n + p^2 - 2pk}\right] \tag{8}$$

with $p = |\boldsymbol{p}|$, $k = |\boldsymbol{k}|$.

2.1. Analytical properties of polarization tensor

We now analyse the analytical property of the polarization tensor. Taking the analytical continuation for the Matsubara frequency ω_n according to $\mathrm{i}\omega_n \to \omega + \mathrm{i}\eta$ with $\eta \to 0^+$, one can have

$$\begin{aligned}\Pi^*(\omega,\boldsymbol{p}) =& \frac{m}{2\pi^2}\int_0^\infty \frac{kf_k\mathrm{d}k}{2p}\left[\ln\frac{2m\omega - p^2 + 2pk}{2m\omega - p^2 - 2pk} - (-p^2 \to +p^2)\right]\\ =& \frac{m}{2\pi^2}\int_0^\infty \frac{kf_k\mathrm{d}k}{2p}\ln b.\end{aligned} \tag{9}$$

Noting that the variable b can be negative or positive, one should carefully take care of the singularity resulting from the b in above expression. It is well known that the negative b will lead to the logarithm singularity. This kind of singularity is related with the famous Landau damping in correlated quantum many-body system.

If $b < 0$, i.e.,

$$\frac{(p^2 - 2pk)^2 - 4m^2\omega^2}{(p^2 + 2pk)^2 - 4m^2\omega^2} < 0, \tag{10}$$

the polarization tensor $\Pi^*(\omega,\boldsymbol{p})$ will be complex. The solution of above inequality is determined by

$$|\frac{p^2 - 2m\omega}{2p}| < k < \frac{p^2 + 2m\omega}{2p}. \tag{11}$$

Therefore, the $\ln b$ can be generally expressed as

$$\ln b = \ln|\frac{(p^2 - 2pk)^2 - 4m^2\omega^2}{(p^2 + 2pk)^2 - 4m^2\omega^2}| + \mathrm{i}\pi\Theta. \tag{12}$$

$$\Theta = \begin{cases} 1, & \left|\dfrac{p^2 - 2m\omega}{2p}\right| < k < \dfrac{p^2 + 2m\omega}{2p} \\ 0, & \text{otherwise.}\end{cases}$$

In this work, the dispersion relation is determined by the real part of the polarization tensor.

2.2. Response susceptibility in static infrared limit

The polarization tensor tells us much information about the many-body collective properties. In the static infrared limit $\omega_n = 0$, $p \to 0$, it will give the general spin-spin or density-density response susceptibility

$$\chi \equiv -\lim_{p\to 0}\Pi^*(0,\boldsymbol{p}) = \frac{m}{2\pi^2}\int_0^\infty f_k\mathrm{d}k. \tag{13}$$

For $T \neq 0$, substituting the quasi-particle Fermi-Dirac distribution function Eq.(7) into Eq.(13) we will get

$$\chi = \frac{m}{2\pi^2}\int_0^\infty \frac{\mathrm{e}^{-(\frac{k^2}{2mT} - \frac{\mu}{T})}}{1 + \mathrm{e}^{-(\frac{k^2}{2mT} - \frac{\mu}{T})}}. \tag{14}$$

Utilizing

$$\frac{1}{1+\mathrm{e}^{-x}} = \sum_{\alpha=0}^\infty (-1)^\alpha \mathrm{e}^{-\alpha x}, \tag{15}$$

and the integral formula

$$\int_0^\infty x^2\mathrm{e}^{-\alpha x^2}\mathrm{d}x = \frac{1}{4\alpha}\sqrt{\frac{\pi}{\alpha}}, \tag{16}$$

one can further simplify the susceptibility as

$$\begin{aligned}\chi =& \frac{m}{2\pi^2}\int_0^\infty \sum_{\alpha=1}^\infty (-1)^{\alpha-1}\mathrm{e}^{-\alpha(\frac{k^2}{2mT} - \frac{\mu}{T})}\mathrm{d}k\\ =& -\frac{1}{T\lambda^3}\sum_{\alpha=1}^\infty (-\mathrm{e}^{\frac{\mu}{T}})^\alpha/\alpha^{\frac{1}{2}}\\ =& -\frac{1}{T\lambda^3}\mathrm{Li}_{1/2}[-\mathrm{e}^{\frac{\mu}{T}}],\end{aligned} \tag{17}$$

where $\lambda = \sqrt{\dfrac{2\pi}{mT}}$ is the thermodynamical de Broglie wavelength and $\mathrm{Li}_{1/2}[x]$ is the standard Fermi integral polylogarithm function.

Equation (17) is identical with $\Pi^*(0, |\boldsymbol{p}| \to 0) = -\dfrac{m_{\mathrm{D}}^2(T,\mu)}{2}$, the $m_{\mathrm{D}}^2(T,\mu)$ is the non-relativistic Debye screening mass squared with unit electric charge.[16]

3. Particle number density

In terms of finite temperature Green function theory, the fermion propagator can give the particle number density $n = N/V$ according to Ref.[18],

$$n = T\int_k \sum \mathrm{e}^{\mathrm{i}\omega_n\eta}\mathrm{tr}\mathcal{G}(\omega_n, \boldsymbol{k}), \tag{18}$$

with

$$\begin{aligned}\mathrm{tr}\mathcal{G}(\omega_n, \boldsymbol{k}) &= \sum_{\beta=\uparrow,\downarrow}\langle\beta|\mathcal{G}(\omega_n, \boldsymbol{k})|\beta\rangle \\ &= 2\mathcal{G}(\omega_n, \boldsymbol{k}).\end{aligned} \tag{19}$$

Here $\beta = \uparrow, \downarrow$ represent the spin projection. Consequently, the particle number density is

$$n = 2T\int_k \sum_{\omega_n} \mathrm{e}^{\mathrm{i}\omega_n\eta}\frac{1}{\mathrm{i}\omega_n - (\varepsilon_{\boldsymbol{k}} - \mu)}. \tag{20}$$

Once again utilizing the residual theorem, one can express the particle number n as

$$\begin{aligned}n &= 2\int_k \frac{1}{\mathrm{e}^{(\frac{k^2}{2mT} - \frac{\mu}{T})} + 1} \\ &= \frac{1}{\pi^2}\int_0^\infty k^2 \frac{1}{\mathrm{e}^{(\frac{k^2}{2mT} - \frac{\mu}{T})} + 1}\mathrm{d}k.\end{aligned} \tag{21}$$

Using Eqs.(15) and (16), one can have the final expression for the particle number density

$$\begin{aligned}n &= \frac{1}{\pi^2}\sum_{\alpha=1}^{\infty}(-1)^{\alpha-1}\mathrm{e}^{\frac{\alpha\mu}{T}}\int_0^\infty k^2\mathrm{e}^{-\frac{\alpha k^2}{2mT}}\mathrm{d}k \\ &= \frac{-2}{\lambda^3}\sum_{\alpha=1}^{\infty}(-\mathrm{e}^{\frac{\mu}{T}})/\alpha^{3/2} \\ &= -\frac{2}{\lambda^3}\mathrm{Li}_{3/2}[-\mathrm{e}^{\frac{\mu}{T}}].\end{aligned} \tag{22}$$

This is the expected result of mean-field theory; i.e., the particle density n is given by the total chemical potential μ and temperature T.

4. Dispersion relation of composite-dimer excitation mode

Furthermore, the dispersion relation of the composite boson/dimer excitation mode can be studied with the polarization tensor according to the response theory. The in-medium two-particle T^*-matrix depends on the frequency ω and the centre-of-mass momentum $\boldsymbol{p}$ according to

$$T^*(\omega, \boldsymbol{p}) = \frac{U_0}{1 - \mathrm{Re}\Pi^*(\omega, \boldsymbol{p})U_0}. \tag{23}$$

U_0 is the effective bare low energy interaction. It usually represents the zero range contact interaction and gives the S-wave scattering length a according to the Born approximation $U_0 = 4\pi a/m$. The dispersion relation of the excitation mode is determined by the pole position of the in-medium T^*-matrix Eq.(23), i.e,

$$\begin{aligned}&\int_0^{|\frac{p^2-2m\omega}{2p}|}\frac{kf_k}{2p}\ln|b|\mathrm{d}k \\ &+\int_{|\frac{p^2-2m\omega}{2p}|}^{\frac{p^2+2pk}{2p}}\frac{kf_k}{2p}\ln|b|\mathrm{d}k \\ &+\int_{\frac{p^2+2pk}{2p}}^{\infty}\frac{kf_k}{2p}\ln|b|\mathrm{d}k = \frac{2\pi}{a}.\end{aligned} \tag{24}$$

The dispersion relation curves can be obtained by numerically solving the above self-consistent equation Eq.(24) for different conditions. It is worth noting that the concrete temperature and density conditions are determined by Eq.(22). The numerical results are displayed in the Figs.1–4. The temperatures are rescaled according to the Fermi characteristic temperature $T_\mathrm{F} = k_\mathrm{F}^2/(2m)$ in all the figures with k_F being the Fermi momentum.

It should be stressed that although the expression is general, the temperature and particle number density in this work is assumed to satisfy the implicit strong degenerate condition $n\lambda^3 \geq 1$ in order to characterize the quantum many-body effects. Otherwise, the dispersion curves will almost coincide with each other and appear as the expected half parabola in the "high temperature" Boltzmann regimes.

In Fig.1, the dispersion curves describe the temperature effect on the dispersion property. For comparison, the interaction dilute parameter $k_\mathrm{F}a$ near the strongly interacting unitary limit $1/(k_\mathrm{F}a) = 0.5\times10^{-3}$ is fixed. From this figure, one can see that the dispersion relation changes significantly according to temperature. With the increase of temperature, the collective dispersion curve is more influenced (upward). As expected, the discrepancy between different temperatures will be more obvious for the high momentum regions.

Figure 2 shows the dispersion relation for the different finite scattering length a-interaction strength with fixed temperature. One can see that the dispersion curves are almost consistent with each other.

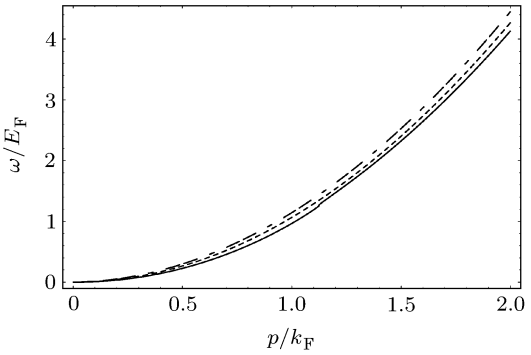


Fig.1. Dispersion relation $\omega/E_{\rm F}$ of the excitation mode as a function of $p/k_{\rm F}$ for $T = 1$ (lower line), $T = 10$ (middle line) and $T = 20$ (upper line) near the unitarity with $1/(k_{\rm F}a) = 0.5 \times 10^{-3}$.

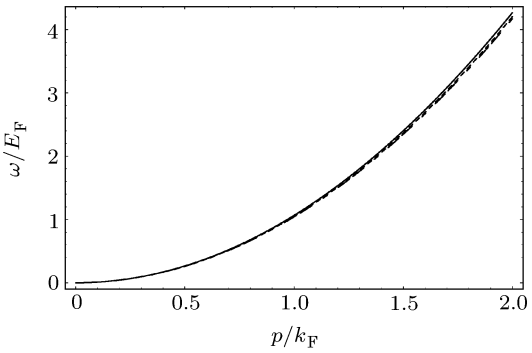


Fig.2. Dispersion relation for $1/(k_{\rm F}a) = 0.2 \times 10^{-3}$ (lower line), $1/(k_{\rm F}a) = 0.3 \times 10^{-3}$ (middle line) and $1/(k_{\rm F}a) = 0.5 \times 10^{-3}$ (upper line) with $T = 10$.

At unitarity with the infinite S-wave scattering length $|a| = \infty$, the dispersion relation is indicated in Fig.3. Although the characteristic is quite similar to the finite contact interaction, the medium effects reflected by the dispersion relation are more obvious in the unitary limit. This can be clearly seen by comparing Fig.3 with Fig.1.

Therefore, in the unitary limit, the collective finite temperature and density effects are very important. In Fig.4, we have given the dispersion relation for the different particle numbers with fixed temperature. As expected, the result characterizes a fundamental physical fact that the denser the particle number, the more important the collective quantum effects are.

It is very interesting to compare Fig.3 with Fig.4. One can see that the temperature and density effects can compete with each other. The temperature effect will drive up the dispersion curve while the density effect plays the inverse role.

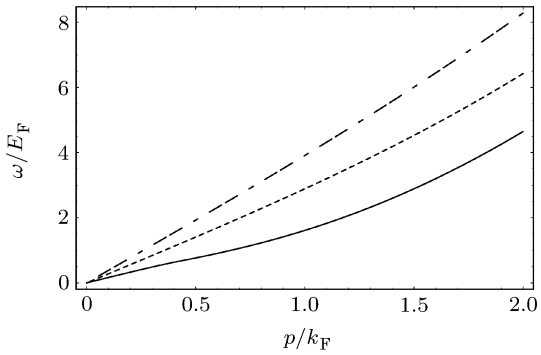


Fig.3. At unitarity, dispersion relation $\omega/E_{\rm F}$ for $T = 0$ (lower line), $T = 10$ (middle line) and $T = 20$ (upper line).

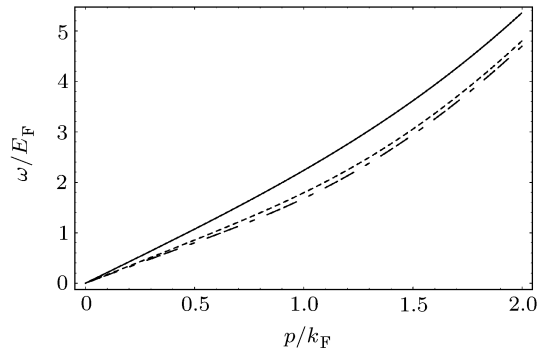


Fig.4. Dispersion relation $\omega/E_{\rm F}$ for particle number density $n = 10$ (lower line), $n = 5$ (middle line) and $n = 1$ (upper line) with fixed $T = 5$ at unitarity.

5. Conclusion

In summary, we have analysed the energy-momentum dispersion relation of the excitation mode at finite temperature and density for the non-relativistic contact interacting fermion system. The analytical property of the polarization tensor has been discussed in terms of the random phase approximation within the finite temperature Green function formalism. From the dispersion relation of the excitation mode, one can see that the unitary fermions property is quite different from the finite contact interaction case. In the strongly interacting limit, the collective temperature and density effects are more important than the finite interaction occasion, which can influence the thermodynamical and transport properties in a substantial way.

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