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On entanglement invariant for a double Jaynes–Cummings model*

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In a recent paper [*Phys. Rev.* A **76** 042313 (2007)], Sainz and Bjork introduced an entanglement invariant $\mathcal{E}$ under evolution for a system of four qubits interacting through two isolated Jaynes–Cummings Hamiltonians. This paper proves that this entanglement invariant $\mathcal{E}$ is closely connected with the linear entropy between two independent subsystems.

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Entanglement is one of the most counter-intuitive features in quantum mechanics which results in correlations stronger than those presented in any classical systems. Over the last two decades, one has witnessed entanglement as an useful (in many cases, irreplaceable) resource for quantum information processing[1,2] and quantum computing.[3] Consequently, the understanding of entangled states has become more and more important. Although the general theory of entanglement has blossomed enormously,[4] only bipartite entanglement is well understood with a good range of quantitative measure of entanglement available.[5] As for multipartite entanglement, the complexity increases greatly with the number of parties involved (see, e.g., Ref.[6]). The classification and quantification of multipartite entanglement is very complicated and challenging. To efficiently understand entanglement, we can formalize the theory of entanglement not only from the mathematical point of view, but also from a more practical approach in order to study its role and manifestations in realistic systems. Recently, many works have been contributed to understand entanglement with respect to the time-dependence (i.e., the entanglement dynamics) for macroscopic physical systems.[7−14] The authors of Refs.[12–14] considered how entanglement dynamics was related to the energy transfer between the system of interest and its environment. Particularly, Ref.[13] showed that, in a two-qubit system coupled with a single-mode cavity field, the entanglement of qubits (measured by concurrence) evolves almost in-step with their energy transfer. Also, in Ref.[14], where pairs of qubits were considered, the entanglement transfer process from one qubit pair to another was accompanied by an energy transfer as well, i.e., entanglement and energy transfer were strictly related. Ref.[9] dealt with a double Jaynes–Cummings model (JCM) in which two initially entangled two-level atoms A and B are independently coupled with separate cavity fields a and b, respectively, but there are no interactions at all between the subsystems Aa and Bb. Focusing on the atomic subsystem, depending on the type of initial $A-B$ entanglement, the authors of Ref.[9] found out that the concurrence $\mathcal{C}_{AB}(t)$[5] may or may not vanish completely in a finite time and remains zero for some time, a phenomenon called entanglement sudden death. They also formulated different forms of entanglement which are invariant for different initial atomic states.

Very recently, Sainz and Björk[15] studied entanglement dynamics of the same double JCM as in Ref.[9] but used the wedge product[16] for entanglement measure. The authors calculated the quantity E_{Aa-Bb} which counts for the entire bipartite entanglement between the atom–field systems Aa and Bb. However, E_{Aa-Bb} is not a universal invariant in general. It depends strongly on the initial state as well as on the local coupling constants (see Figs.1 and 2 in Ref.[15]). The main contribution of Ref.[15] is the finding of an entanglement invariant $\mathcal{E}$ under evolution, which demonstrates that entanglement spreads

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out over all of the system's degrees of freedom that become entangled through the local interactions

$$\begin{aligned}\mathcal{E} =&2E_{Aa-Bb}+E_{AB-ab}+E_{Ab-Ba}\\&-(E_{A-Bab}+E_{B-Aab}\\&+E_{a-ABb}+E_{b-ABa}).\end{aligned}\tag{1}$$

The quantity $\mathcal{E}$ remains unchanged in the course of time evolution for all system parameters. Although some explanations were given, a detailed derivation of the form of $\mathcal{E}$ was not provided in Ref.[15] (the authors themselves recognized that the interdependence of the various kinds of entanglement of the state is not very intuitive and a rigorous derivation of $\mathcal{E}$ is probably rather difficult). It is worth noting, however, that though $\mathcal{E}$ in Ref.[15] is time-independent it is so only for the initial pure X states of the atoms.

In the following, we shall show that the invariant $\mathcal{E}$ is strictly related to the linear entropy of the partition $Aa-Bb$, for an arbitrary initial state of the two atoms A and B, no matter it is a pure X state or not. The linear entropy is defined as[17]

$$E_{\rm L}(\rho_S)=\frac{d}{d-1}[1-{\rm Tr}(\rho_S^2)],\tag{2}$$

where ρ_S is a reduced density matrix obtained by tracing out the subsystem $\overline{S}$ (the complement of S) and $d=\min\{{\rm dim}S,{\rm dim}\overline{S}\}$, where ${\rm dim}S$ and ${\rm dim}\overline{S}$ are, respectively, the Hilbert space dimension of the subsystem S and its complement $\overline{S}$.

The double JCM is described by the Hamiltonian

$$H=\sum_{k=A,B}H^k,\tag{3}$$

where ($\hbar=1$)

$$H^k=\nu_k a_k^+a_k+\frac{\omega_k}{2}\sigma_z^k+g_k(a_k^+\sigma_-^k+a_k\sigma_+^k),\tag{4}$$

with ν_k the field frequency, ω_k the energy difference between the two states of an atom, g_k the coupling constant between the atom and the cavity field, a_k^+ (a_k) the creation (annihilation) operator for the field, $\sigma_z^k=|1\rangle_{kk}\langle 1|-|0\rangle_{kk}\langle 0|$, $\sigma_+^k=|1\rangle_{kk}\langle 0|$ and $\sigma_-^k=|0\rangle_{kk}\langle 1|$.

Instead of the pure atomic X states we consider an arbitrary initial state of the two atoms A and B in the general form

$$|\Psi(0)\rangle_{AB}=\sum_{i,j=0}^{1}x_{ij}|ij\rangle_{AB},\tag{5}$$

where $\sum_{i,j=0}^{1}|x_{ij}|^2=1$, $|ij\rangle_{AB}=|i\rangle_A\otimes|j\rangle_B$, assume the cavities to be in the vacuum $|00\rangle_{ab}=|0\rangle_a\otimes|0\rangle_b$. The total initial state of the whole system is thus

$$|\Phi(0)\rangle_{ABab}=\sum_{i,j=0}^{1}x_{ij}|ij\rangle_{AB}|00\rangle_{ab}.\tag{6}$$

At time $t>0$ the state (6) will evolve into

$$\begin{aligned}|\Phi(t)\rangle_{ABab}=&[C_1|0000\rangle+C_2|0100\rangle+C_3|0001\rangle\\&+C_4|1000\rangle+C_5|0010\rangle.\\&+C_6|1100\rangle+C_7|1001\rangle\\&+C_8|0110\rangle+C_9|0011\rangle]_{ABab},\end{aligned}\tag{7}$$

where C_n are some time-dependent coefficients. After tracing over the degrees of freedom of the subsystem Bb in state (7), we get a mixed state of the subsystem Aa characterized by the reduced density matrix

$$\rho_{Aa}=\begin{pmatrix}\varrho_{11}&\varrho_{12}&\varrho_{13}&0\\\varrho_{21}&\varrho_{22}&\varrho_{23}&0\\\varrho_{31}&\varrho_{32}&\varrho_{33}&0\\0&0&0&0\end{pmatrix},\tag{8}$$

where

$$\begin{aligned}\varrho_{11}=&|x_{00}|^2+|x_{01}|^2,\\\varrho_{12}=&\varrho_{21}^*=i(x_{00}x_{10}^*+x_{01}x_{11}^*)\sin g_At,\\\varrho_{13}=&\varrho_{31}^*=(x_{00}x_{10}^*+x_{01}x_{11}^*)\cos g_At,\\\varrho_{22}=&(|x_{10}|^2+|x_{11}|^2)\sin^2 g_At,\\\varrho_{23}=&\varrho_{32}^*=-i(|x_{10}|^2+|x_{11}|^2)\sin g_At\cos g_At,\\\varrho_{33}=&(|x_{10}|^2+|x_{11}|^2)\cos^2 g_At.\end{aligned}\tag{9}$$

By the definition given in Eq.(2) and using the normalization constraint $\sum_{i,j=0}^{1}|x_{ij}|^2=1$, we obtain the linear entropy $E_{\rm L}(\rho_{Aa})$ of ρ_{Aa} as

$$E_{\rm L}(\rho_{Aa})=\frac{8}{3}|x_{00}x_{11}-x_{01}x_{10}|^2.\tag{10}$$

Clearly, $E_{\rm L}(\rho_{Aa})$ is fully determined only by the initial conditions via the coefficients x_{ij}, i.e., it is independent of both the time t and the local coupling constants g_k. It is well known that the concurrence of an arbitrary initial two-qubit state (6) is $\mathcal{C}_{AB}(0)=2|x_{00}x_{11}-x_{01}x_{10}|$, so

$$E_{\rm L}(\rho_{Aa})=\frac{2}{3}\mathcal{C}_{AB}^2(0).\tag{11}$$

In particular, if the initial states of the atoms A and B are X states of the forms given by Eqs.(5) and (6) in Ref.[15], i.e., $x_{00}(x_{01})=\cos\alpha$, $x_{11}(x_{10})=\sin\alpha$ and $x_{01}(x_{00})=x_{10}(x_{11})=0$, then we get

$$E_{\rm L}(\rho_{Aa})=\frac{8}{3}\sin^2\alpha\cos^2\alpha=\frac{2}{3}\mathcal{C}_{AB}^2(0).\tag{12}$$

As shown in Ref.[15], the quantity $\mathcal{E}$ in Eq.(1) is also independent of time for these two types of initial states and the value of $\mathcal{E}$ in both cases is $\mathcal{E} = \sin^2\alpha\cos^2\alpha$. From this it follows an useful relation between their $\mathcal{E}$ and our $E_{\rm L}(\rho_{Aa})$:

$$\mathcal{E} = \frac{3}{8}E_{\rm L}(\rho_{Aa}). \tag{13}$$

In conclusion, we have studied the double JCM and proved that the linear entropy $E_{\rm L}(\rho_{Aa})$ is closely connected to the invariant $\mathcal{E}$ introduced in Ref.[15] via the relation (13). Provided that the cavities are initially in the vacuum, the linear entropy $E_{\rm L}(\rho_{Aa})$ is conserved in time. The conservation is universal in the sense that it is not influenced by the local atom-field coupling constants g_k and by the initial state of the atomic system $|\Psi(0)\rangle$. It can be shown that in the most generalized scenario of the multiple JCM with $H = \sum_{k=A_1,\ A_2,\ldots,A_N} H^k$ the linear entropy $E_{\rm L}(\rho_{A_1a_1})$ also remains a universal invariant. Quantitatively, the net value of $E_{\rm L}(\rho_{A_1a_1})$ is determined by the entanglement amount of the initial atomic state. The physical reason for variation of the initial atomic entanglement rests in the spreading out of entanglement over all of the system's degrees of freedom that may become entangled with each other in all possible ways through the underlying local interactions. Though the authors of Ref.[15] introduced a unique form of entanglement invariant which is independent of the initial (pure) atomic states, a solid derivation of their form was not given. Hence, our study would shed some light on the origin of entanglement invariant in the so-called multiple JCM.

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