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Routh method of reduction for Birkhoffian systems in the event space*

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For a Birkhoffian system in the event space, this paper presents the Routh method of reduction. The parametric equations of the Birkhoffian system in the event space are established, and the definition of cyclic coordinates for the system is given and the corresponding cyclic integral is obtained. Through the cyclic integral, the order of the system can be reduced. The Routh functions for the Birkhoffian system in the event space are constructed, and the Routh method of reduction is successfully generalized to the Birkhoffian system in the event space. The results show that if the system has a cyclic integral, then the parametric equations of the system can be reduced at least by two degrees and the form of the equations holds. An example is given to illustrate the application of the results.

Keywords: event space, Birkhoffian system, reduction, cyclic integral
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1. Introduction

For a holonomic conservative mechanical system, the Lagrangian equations of the system can be reduced by cancelling the cyclic coordinates through the construction of Routh function using cyclic integrals, and the form of Lagrangian equations holds after reduction, which is the famous Routh method of reduction in analytical mechanics.[1,2] In last 20 years, more attentions have been paid to the Routh method of reduction for constrained mechanical systems, and there have been some important results of research on Routh reduction.[3−8] But the research on Routh reduction for Birkhoffian systems in the event space has not been reported yet. In this paper, the Routh method of reduction for Lagrangian system is successfully generalized to the Birkhoffian system in the event space. By construction of a series of new dynamical functions called Routh functions of the Birkhoffian system, we study the reduction problem for Birkhoffian systems in the event space.

2. The cyclic integral for a Birkhoffian system in the event space

We consider a Birkhoffian system which is determined by the Birkhoff variables $a^{\mu}\,(\mu=1,\ldots,2n)$. Now we construct an event space of $(2n+1)$ dimensions. The coordinates of space points are $a^{\mu}\,(\mu=1,\ldots,2n)$ and t, which may be regarded as totally independent variables. Introducing the notation

$$x_1=t,\quad x_{\mu+1}=a^{\mu},\quad (\mu=1,\ldots,2n) \tag{1}$$

we can give all the variables $x_{\alpha}\,(\alpha=1,\ldots,2n+1)$ as functions of some parameter τ. Let $x_{\alpha}=x_{\alpha}\,(\tau)$ be some curves of class C^2 such that

$$\frac{\mathrm{d}x_{\alpha}}{\mathrm{d}\tau}=x'_{\alpha} \tag{2}$$

are not all zero at the same time, and we have

$$\dot{x}_{\alpha}=\frac{\mathrm{d}x_{\alpha}}{\mathrm{d}t}=\frac{x'_{\alpha}}{x'_1}. \tag{3}$$

For a given Birkhoffian $B=B\,(t,a^{\nu})$ and given Birkhoff's functions $R_{\mu}=R_{\mu}\,(t,a^{\nu})\,(\mu=1,\ldots,2n)$, the Birkhoff's functions $B_{\alpha}\,(x_{\beta})\,(\alpha=1,\ldots,2n+1)$ of parametric form in the event space are defined by the equalities

$$\begin{aligned}&B_1\,(x_{\beta})=-B\,(x_1,x_2,\ldots,x_{2n+1})\,,\\&B_{\mu+1}\,(x_{\beta})=R_{\mu}\,(x_1,x_2,\ldots x_{2n+1})\,,\\&(\mu=1,\ldots,2n)\,.\end{aligned} \tag{4}$$

The parametric equations in the event space for the Birkhoffian system can be written in the following form

$$\left(\frac{\partial B_{\beta}}{\partial x_{\alpha}}-\frac{\partial B_{\alpha}}{\partial x_{\beta}}\right)x'_{\beta}=0,\quad (\alpha=1,\ldots,2n+1)\,. \tag{5}$$

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If the Birkhoff's functions B_α do not explicitly include some variable $x_k\,(1 \le k \le 2n+1)$ in the event space, i.e.

$$\frac{\partial B_\alpha}{\partial x_k} = 0, \quad (\alpha = 1, \ldots, 2n+1; 1 \le k \le 2n+1), \tag{6}$$

then x_k is called a cyclic coordinate of the Birkhoffian system in event space.

Since x_k is a cyclic coordinate, the k-th equation of the parametric equations (5) takes the form

$$\left(-\frac{\partial B_k}{\partial x_\beta}\right) x'_\beta = 0, \tag{7}$$

that is

$$-\frac{\mathrm{d}B_k}{\mathrm{d}\tau} = 0, \tag{8}$$

therefore

$$B_k\left(x_1, x_2, \ldots, x_{k-1}, x_{k+1}, \ldots, x_{2n+1}\right) = c_k, \tag{9}$$

here c_k is an arbitrary constant. Obviously, the integral (9) is a first integral of Eq.(5), called a cyclic integral corresponding to the cyclic coordinate x_k for the Birkhoffian system in the event space.

3. The Routh method of reduction for a Birkhoffian system in the event space

In 1877, Routh presented a method for reduction of the Lagrange equations which may cancel cyclic variables by using cyclic integrals.[1,2] The Routh method not only realize the aim of reduction but also make the reduced equations to keep the form of Lagrange equations. Now we try to generalize the work of Routh to the Birkhoffian system in the event space.

If x_k is a cyclic coordinate, then the system has a cyclic integral (9). Assume that some x_j can be expressed as a function of variables $x_\beta\,(\beta \ne k, j)$, c_k from Eq.(9), that is

$$x_j = f_j\left(x_\beta, c_k\right), \quad (\beta \ne k, j). \tag{10}$$

Now we construct the Routh functions of the Birkhoffian system in the event space as

$$B^*_\alpha = B_\alpha - \frac{\partial}{\partial x_\alpha}\left(B_j f_j\right), \quad (\alpha = 1, \ldots, 2n+1). \tag{11}$$

Substituting Eq.(10) into Eq.(11), we have

$$B^*_\alpha = \hat{B}_\alpha - \frac{\partial}{\partial x_\alpha}\left(\hat{B}_j f_j\right) = B^*_\alpha\left(x_\beta, c_k\right), \quad (\alpha = 1, \ldots, 2n+1; \beta \ne k, j) \tag{12}$$

From Eq.(12), we get

$$\frac{\partial B^*_\alpha}{\partial x_\beta} = \frac{\partial \hat{B}_\alpha}{\partial x_\beta} - \frac{\partial^2\left(\hat{B}_j f_j\right)}{\partial x_\alpha \partial x_\beta}. \tag{13}$$

Since

$$\frac{\partial \hat{B}_\alpha}{\partial x_\beta} = \frac{\partial B_\alpha}{\partial x_\beta} + \frac{\partial B_\alpha}{\partial x_j}\frac{\partial f_j}{\partial x_\beta}, \quad (\alpha, \beta = 1, \ldots, 2n+1; \beta \ne k, j), \tag{14}$$

substituting Eq.(14) into Eq.(13), we have

$$\frac{\partial B_\alpha}{\partial x_\beta} = \frac{\partial B^*_\alpha}{\partial x_\beta} - \frac{\partial B_\alpha}{\partial x_j}\frac{\partial f_j}{\partial x_\beta} + \frac{\partial^2\left(\hat{B}_j f_j\right)}{\partial x_\alpha \partial x_\beta}, \quad (\alpha, \beta = 1, \ldots, 2n+1; \beta \ne k, j). \tag{15}$$

Using Eq.(15), and considering the relation (10), we obtain

$$\begin{aligned}
&\left(\frac{\partial B_\beta}{\partial x_\alpha} - \frac{\partial B_\alpha}{\partial x_\beta}\right) x'_\beta \\
&= \left(\frac{\partial B^*_\beta}{\partial x_\alpha} - \frac{\partial B^*_\alpha}{\partial x_\beta}\right) x'_\beta + \frac{\partial B_\alpha}{\partial x_j}\left(\frac{\partial f_j}{\partial x_\beta} x'_\beta - x'_j\right) \\
&\quad - \frac{\partial f_j}{\partial x_\alpha}\left(\frac{\partial B_\beta}{\partial x_j} x'_\beta\right) \\
&= \left(\frac{\partial B^*_\beta}{\partial x_\alpha} - \frac{\partial B^*_\alpha}{\partial x_\beta}\right) x'_\beta - \frac{\partial f_j}{\partial x_\alpha}\left(\frac{\partial B_\beta}{\partial x_j} x'_\beta\right).
\end{aligned} \tag{16}$$

From Eq.(11), we get

$$\frac{\partial B^*_\alpha}{\partial c_k} = -B_j \frac{\partial^2 f_j}{\partial x_\alpha \partial c_k} - \frac{\partial B_j}{\partial x_\alpha}\frac{\partial f_j}{\partial c_k}. \tag{17}$$

Multiplying both sides of Eq.(17) by x'_α, and summing up relative to α, we have

$$\begin{aligned}
\frac{\partial B^*_\alpha}{\partial c_k} x'_\alpha &= -B_j \frac{\partial^2 f_j}{\partial x_\alpha \partial c_k} x'_\alpha - \frac{\partial B_j}{\partial x_\alpha}\frac{\partial f_j}{\partial c_k} x'_\alpha \\
&= -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(B_j \frac{\partial f_j}{\partial c_k}\right).
\end{aligned} \tag{18}$$

Taking notice of Eq.(11) and Eq.(9), we have

$$B^*_k = B_k = c_k. \tag{19}$$

Substituting Eq.(19) into Eq.(18), we get

$$x'_k = -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(B_j\frac{\partial f_j}{\partial c_k}\right) - \frac{\partial B^*_\alpha}{\partial c_k}x'_\alpha,$$
$$(\alpha = 1,\ldots,2n+1;\alpha\neq k)\,. \tag{20}$$

Integrating Eq.(20) with respect to the parameter τ, we obtain

$$x_k = -B_j\frac{\partial f_j}{\partial c_k} - \int\frac{\partial B^*_\alpha}{\partial c_k}x'_\alpha\mathrm{d}\tau,$$
$$(\alpha = 1,\ldots,2n+1;\quad \alpha\neq k)\,. \tag{21}$$

From Eqs.(5), (16) and (21), we can obtain the following proposition.

Proposition 1 For a Birkhoffian system (5) in the event space, if $x_k\,(1\leq k\leq 2n+1)$ is a cyclic coordinate, and satisfies the following conditions

$$\frac{\partial f_j}{\partial x_\alpha}\left(\frac{\partial B_\beta}{\partial x_j}x'_\beta\right) = 0,\quad (\alpha = 1\ldots 2n+1)\,, \tag{22}$$

then the $(2n+1)$ Birkhoffian equations (5) can be reduced to $(2n-1)$ equations as follows:

$$\left(\frac{\partial B^*_\beta}{\partial x_\alpha} - \frac{\partial B^*_\alpha}{\partial x_\beta}\right)x'_\beta = 0,$$
$$(\alpha = 1,\ldots,2n+1;\alpha\neq k,j)\,, \tag{23}$$

where B^*_α are determined by Eq.(12). Equation (23) may be called Routh equations of the Birkhoffian system in the event space. In the meanwhile, x_j, x_k are given by Eq.(10) and Eq.(21) respectively.

From Proposition 1, we can obtain the following propositions.

Proposition 2 For a Birkhoffian system (5) in the event space, if $x_k\,(1\leq k\leq 2n+1)$ is a cyclic coordinate, and from the cyclic integral $B_k = c_k$ some x_j can be expressed as

$$x_j = f_j\,(c_k)\,, \tag{24}$$

then the $(2n+1)$ Birkhoffian equations (5) can be reduced to $(2n-1)$ Routh equations (23), and x_k is given by Eq.(21).

Proposition 3 For a Birkhoffian system (5) in the event space, if x_k, x_j are cyclic coordinates, then the $(2n+1)$ Birkhoffian equations (5) can be reduced to $(2n-1)$ Routh equations (23), and x_j, x_k are given by Eq.(10) and Eq.(21) respectively.

4. Example

As an example, we study the reduction problem of a fourth-order Birkhoffian system in the event space, whose Birkhoffian and Birkhoff's functions are[9]

$$B = a^2 + \frac{1}{2}\left[\left(a^3\right)^2 + \left(a^4\right)^2\right],$$
$$R_1 = a^3, R_2 = a^4, R_3 = R_4 = 0. \tag{25}$$

Firstly, we establish the parametric equations of the system in the event space. Equation (4) gives

$$B_1 = -x_3 - \frac{1}{2}\left(x_4^2 + x_5^2\right),\quad B_2 = x_4,$$
$$B_3 = x_5,\quad B_4 = B_5 = 0. \tag{26}$$

The parametric equations represented by Eq.(5) give

$$x'_3 + x_4x'_4 + x_5x'_5 = 0,$$
$$-x'_4 = 0,\quad -x'_1 - x'_5 = 0,$$
$$-x_4x'_1 + x'_2 = 0,$$
$$-x_5x'_1 + x'_3 = 0. \tag{27}$$

Obviously, x_2 is a cyclic coordinate, therefore, the system has a cyclic integral

$$B_2 = c_2 = \text{const.} \tag{28}$$

From Eq.(28), we get

$$x_4 = f_4 = c_2. \tag{29}$$

From Eq.(12) we have

$$B^*_1 = -x_3 - \frac{1}{2}\left(c_2^2 + x_5^2\right),$$
$$B^*_2 = c_2,\quad B^*_3 = x_5,\quad B^*_4 = B^*_5 = 0. \tag{30}$$

Hence the Routh equations represented by Eq.(23) give

$$x'_3 + x_5x'_5 = 0,$$
$$-x'_1 - x'_5 = 0,$$
$$-x_5x'_1 + x'_3 = 0. \tag{31}$$

Equation (31) is a set of three first order differential equations, comparing with Eq.(27), we find that the number of equations is reduced by 2.

According to Eq.(21), we get

$$x_2 = \int c_2x'_1\mathrm{d}\tau = c_2x_1 + c, \tag{32}$$

where c is an arbitrary constant. The motion of the system is given by Eqs.(31), (29) and (32).

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