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Lie symmetry and Hojman conserved quantity of Nambu system

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This paper studies the Lie symmetry and Hojman conserved quantity of the Nambu system. The determining equations of Lie symmetry for the system are given. The conditions for existence and the form of the Hojman conserved quantity led by the Lie symmetry for the system are obtained. Finally, an example is given to illustrate the application of the results.

Keywords: Nambu system, Lie symmetry, Hojman conserved quantity
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1. Introduction

The conservation laws for a dynamical system play an important role in the fields of mathematics, mechanics and physics. A recent development direction is to study the conserved quantities by using symmetrical methods of the systems in analytical mechanics, which is very important to learn the physical states and properties of the systems. There are three main symmetrical methods used to seek the conserved quantity of the dynamical system: Noether symmetry,[1] Lie symmetry,[2] and Mei symmetry.[3−5] Recently, many important achievements have been obtained in this popular research field.[3−11]

In 1973, Nambu proposed a generalization of ordinary Hamilton mechanics which is now called Nambu mechanics.[12] Mukunda *et al* studied the connections between the Nambu system and Hamilton system.[13−15] Ogawa *et al* presented a Lagrangian formulation for Nambu mechanics and advanced an application Euler equation.[16] Now, Nambu mechanics system is widely used in particle physics, cosmography, string theory, hydrodynamics, mathematical physics, and computational mathematics, etc.

The symmetries and conserved quantities of the Nambu system with high dimensions has been rarely studied so far. In this paper, we study the Lie symmetry and Hojman conserved quantity of the Nambu system. We construct the determining equations of Lie symmetry for the Nambu system. The conditions for existence and the form of the Hojman conserved quantity led by the Lie symmetry for the system are obtained.

2. The Nambu mechanical system

Suppose that the configuration of a mechanical system in n-dimensional phase space is determined by n generalized coordinates q_i $(i = 1, 2, \ldots, n)$. For any functions $A_i(\boldsymbol{q})$, we define the Nambu-Poisson bracket:

$$\{A_1(\boldsymbol{q}), A_2(\boldsymbol{q}), \ldots, A_n(\boldsymbol{q})\} \equiv \begin{pmatrix} \dfrac{\partial A_1}{\partial q_1} & \dfrac{\partial A_1}{\partial q_2} & \cdots & \dfrac{\partial A_1}{\partial q_n} \\ \dfrac{\partial A_2}{\partial q_1} & \dfrac{\partial A_2}{\partial q_2} & \cdots & \dfrac{\partial A_2}{\partial q_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial A_n}{\partial q_1} & \dfrac{\partial A_n}{\partial q_2} & \cdots & \dfrac{\partial A_n}{\partial q_n} \end{pmatrix}. \tag{1}$$

Introducing $n-1$ Nambu–Hamilton functions $H_k(\boldsymbol{q})$ $(k = 1, 2, \ldots, n-1)$, we can write the equations of motion of the Nambu mechanical system as

$$\dot{q}_i = \{q_i, H_1(\boldsymbol{q}), \ldots, H_{n-1}(\boldsymbol{q})\}, \quad (i = 1, 2, \ldots, n). \tag{2}$$

For any function $F(q_1, \ldots, q_n)$ in which the time is invariable, we have

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$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial(F, H_1, \ldots, H_{n-1})}{\partial(q_1, q_2, \ldots, q_n)} = \{F, H_1, \ldots, H_{n-1}\}.$$

The action of Nambu mechanical system is

$$\int_{C_{n-1}} \Omega^{(n-1)} = \int_{C_{n-1}} q_1 \mathrm{d}q_2 \wedge \ldots \wedge \mathrm{d}q_n - H_1 \mathrm{d}H_2 \wedge \ldots \wedge \mathrm{d}H_{n-1} \wedge \mathrm{d}t. \quad (3)$$

Apparently, when we take $n = 2$, this system degenerates to a Hamilton system.

3. Lie symmetry of Nambu system

The Lie symmetry is an invariance of differential equations under the infinitesimal transformations.[7]

Introducing the infinitesimal transformations

$$t^* = t + \Delta t, \quad q_i^*(t^*) = q_i(t) + \Delta q_i, \quad (i = 1, 2, \ldots, n), \quad (4)$$

we get the expanded formulae

$$t^* = t + \varepsilon \xi_0(t, \boldsymbol{q}), \quad q_i^*(t^*) = q_i(t) + \varepsilon \xi_i(t, \boldsymbol{q}), \quad (5)$$

where ε is an infinitesimal parameter, and ξ_0, ξ_i are the generators of infinitesimal transformations.

Expanding Eq.(2), we get

$$\dot{q}_i = g_i(t, \boldsymbol{q}), \quad (i = 1, 2, \ldots, n). \quad (6)$$

According to the invariance of differential equations under the infinitesimal transformations, the invariance of the Nambu differential equations (6) is

$$X^{(1)}(\dot{q}_i - g_i(t, \boldsymbol{q})) = 0, \quad (i = 1, 2, \ldots, n), \quad (7)$$

where

$$X^{(1)} = \xi_0 \frac{\partial}{\partial t} + \xi_i \frac{\partial}{\partial q_i} + \left(\frac{\bar{\mathrm{d}}\xi_i}{\mathrm{d}t} - g_i \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) \frac{\partial}{\partial \dot{q}_i}, \quad (8)$$

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + g_i \frac{\partial}{\partial q_i}.$$

Equation (7) can be expanded as

$$\frac{\bar{\mathrm{d}}\xi_i}{\mathrm{d}t} - g_i \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} = \xi_0 \frac{\partial g_i}{\partial t} + \xi_j \frac{\partial g_i}{\partial q_j}, \quad (i, j = 1, 2, \ldots, n). \quad (9)$$

We call Eq.(9) the set of determining equations of Lie symmetry of the Nambu system.

4. The Hojman conserved quantity of Nambu system

The Lie symmetry of the Nambu system can lead to the Hojman conserved quantity under certain conditions.

Theorem In the Nambu system, if the generators ξ_0, ξ_i satisfy the conditions

$$(\xi_j - g_j \xi_0) \frac{\partial^2 g_i}{\partial q_j \partial q_i} = 0, \quad (i, j = 1, 2, \ldots, n), \quad (10)$$

the system possesses the following Hojman conserved quantity:

$$I_{\mathrm{H}} = \frac{\partial}{\partial q_i}(\xi_i - \xi_0 g_i) = \text{const.} \quad (11)$$

Proof We have

$$\begin{aligned} \frac{\bar{\mathrm{d}} I_{\mathrm{H}}}{\mathrm{d}t} &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial \xi_i}{\partial q_i} - g_i \frac{\partial \xi_0}{\partial q_i} - \xi_0 \frac{\partial g_i}{\partial q_i}\right) \\ &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial \xi_i}{\partial q_i} + \frac{\partial \xi_0}{\partial t}\right) - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t} \\ &\quad - \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\xi_0 \frac{\partial g_i}{\partial q_i}\right). \end{aligned} \quad (12)$$

Noticing the following equations:

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_0}{\partial t} &= \frac{\partial}{\partial t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - \frac{\partial \xi_0}{\partial q_i}\frac{\partial g_i}{\partial t}, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\frac{\partial \xi_i}{\partial q_i} &= \frac{\partial}{\partial q_i}\left(\frac{\bar{\mathrm{d}}\xi_i}{\mathrm{d}t}\right) - \frac{\partial \xi_s}{\partial q_i}\frac{\partial g_i}{\partial q_s}, \end{aligned} \quad (13)$$

we get

$$\begin{aligned} &\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_i}{\partial q_i}\right) \\ &= \frac{\partial}{\partial t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) - \frac{\partial \xi_0}{\partial q_i}\frac{\partial g_i}{\partial t} + \frac{\partial}{\partial q_i}\left(\frac{\bar{\mathrm{d}}\xi_i}{\mathrm{d}t}\right) - \frac{\partial \xi_s}{\partial q_i}\frac{\partial g_i}{\partial q_s} \\ &= \frac{\partial}{\partial t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) + \frac{\partial}{\partial q_i}\left(\frac{\bar{\mathrm{d}}\xi_i}{\mathrm{d}t}\right) - \frac{\partial}{\partial q_i}\left(\xi_0 \frac{\partial g_i}{\partial t} + \xi_s \frac{\partial g_i}{\partial q_s}\right) \\ &\quad + \xi_0 \frac{\partial^2 g_i}{\partial q_i \partial t} + \xi_s \frac{\partial^2 g_i}{\partial q_i \partial q_s} \\ &= \frac{\partial}{\partial t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) + \frac{\partial}{\partial q_i}\left(\frac{\bar{\mathrm{d}}\xi_i}{\mathrm{d}t}\right) - \frac{\partial}{\partial q_i}\left(\frac{\bar{\mathrm{d}}\xi_i}{\mathrm{d}t} - g_i \frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) \\ &\quad + \xi_0 \frac{\partial^2 g_i}{\partial q_i \partial t} + \xi_s \frac{\partial^2 g_i}{\partial q_i \partial q_s} \\ &= \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right) + \frac{\partial g_i}{\partial q_i}\left(\frac{\partial \xi_0}{\partial t} + g_s \frac{\partial \xi_0}{\partial q_s}\right) \end{aligned}$$

$$
\begin{aligned}
&+\xi_0\frac{\partial^2 g_i}{\partial q_i\partial t}+\xi_s\frac{\partial^2 g_i}{\partial q_i\partial q_s}\\
=&\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)+\frac{\partial}{\partial t}\left(\xi_0\frac{\partial g_i}{\partial q_i}\right)-\xi_0\frac{\partial^2 g_i}{\partial q_i\partial t}\\
&+g_s\frac{\partial}{\partial q_s}\left(\xi_0\frac{\partial g_i}{\partial q_i}\right)-g_s\xi_0\frac{\partial^2 g_i}{\partial q_i\partial q_s}\\
&+\xi_0\frac{\partial^2 g_i}{\partial q_i\partial t}+\xi_s\frac{\partial^2 g_i}{\partial q_i\partial q_s}\\
=&\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}\right)+\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\left(\xi_0\frac{\partial g_i}{\partial q_i}\right)\\
&+(\xi_s-g_s\xi_0)\frac{\partial^2 g_i}{\partial q_s\partial q_i}. \qquad (14)
\end{aligned}
$$

Using Eq.(14) and the conditions (10), we can obtain

$$\frac{\bar{\mathrm{d}}I_{\mathrm{H}}}{\mathrm{d}t}=0. \qquad (15)$$

5. Argument

The Hojman conserved quantity of the Nambu system has more generalized significance. It can degenerate to the Hojman conserved quantity of the Hamilton system.

When the dimension of the Nambu system is two, equations represented by Eq.(2) are just the canonical equations of the Hamilton system, and equations represented by Eq.(6) become

$$g_s=\frac{\partial H}{\partial p_s},\quad h_s=-\frac{\partial H}{\partial q_s}. \qquad (16)$$

The variables q_i of the Nambu system ($i=1,2$) include the canonical variables q_s and p_s of Hamilton system, and ξ_i include the generators ξ_s and η_s. So, in Hamilton system, the conditions represented by Eq.(10) for the existence of the Hojman conserved quantity for the Nambu system become

$$
\begin{aligned}
&(\xi_j-g_j\xi_0)\frac{\partial^2 g_i}{\partial q_j\partial q_i}\\
=&\ (\xi_j-g_j\xi_0)\frac{\partial}{\partial q_j}\left(\frac{\partial g_s}{\partial q_s}+\frac{\partial h_s}{\partial p_s}\right)\\
=&\ (\xi_j-g_j\xi_0)\frac{\partial}{\partial q_j}\left[\frac{\partial}{\partial q_s}\left(\frac{\partial H}{\partial p_s}\right)+\frac{\partial}{\partial p_s}\left(-\frac{\partial H}{\partial q_s}\right)\right]\\
=&\ 0. \qquad (17)
\end{aligned}
$$

For the Hamilton system, we can see that these conditions can always be achieved no matter what the generators are chosen. So Lie symmetry can always lead to the Hojman conserved quantity for Hamilton system. Then the Hojman conserved quantity (11) of the Nambu system becomes

$$I_{\mathrm{H}}=\frac{\partial\xi_s}{\partial q_s}+\frac{\partial\eta_s}{\partial p_s}+\frac{\partial\xi_0}{\partial t}-\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}, \qquad (18)$$

which is just the Hojman conserved quantity led by Lie symmetry of the Hamilton system.[17]

6. Example

Suppose that the two Nambu–Hamilton functions of the Nambu system with three dimensions are:

$$
\begin{aligned}
H_1&=\frac{1}{2}(x_1^2+x_2^2),\\
H_2&=\frac{1}{2}(x_1^2-x_2^2+x_3). \qquad (19)
\end{aligned}
$$

We get the equations of motion of the system:

$$
\begin{aligned}
\dot{x}_1&=x_2,\\
\dot{x}_2&=-x_1,\\
\dot{x}_3&=-4x_1x_2. \qquad (20)
\end{aligned}
$$

The determining equations of Lie symmetry are:

$$
\begin{aligned}
&\frac{\bar{\mathrm{d}}\xi_1}{\mathrm{d}t}-g_1\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}=\xi_2,\\
&\frac{\bar{\mathrm{d}}\xi_2}{\mathrm{d}t}-g_2\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}=-\xi_1,\\
&\frac{\bar{\mathrm{d}}\xi_3}{\mathrm{d}t}-g_3\frac{\bar{\mathrm{d}}\xi_0}{\mathrm{d}t}=-4x_2\xi_1-4x_1\xi_2. \qquad (21)
\end{aligned}
$$

Using Eq.(20), we can obtain the following solutions:

$$\xi_0=\xi_1=\xi_2=0,\xi_3=\ln(2x_1^2+x_3). \qquad (22)$$

Apparently they satisfy the conditions Eq.(10). Substituting them into Eq.(11), we can get the Hojman conserved quantity led by the Lie symmetry of the Nambu system

$$I_{\mathrm{H}}=\frac{1}{2x_1^2+x_3}=\text{const.} \qquad (23)$$

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