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New periodic wave solutions, localized excitations and their interaction for (2+1)-dimensional Burgers equation*

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Based on the Bäcklund method and the multilinear variable separation approach (MLVSA), this paper finds a general solution including two arbitrary functions for the (2+1)-dimensional Burgers equations. Then a class of new doubly periodic wave solutions for (2+1)-dimensional Burgers equations is obtained by introducing appropriate Jacobi elliptic functions, Weierstrass elliptic functions and their combination in the general solutions (which contains two arbitrary functions). Two types of limit cases are considered. Firstly, taking one of the moduli to be unity and the other zero, it obtains particular wave (called semi-localized) patterns, which is periodic in one direction, but localized in the other direction. Secondly, if both moduli are tending to 1 as a limit, it derives some novel localized excitations (two-dromion solution).

Keywords: (2+1)-dimensional Burgers equation, mutilinear variable separation approach, periodic wave solutions, localized excitation

PACC: 0290, 0340K, 0365G

1. Introduction

Solitons, chaos and fractals are three most important aspects of nonlinear science.[1−5] They are widely applied in many branches of natural science such as chemistry, biology, mathematics, communication and particularly in almost all branches of physics like the fluid dynamics, plasma physics, field theory, optics, condensed matter physics, etc.[6−8] Usually, these three aspects are treated independently since one often considers that solitons are the basic excitations of the integrable models while the chaos and fractals are the elementary behaviours of the nonintegrable systems. In other words, one does not analyse the possibility of the existence of chaos and fractals in a soliton system. However, the above consideration may not be complete especially in some higher dimensions. In recent study of soliton systems, one has found that some characteristic lower-dimensional arbitrary functions are existed in the exact excitations of some two-dimensional integrable models. This means that any lower-dimensional chaotic and fractal solutions can be used to construct the exact solutions of the higher dimensional integrable models, which imply that any exotic behaviour may propagate along this characteristics. In order to solve higher-dimensional nonlinear evolution equation, two kinds of 'variables separating' procedure, which are an extension of the variable separation method in the linear case, have been established recently. The first kind of method is called the symmetry constraint or nonlinearization of the Lax pairs because this type of procedure is used only for integrable models which possess Lax pairs so that it is called equivalently the 'formal variable separation approach' (FVSA).[9] The independent variables of a reduced field in FVSA have not totally been separated though the reduced field satisfies some lower-dimensional equations. The second kind of variable separation method is established first by Lou and Lu early in 1996.[10] Lou and Lu successfully applied a kind of variable separation method firstly for the Davey–Stewartson (DS) equation.[10] Quite recently, the method, now called multilinear variable separation approach (MLVSA), has been revisited and developed well for various (2+1)-dimensional models[11−20] including the differential-difference equations.[21] As a matter of fact, a nearly systematic process of the

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MLVSA to solve (2+1)-dimensional non-linear partial differential and differential-difference equations has been established. Furthermore the MLVSA is still in progress, it aims at obtaining more general solutions in the sense that it admits more arbitrary separation functions entering into the solutions.[22,23]

Recently, the doubly periodic wave solutions in terms of the Jacobi elliptic functions for the nonlinear partial differential equations (PDEs) attract considerable interest[24−34] because of the elegant properties of the elliptic functions. The Jacobi elliptic functions

$$\mathrm{sn}\xi = \mathrm{sn}(\xi|m), \quad \mathrm{cn}\xi = \mathrm{cn}(\xi|m), \quad \mathrm{dn}\xi = \mathrm{dn}(\xi|m),$$

where m $(0 < m < 1)$ is the modulus of the elliptic function, are doubly periodic and possess properties of trigonometric functions, namely,

$$\mathrm{sn}^2\xi + \mathrm{cn}^2\xi = 1, \quad \mathrm{dn}^2\xi + m^2\mathrm{sn}^2\xi = 1,$$

$$\begin{aligned}(\mathrm{sn}\xi)' &= \mathrm{cn}\xi\mathrm{dn}\xi,\\ (\mathrm{cn}\xi)' &= -\mathrm{sn}\xi\mathrm{dn}\xi,\\ (\mathrm{dn}\xi)' &= -m^2\mathrm{sn}\xi\mathrm{cn}\xi.\end{aligned}$$

When $m \to 0$, the Jacobi elliptic functions degenerate to the trigonometric functions, i.e.,

$$\mathrm{sn}\xi \to \sin\xi, \quad \mathrm{cn}\xi \to \cos\xi, \quad \mathrm{dn}\xi \to 1.$$

When $m \to 1$, the Jacobi elliptic functions degenerate to the hyperbolic functions, i.e.

$$\mathrm{sn}\xi \to \tanh\xi, \quad \mathrm{cn}\xi \to \mathrm{sech}\xi, \quad \mathrm{dn}\xi \to \mathrm{sech}\xi.$$

Detailed explanations about Jacobi elliptic functions can be found in Refs.[35] and [36]. However, the interaction properties of the Jacobi elliptic waves have not yet been reported in the literatures. In order to obtain the Jacobi elliptic wave solutions for a given nonlinear PDE, some elegant techniques are invented, e.g., Legendre polynomial and Lamé equation method,[27] theta function method,[28] the (modified) mapping method,[29−34] etc.

One of our topics in the present work is devoted to demonstrating that MLVSA, such a power method for obtaining the solutions of the nonlinear PDEs, can also be used to obtain the doubly periodic wave solutions for the nonlinear PDE under consideration. And another is devoted to studying a class of new doubly periodic wave solutions and the interaction properties of the Jacobi elliptic waves. Especially, we consider the (2+1)-dimensional Burgers equation as a simple illustrative example in this paper. The structure of this paper is elucidated as follows. In Section 2, a general solution including two arbitrary functions is obtained for the Burgers equation by means of the MLVSA. By choosing appropriately these arbitrary functions, a class of doubly periodic wave solutions for the equation in question are given in Section 3. Section 4 is devoted to studying the interaction properties of the periodic waves by virtue of the numerical method. Some conclusions and discussion are given in the last section.

2. A general solution to the Burgers equation

Let us consider the (2+1)-dimensional Burgers equation

$$u_t = uu_y + avu_x + bu_{yy} + abu_{xx}, \tag{1}$$

$$u_x = v_y. \tag{2}$$

An equivalent form of the Burgers equations is derived from the generalized Painlevé integrability classification in Ref.[37]. One can obtain some special solutions of Eqs.(1) and (2) by introducing the Bäcklund transformation

$$u = 2b(\ln f)_y + u_0, \tag{3}$$

$$v = 2b(\ln f)_x + v_0, \tag{4}$$

where u_0, v_0 are two arbitrary seed solutions. For simplicity, we fix the seed solutions as $u_0 = 0$, $v_0 = q(x,t)$.

Substituting Eqs.(3) and (4) into Eqs.(1) and (2) one arrives at

$$\begin{aligned}&(-f_{yt} + av_0f_{xy} + bf_{yyy} + abf_{xxy})f + f_yf_t\\ &- av_0f_xf_y + bf_{yy}f_y - abf_{xx}f_y = 0,\end{aligned} \tag{5}$$

which can be changed into bilinear form as follows:

$$\begin{aligned}&[-D_yD_t + av_0D_xD_y]f\cdot f\\ &+ b[D_y^3 + aD_x^2D_y]f\cdot f = 0.\end{aligned}$$

Here, mark D is a bilinear operator under the Hiröta significance

$$\begin{aligned}D_x^mD_y^nD_t^kgf \equiv&(\partial x - \partial x')^m(\partial y - \partial y')^n\\ &\times(\partial t - \partial t')^k g\cdot f|_{(x=x',y=y',t=t')}.\end{aligned}$$

Then the following equation can be obtained:

$$(f_y - f\partial y)(f_t - av_0f_x - bf_{yy} - abf_{xx}) = 0.$$

With the help of the above equation, Eq.(5) can be simplified into a linear PDE

$$f_t - av_0 f_x - bf_{yy} - abf_{xx} = 0. \tag{6}$$

By Eq.(6) the equation (5) is satisfied identically.

Because of Eq.(6) being a linear PDE, one can use linear iteration principle such that f takes the form

$$f = \lambda + \sum_{k=0}^{N} p_k(x,t)q_k(y,t), \tag{7}$$

where λ is an arbitrary constant, $p_k(x,t) \equiv p_k$ and $q_k(y,t) \equiv q_k$ are variable separation functions of $\{x,t\}$ and $\{y,t\}$ respectively.

Substituting Eq.(7) into Eq.(6) one can derive the following variable separation system of equations

$$p_{kt} - av_0 p_{kx} - abp_{kxx} + \sum_{k=0}^{N}\sum_{l=0}^{M} \tau_{kl} p_k = 0, \tag{8}$$

$$q_{kt} - bq_{kyy} - \sum_{k=0}^{N}\sum_{l=0}^{M} \tau_{kl} q_k = 0, \tag{9}$$

where τ_{kl} $(k = 1,2,...,N; l = 1,2,...,M)$ are all arbitrary functions of variable t.

The corresponding variable separation solutions of the (2+1)-dimensional Burgers equations (1) and (2) are given by

$$u = \frac{2b\sum\limits_{k=0}^{N} p_k q_{ky}}{\lambda + \sum\limits_{k=0}^{N} p_k q_k},$$

$$v = \frac{2b\sum\limits_{k=0}^{N} p_{kx} p_k}{\lambda + \sum\limits_{k=0}^{N} p_k q_k} + v_0,$$

where v_0, p_k, q_k satisfy Eqs.(8) and (9).

Now we shall discuss two special cases.

Case 1 $M = N = 1, \{p_1, q_1\} = \{p, q\}, \tau_{11} = \tau(t)$. Equations (7), (8) and (9) can be changed into

$$\begin{aligned} &f = \lambda + pq, \\ &p_t - av_0 p_x - abp_{xx} + \tau(t)p = 0, \\ &q_t - bq_{yy} - \tau(t)q = 0. \end{aligned} \tag{10}$$

Because of v_0 being an arbitrary function of $\{x,t\}$, we can take p as an arbitrary function of $\{x,t\}$. Then the seed solution v_0 can be determined by Eq.(10),

$$v_0 = \frac{p_t - abp_{xx} + \tau(t)p}{ap_x}.$$

The variable-separable solutions of Eqs.(1) and (2) possess the form

$$\begin{aligned} u &= \frac{2bpq_y}{\lambda + pq}, \\ v &= \frac{2bp_x q}{\lambda + pq} + \frac{p_t - abp_{xx} + \tau_t p}{ap_x}. \end{aligned}$$

Case 2 $M = 1, N = 3, \lambda = a_0$, $\{p_1, q_1\} = \{p(x,t), a_1\}$, $\{p_2, q_2\} = \{a_2, q(y,t)\}$, $\{p_3, q_3\} = \{p(x,t), a_3 q(y,t)\}$, $\tau_{kl}(t) = 0$, where a_i $(i = 0..., 3)$ are arbitrary constants.

Equations (7), (8) and (9) can be transformed into

$$f = a_0 + a_1 p + a_2 q + a_3 pq, \tag{11}$$

$$p_t - av_0 p_x - abp_{xx} = 0, \tag{12}$$

$$q_t - bq_{yy} = 0. \tag{13}$$

According to Eqs.(11)–(13), we can yield another variable-separable solution of Eqs.(1) and (2):

$$\begin{aligned} u =& \frac{2b(a_2 + a_3 p)q_y}{a_0 + a_1 p + a_2 q + a_3 pq}, \\ v =& \frac{2b(a_1 + a_3 q)p_x}{a_0 + a_1 p + a_2 q + a_3 pq} \\ &+ \frac{p_t - abp_{xx}}{ap_x}. \end{aligned} \tag{14}$$

3. New periodic wave solutions

The arbitrariness of functions $p(x,t)$ and $q(y,t)$ included in the above solutions implies that the physical quantities u and v possess rich structures. For example, based on the variable separation solutions of Case 1, one can find abundant stable localized excitations such as dromions, breathers, lumps, peakons, campactons and foldons, as well as fractal patterns and chaotic patterns, etc., which are neglected in the present paper since some similar situations have been reported in proceeding literature. In terms of the variable separation solutions of Case 2, we will discuss some significant localized excitations derived from the variable separation solutions, namely Eq.(14), by choosing appropriately the arbitrary functions p and q, one can obtain a diversity of exact solutions to the (2+1)-dimensional Burgers equation.

However, we are only interested in the periodic wave solutions in terms of the Jacobi elliptic functions in this paper. It is worth while noticing that the Jacobi transformation $\mathrm{dn}(\xi|m) = \mathrm{cn}(\sqrt[2]{m}\xi|m^{-1})$[24] implies that any solution found by the dn-function may be transformed into an equivalent one which can be obtained by cn-function. Moreover, since other Jacobi elliptic functions have singularities, we consider only

the periodic wave solutions in terms of sn-functions and cn-functions. There are several cases to be considered.

Case 1 For $p = \mathrm{sn}^3(kx - v_1 t) = \mathrm{sn}^3\xi$, $q = \mathrm{sn}^3(ly - v_2 t) = \mathrm{sn}^3\eta$, we have

$$u = \frac{6b(a_2 + a_3\mathrm{sn}^3\xi)\mathrm{sn}^2\eta\mathrm{cn}\eta\mathrm{dn}\eta}{a_0 + a_1\mathrm{sn}^3\xi + a_2\mathrm{sn}^3\eta + a_3\mathrm{sn}^3\xi\mathrm{sn}^3\eta}, \qquad (15)$$

whose typical spatial structure is shown in Fig.1. This is a periodic wave solution and the parameters are $a_0 = -6, a_1 = -1, a_2 = -1, a_3 = 0.5, b = 1, k = 1, l = 1, v_1 = 0, v_2 = 0, m_1 = 0.9, m_2 = 0.6$ and $t = 0$.

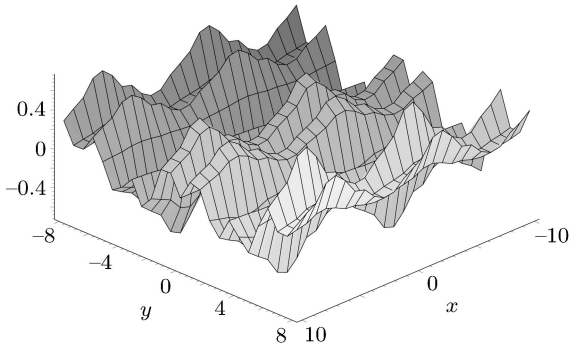


Fig.1. $m_1 = 0.9,\ m_2 = 0.6,\ t = 0.$

As $m_1 \to 1$ and $m_2 \to 1$, equation (15) degenerates to

$$u = \frac{6b(a_2 + a_3\tanh^3\xi)\tanh^2\eta\mathrm{sech}^2\eta}{a_0 + a_1\tanh^3\xi + a_2\tanh^3\eta + a_3\tanh^3\xi\tanh^3\eta}.$$

The solution is plotted in Figs.2, 3 and 4. It is a new solution to Eqs.(1) and (2).

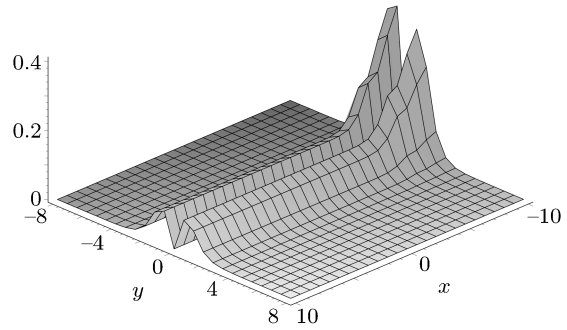


Fig.2. $m_1 = 1,\ m_2 = 1,\ t = -4.$

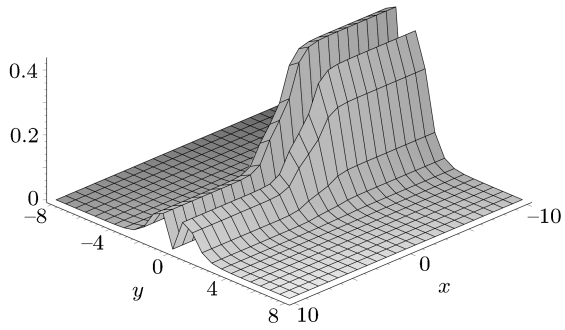


Fig.3. $m_1 = 1,\ m_2 = 1,\ t = 0.$

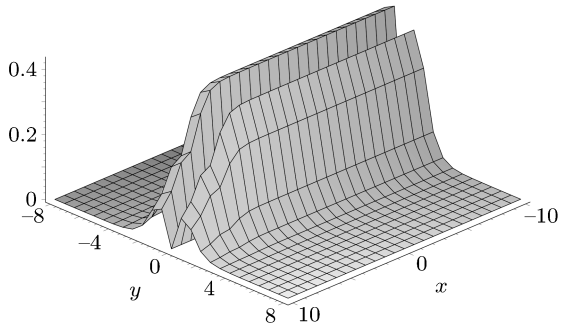


Fig.4. $m_1 = 1,\ m_2 = 1,\ t = 4.$

As $m_1 \to 1$ and $m_2 \to 0$, or $m_1 \to 1$ and $m_2 \to 0$ we can obtain a half package of defined structures to Eqs.(1) and (2), which is illustrated in Figs.5 and 6.

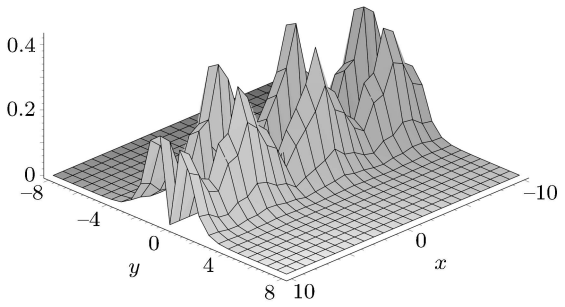


Fig.5. $m_1 = 0,\ m_2 = 1,\ t = 0.$

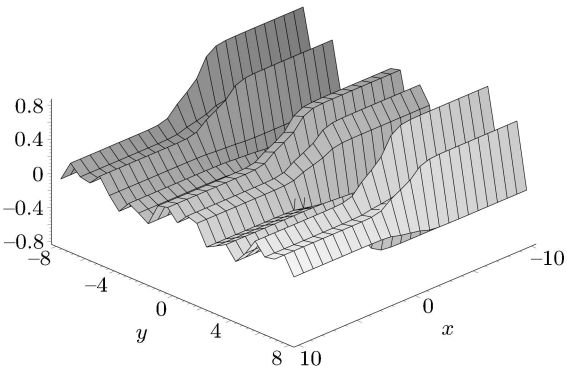


Fig.6. $m_1 = 1,\ m_2 = 0,\ t = 0.$

Case 2 For $p = \mathrm{cn}^3(kx - v_1t) = \mathrm{cn}^3\xi$, $q = \mathrm{sn}^3(ly - v_2t) = \mathrm{sn}^3\eta$, with the help of Eq.(14), we obtain a doubly periodic wave solution to Eqs.(1) and (2)

$$u = \frac{-6b(a_2 + a_3\mathrm{cn}^3\xi)\mathrm{cn}^2\eta\mathrm{sn}\eta\mathrm{dn}\eta}{a_0 + a_1\mathrm{cn}^3\xi + a_2\mathrm{cn}^3\eta + a_3\mathrm{cn}^3\xi\mathrm{cn}^3\eta}. \quad (16)$$

Its typical spatial structure is depicted in Fig.7. As $m_1 \to 1$ and $m_2 \to 1$, equation becomes

$$u = \frac{-6b(a_2 + a_3\mathrm{sech}^3\xi)\mathrm{sech}^3\eta\tanh\eta}{a_0 + a_1\mathrm{sech}^3\xi + a_2\tanh^3\eta + a_3\mathrm{sech}^3\xi\mathrm{sech}^3\eta},$$

which is illustrated in Fig.8. It is another novel solution to Eqs.(1) and (2). In the same way, as $m_1 \to 1$ and $m_2 \to 0$, or $m_1 \to 1$ and $m_2 \to 0$ we can also get half package of defined structures to Eqs.(1) and (2).

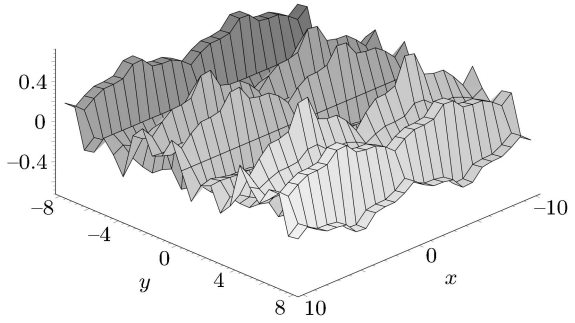


Fig.7. $m_1 = 0.9,\ m_2 = 0.6,\ t = 0.$

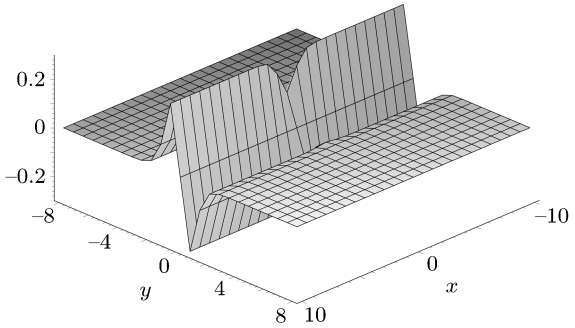


Fig.8. $m_1 = 1,\ m_2 = 1,\ t = 0.$

Case 3 For $p = \mathrm{sn}^3(kx - v_1t) = \mathrm{sn}^3\xi$, $q = \mathrm{cn}^3(ly - v_2t) = \mathrm{cn}^3\eta$, the exact periodic wave solution to Eqs.(1) and (2) is

$$u = \frac{-6b(a_2 + a_3\mathrm{sn}^3\xi)\mathrm{cn}^2\eta\mathrm{sn}\eta\mathrm{dn}\eta}{a_0 + a_1\mathrm{sn}^3\xi + a_2\mathrm{cn}^3\eta + a_3\mathrm{sn}^3\xi\mathrm{cn}^3\eta}, \quad (17)$$

which is shown in Fig.9, a new multisoliton solution with two pits and two peaks. As $m_1 \to 1$ and $m_2 \to 1$, one has

$$u = \frac{-6b(a_2 + a_3\tanh^3\xi)\tanh^2\eta\mathrm{sech}^2\eta}{a_0 + a_1\tanh^3\xi + a_2\tanh^3\eta + a_3\tanh^3\xi\tanh^3\eta},$$

which is illustrated in Fig.10.

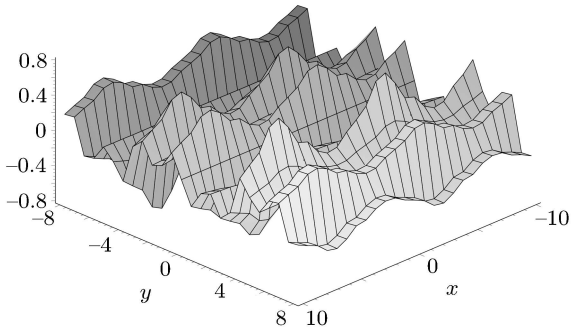


Fig.9. $m_1 = 0.9,\ m_2 = 0.6,\ t = 0.$

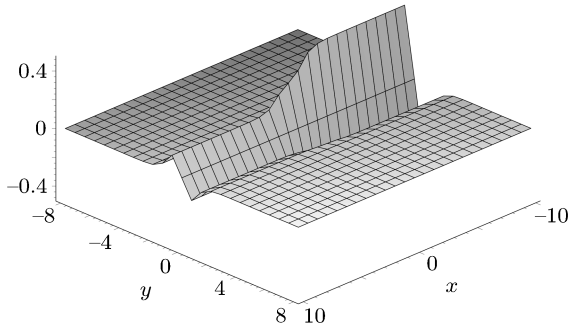


Fig.10. $m_1 = 1,\ m_2 = 1,\ t = 0.$

Case 4 For $p = \boldsymbol{P}(kx - v_1t, g_2, g_3) = \boldsymbol{P}(\xi)$, $q = \boldsymbol{P}(ly - v_2t, g_2, g_3) = \boldsymbol{P}(\eta)$, it follows from Eq.(14) that

$$u = \frac{6b(a_2 + a_3\boldsymbol{P}(\xi))\boldsymbol{P}'(\eta)}{a_0 + a_1\boldsymbol{P}(\xi) + a_2\boldsymbol{P}(\eta) + a_3\boldsymbol{P}(\xi)\boldsymbol{P}(\eta)}, \quad (18)$$

where $\boldsymbol{P}(\xi)$ is Weierstrass elliptic function, real parameters g_2, g_3 are called invariant.[25] And $\boldsymbol{P}(\xi)$ should satisfy the following equation:

$$\boldsymbol{P}'^2(\xi) = 4\boldsymbol{P}^3(\xi) - g_2\boldsymbol{P}(\xi) - g_3,$$
$$\boldsymbol{P}''(\xi) = 6\boldsymbol{P}^2(\xi) - \frac{1}{2}g_2, \quad (19)$$

where $\boldsymbol{P}'(\xi) = \mathrm{d}\boldsymbol{P}(\xi)/\mathrm{d}\xi, \boldsymbol{P}''(\xi) = \mathrm{d}\boldsymbol{P}^2(\xi)/\mathrm{d}\xi^2$. The solution (18) is a doubly periodic wave solution.

Case 5 For $p = \boldsymbol{P}(kx - v_1t, g_2, g_3) = \boldsymbol{P}(\xi)$, $q = \boldsymbol{P}'(ly - v_2t, g_2, g_3) = \boldsymbol{P}'(\eta)$, we have

$$u = \frac{6b(a_2 + a_3\boldsymbol{P}(\xi))\boldsymbol{P}''(\eta)}{a_0 + a_1\boldsymbol{P}(\xi) + a_2\boldsymbol{P}'(\eta) + a_3\boldsymbol{P}(\xi)\boldsymbol{P}'(\eta)}. \quad (20)$$

It is also a doubly periodic wave solution. Using the combination of Jacobi elliptic function and Weierstrass elliptic function, we can change Eqs.(18) and (20) into the form of Jacobi elliptic functions.

4. Localized excitations and their interaction for (2+1)-dimensional Burgers equation

Case 1 $p = \mathrm{sn}^3(k_1x - v_1t) + \mathrm{sn}^3(k_2x - v_2t) = \mathrm{sn}^3\xi_1 + \mathrm{sn}^3\xi_2$, $q = \mathrm{sn}^3(ly - v_3t) = \mathrm{sn}^3\eta$. It follows from Eq.(14) that

$$u = \frac{6b(a_2 + a_3(\mathrm{sn}^3\xi_1 + \mathrm{sn}^3\xi_2))\mathrm{sn}^2\eta\mathrm{cn}\eta\mathrm{dn}\eta}{a_0 + a_1(\mathrm{sn}^3\xi_1 + \mathrm{sn}^3\xi_2) + a_2\mathrm{sn}^3\eta + a_3(\mathrm{sn}^3\xi_1 + \mathrm{sn}^3\xi_2)\mathrm{sn}^3\eta}. \tag{21}$$

Its typical spatial structure is shown in Figs.11, 12 and 13, the parameters are $a_0 = -6, a_1 = -1, a_2 = -1, a_3 = 0.5, b = 1, k_1 = 1, k_2 = 2, l = 1, v_1 = 2, v_2 = 0, v_3 = 0, m_1 = 0.9, m_2 = 0.8, m_3 = 0.6$, and they are valid throughout this section, unless explicitly stated otherwise. The plot represents the interaction between the Jacobi elliptic wave with different speeds. As $m_1 \to 1$, $m_2 \to 1$, and $m_3 \to 1$, equation (14) degenerates to

$$u = \frac{6b(a_2 + a_3(\tanh^3\xi_1 + \tanh^3\xi_2))\tanh^2\eta\mathrm{sech}^2\eta}{a_0 + a_1(\tanh^3\xi_1 + \tanh^3\xi_2) + a_2\tanh^3\eta + a_3(\tanh^3\xi_1 + \tanh^3\xi_2)\tanh^3\eta},$$

which is a new two-dromion solution to Eqs.(1) and (2). The evolution of the two dromions is depicted in Figs.14, 15 and 16. As $m_1 \to 1$ and $m_2 \to 0$, we can obtain a half package of defined structures to Eqs.(1) and (2), which is illustrated in Figs.17, 18 and 19. When $m_1 \to 1$ and $m_2 \to 0$ a half package of defined structures to Eqs.(1) and (2) can also be obtained.

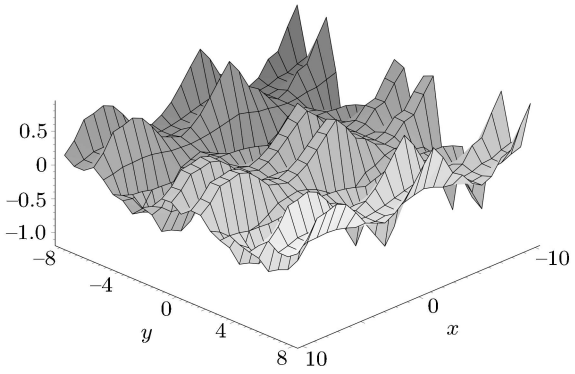


Fig.11. $m_1 = 0.9,\ m_2 = 0.8,\ m_3 = 0.6,\ t = -4.$

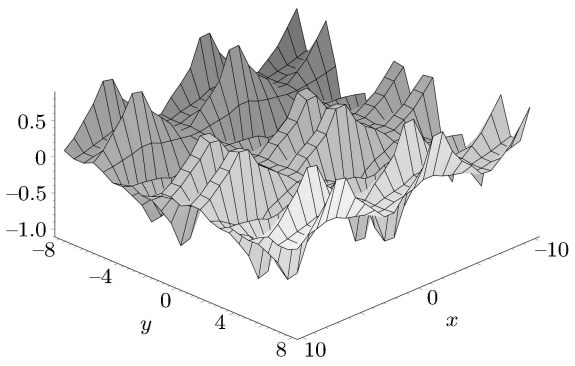


Fig.12. $m_1 = 0.9,\ m_2 = 0.8,\ m_3 = 0.6,\ t = 0.$

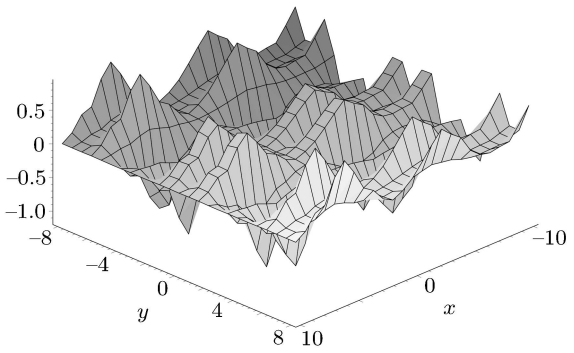


Fig.13. $m_1 = 0.9,\ m_2 = 0.8,\ m_3 = 0.6,\ t = 4.$

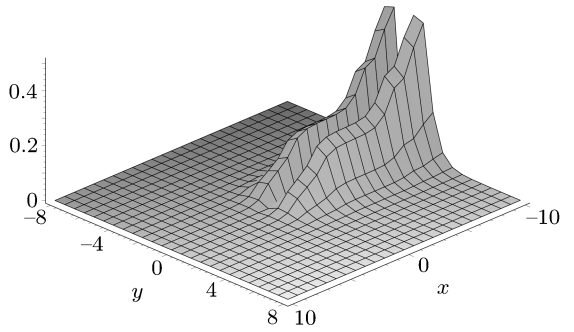


Fig.14. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = -4.$

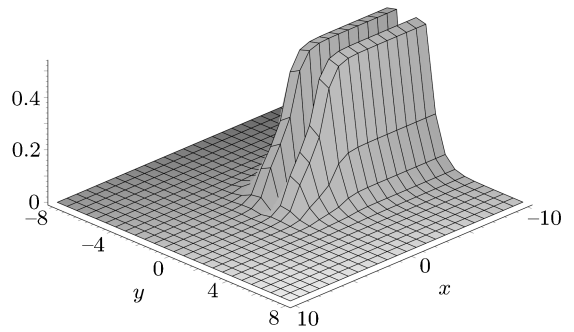


Fig.15. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = 0.$

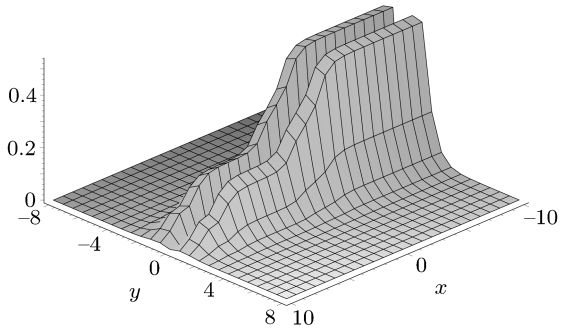


Fig.16. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = 4.$

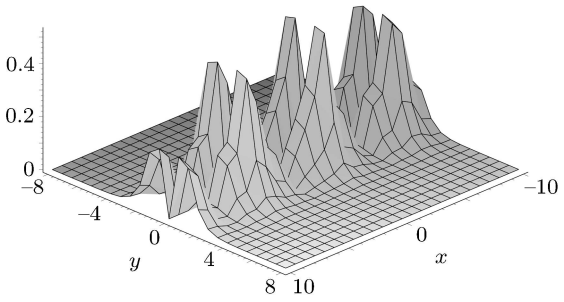


Fig.18. $m_1 = 0,\ m_2 = 0,\ m_3 = 1,\ t = 0.$

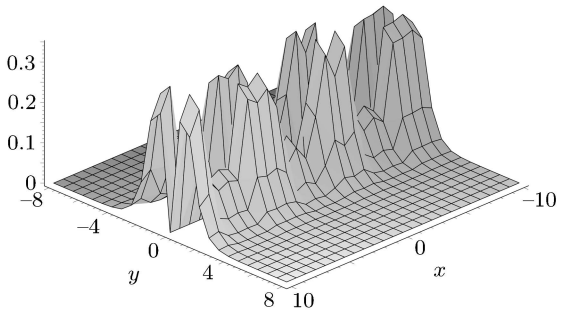


Fig.17. $m_1 = 0,\ m_2 = 0,\ m_3 = 1,\ t = -4.$

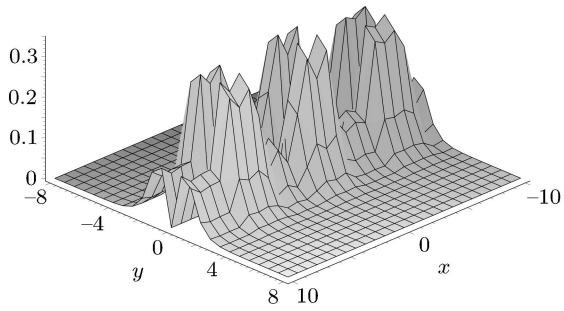


Fig.19. $m_1 = 0,\ m_2 = 0,\ m_3 = 1,\ t = 4.$

Case 2 $p = \mathrm{cn}^3(k_1x - v_1t) + \mathrm{cn}^3(k_2x - v_2t) = \mathrm{cn}^3\xi_1 + \mathrm{cn}^3\xi_2$, $q = \mathrm{cn}^3(ly - v_3t) = \mathrm{cn}^3\eta$. It follows from Eq.(14) that

$$u = \frac{-6b(a_2 + a_3(\mathrm{cn}^3\xi_1 + \mathrm{cn}^3\xi_2))\mathrm{cn}^2\eta\mathrm{sn}\eta\mathrm{dn}\eta}{a_0 + a_1(\mathrm{cn}^3\xi_1 + \mathrm{cn}^3\xi_2) + a_2\mathrm{cn}^3\eta + a_3(\mathrm{cn}^3\xi_1 + \mathrm{cn}^3\xi_2)\mathrm{cn}^3\eta}, \tag{22}$$

which is another new two-dromion solution to Eqs.(1) and (2). As $m_1 \to 1$, $m_2 \to 1$, and $m_3 \to 1$, equation (14)degenerates to

$$u = \frac{-6b(a_2 + a_3(\mathrm{sech}^3\xi_1 + \mathrm{sech}^3\xi_2))\mathrm{sech}^3\eta\tanh\eta}{a_0 + a_1(\mathrm{sech}^3\xi_1 + \mathrm{sech}^3\xi_2) + a_2\mathrm{sech}^3\eta + a_3(\mathrm{sech}^3\xi_1 + \mathrm{sech}^3\xi_2)\mathrm{sech}^3\eta}.$$

It is a new dromion solution to Eqs.(1) and (2). The evolution of the new dromion solution is shown in Figs.(20), (21) and (22).

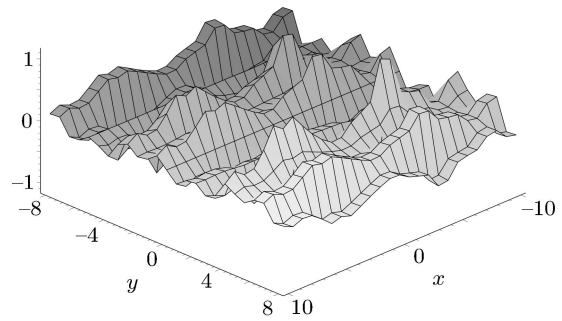


Fig.20. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = -4.$

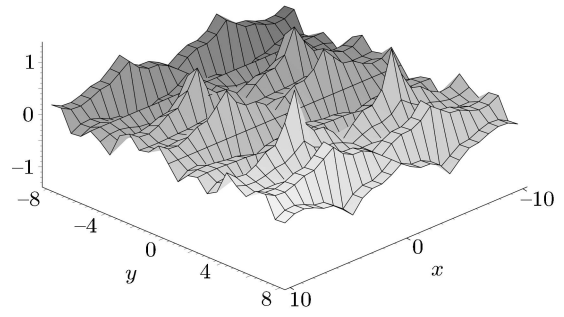


Fig.21. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = 0.$

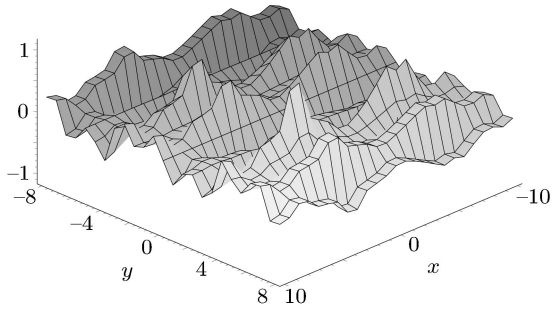


Fig.22. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = 4.$

Case 3 $p = \mathrm{sn}^3(k_1x - v_1t) + \mathrm{sn}^3(k_2x - v_2t) = \mathrm{sn}^3\xi_1 + \mathrm{sn}^3\xi_2$, $q = \mathrm{cn}^3(ly - v_3t) = \mathrm{cn}^3\eta$. It follows from Eq.(14) that

$$u = \frac{-6b(a_2 + a_3(\mathrm{sn}^3\xi_1 + \mathrm{sn}^3\xi_2))\mathrm{cn}^2\eta\mathrm{sn}\eta\mathrm{dn}\eta}{a_0 + a_1(\mathrm{sn}^3\xi_1 + \mathrm{sn}^3\xi_2) + a_2\mathrm{cn}^3\eta + a_3(\mathrm{sn}^3\xi_1 + \mathrm{sn}^3\xi_2)\mathrm{cn}^3\eta}. \tag{23}$$

This is periodic wave solution, too. As $m_1 \to 1$, $m_2 \to 1$, and $m_3 \to 1$, equation (14) degenerates to

$$u = \frac{-6b(a_2 + a_3(\tanh^3\xi_1 + \tanh^3\xi_2))\tanh\eta\mathrm{sech}^3\eta}{a_0 + a_1(\tanh^3\xi_1 + \tanh^3\xi_2) + a_2\mathrm{sech}^3\eta + a_3(\tanh^3\xi_1 + \tanh^3\xi_2)\mathrm{sech}^3\eta},$$

which is depicted in Figs.23, 24 and 25. It is another new dromion solution to Eqs.(1) and (2).

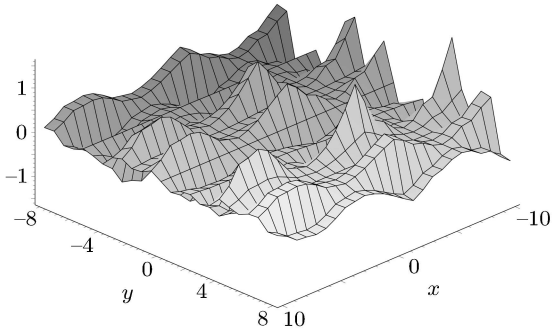


Fig.23. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = -4.$

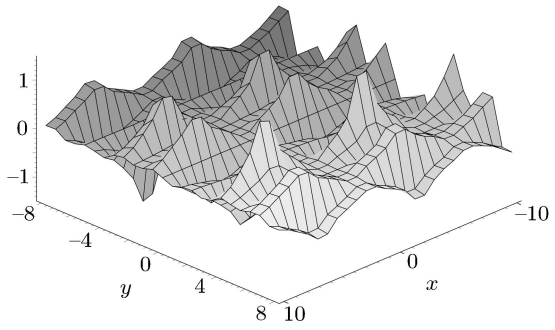


Fig.24. $m_1 = 1\ m_2 = 1,\ m_3 = 1,\ t = 0.$

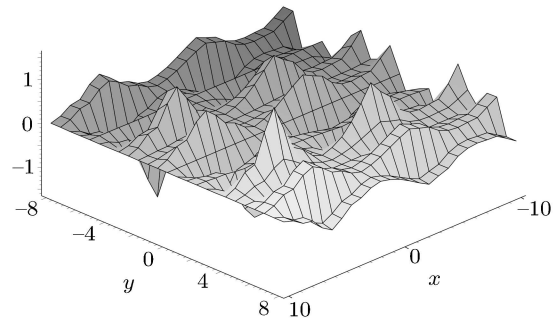


Fig.25. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = 4.$

Case 4 $p = \mathrm{cn}^3(k_1x - v_1t) + \mathrm{cn}^3(k_2x - v_2t) = \mathrm{cn}^3\xi_1 + \mathrm{cn}^3\xi_2$, $q = \mathrm{sn}^3(ly - v_3t) = \mathrm{sn}^3\eta$. It follows from Eq.(14) that

$$u = \frac{6b(a_2 + a_3(\mathrm{cn}^3\xi_1 + \mathrm{cn}^3\xi_2))\mathrm{sn}^2\eta\mathrm{sn}\eta\mathrm{dn}\eta}{a_0 + a_1(\mathrm{cn}^3\xi_1 + \mathrm{cn}^3\xi_2) + a_2\mathrm{sn}^3\eta + a_3(\mathrm{cn}^3\xi_1 + \mathrm{cn}^3\xi_2)\mathrm{sn}^3\eta}. \tag{24}$$

This is also a periodic wave solution. As $m_1 \to 1$, $m_2 \to 1$, and $m_3 \to 1$, equation (14) degenerates to

$$u = \frac{6b(a_2 + a_3(\mathrm{sech}^3\xi_1 + \mathrm{sech}^3\xi_2))\tanh^2\eta\mathrm{sech}^2\eta}{a_0 + a_1(\mathrm{sech}^3\xi_1 + \mathrm{sech}^3\xi_2) + a_2\tanh^3\eta + a_3(\mathrm{sech}^3\xi_1 + \mathrm{sech}^3\xi_2)\tanh^3\eta},$$

whose typical spatial structure is shown in Figs.26, 27, and 28. This is also a new dromion solution to Eqs.(1) and (2).

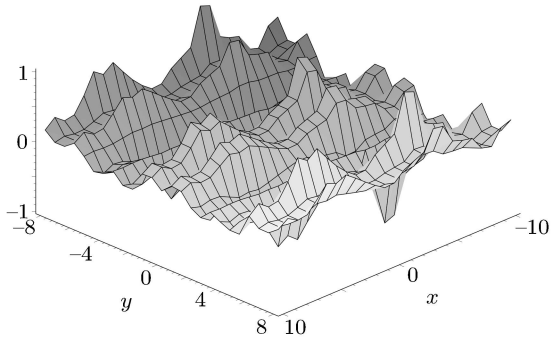


Fig.26. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = -4.$

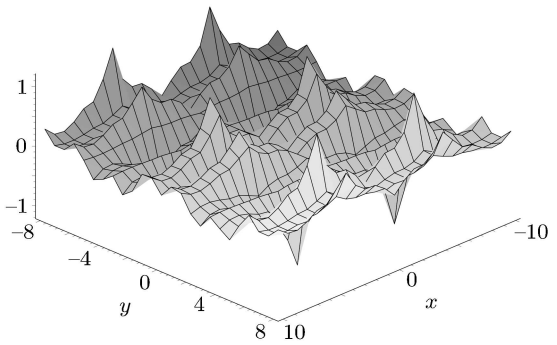


Fig.27. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = 0.$

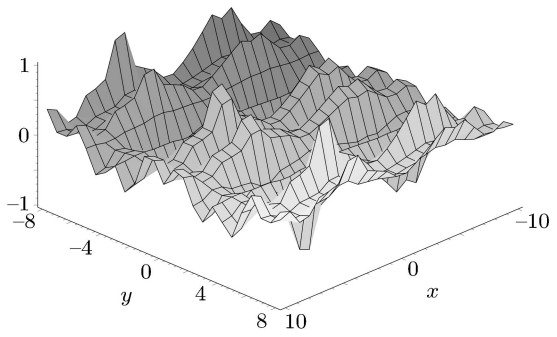


Fig.28. $m_1 = 1,\ m_2 = 1,\ m_3 = 1,\ t = 4.$

5. Conclusion and discussion

In conclusion, using the MLVSA for the (2+1)-dimensional Burgers equations, we can construct many new types of the multisoliton solutions. The method here used is standard and simple, so these types of multisoliton solutions may exist for other (2+1)-dimensional nonlinear evolution systems. Further studies on this method are worthwhile. Based on the quantity u, some novel interactions are investigated both analytically and graphically. The interaction between two special dromions possesses novel property, that is, there exists a multi-valued foldon in the process of their collision, which is different from the reported cases in previous literatures. Similarly, based on the quantity v, we may obtain many localized excitations and their interaction. For the sake of saving space, we do not discuss them here. In our future work, we will be devoted to generalizing this method to other (2+1)-dimensional equations, (1+1)-dimensional and (3+1)-dimensional non-linear systems, and the differential-difference equations. Although experimental physicists have not yet found real dromion-type excitations (some patterns localized exponentially in all directions) in any real physical systems, we still believe that there exist many possible applications for this interesting phenomenon. Anyway, we hope these localized excitation patterns and their interaction related to Jacobi functions and Weierstrass functions, etc., would be helpful to some applications in reality.

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