Vol 17 No 11, November 2008
1674-1056/2008/17(11)/4232-10

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Cumulative second-harmonic generation of Lamb waves propagating in a two-layered solid plate*

Xiang Yan-Xun(项延训)[a)†] and Deng Ming-Xi(邓明晰)[b)]

[a)] *School of Mechanical and Power Engineering, East China University of Science and Technology, Shanghai* 200237, *China*

[b)] *Department of Physics, Logistics Engineering University, Chongqing* 400016, *China*

(Received 2 April 2008; revised manuscript received 14 May 2008)

The physical process of cumulative second-harmonic generation of Lamb waves propagating in a two-layered solid plate is presented by using the second-order perturbation and the technique of nonlinear reflection of acoustic waves at an interface. In general, the cumulative second-harmonic generation of a dispersive guided wave propagation does not occur. However, the present paper shows that the second-harmonic of Lamb wave propagation arising from the nonlinear interaction of the partial bulk acoustic waves and the restriction of the three boundaries of the solid plates does have a cumulative growth effect if some conditions are satisfied. Through boundary condition and initial condition of excitation, the analytical expression of cumulative second-harmonic of Lamb waves propagation is determined. Numerical results show the cumulative effect of Lamb waves on second-harmonic field patterns.

Keywords: Lamb waves, second harmonics, cumulative effect, layered plates
PACC: 4325, 4335

1. Introduction

Composite materials have been widely used in many fields since the 1980s because of their special properties such as high intensity, high rigidity and low density. Most of composite materials are of multilayer plate structures, it is very necessary to investigate such a plate structure to guarantee the quality of the material. Lamb wave nondestructive testing technique receives more and more attention because of its higher inspection efficiency than traditional bulk wave technique.[1−3] In recent years, the study shows that it is more accurate to evaluate the properties of materials by considering the elastic nonlinearity of Lamb waves.[4−6] Usually, the interest is focused on second-harmonic generation with a cumulative effect. Thus, it is essential to analyse the physical process of cumulative second-harmonic generation of Lamb wave propagating in a two-layered solid waveguide. Although the second-harmonic of Lamb wave propagation, in general, does not have a cumulative effect because of its dispersion, there may exist certain particular cases, similar to the Lamb waves propagating in a solid plate,[7,8] in which the Lamb wave second-harmonic does accumulate within a propagation distance. In practice, the existence condition of cumulative second-harmonic generation is of significance due to the possible application of nonlinearity of Lamb waves to nondestructive testing. In addition, the analytical description is needed for the quantitative analysis of second-harmonic behaviour of Lamb wave propagation. The purpose of this paper is to obtain the existence condition of Lamb wave cumulative second-harmonic generation, the corresponding analytical solution as well as a clear physical picture of process of cumulative second-harmonic generation of Lamb wave propagating in a two-layered solid plate.

Lamb wave propagating in a two-layered solid plate consists of eight partial bulk waves (including two longitudinal waves and two transverse waves in each layer) in view of the partial bulk wave analysis method of a guided wave (i.e. several partial bulk waves constitute a guided wave).[9] Due to the bulk nonlinearity of the two materials of the solid waveguide, the nonlinear interactions among the four partial bulk waves in each of the solid plate generate a driven second-harmonic which can be analysed by using the second-order approximation.[10−12] We also employ the technique of analysing the nonlinear acoustic reflection at an interface and the method of

*Project supported by the Shanghai Leading Academic Discipline Project, China (Grant No B503).

†E-mail: yxxiang@ecust.edu.cn

analysing the cumulative second-harmonic generation of Lamb wave propagating in a solid plate to determine the existence condition of Lamb wave cumulative second-harmonic propagation and the corresponding analytical solution.[7,8,13]

For simplicity, the analysis is based on the assumptions that the two materials of the solid plates are isotropic and homogeneous with no attenuation and no dispersion, that the interface between the two materials is an ideal one (i.e. rigidly adhesive), that the higher harmonic energy is much less than the fundamental one, and that there is no exchange of energy between the fundamental waves (i.e. the eight partial bulk acoustic waves of Lamb waves propagation) and the higher harmonics, that the solution is physically relevant at a distance less than the shock formation distance.

2. Theoretical fundamental

As shown in Fig.1, Lamb wave propagation occurs in a two-layered solid plate, where one layer is of material M_1 with a thickness d_1, and the other is of material M_2 with a thickness d_2. Figure 1 shows a Cartesian coordinate system. The coordinate origin is chosen at the interface between M_1 and M_2, and the *oy*-axis is perpendicular to the interface and the *oz*-axis is along the interface.

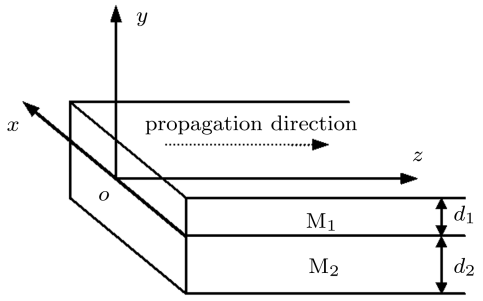


Fig.1. The structure of two-layered solid plate.

According to the method of analysing the partial bulk acoustic waves of guided wave propagating in a waveguide, eight partial bulk waves constitute a Lamb waves propagation. For a homogeneous solid with no attenuation and no dispersion, the fundamental and the second-order wave equations associated with the displacement vector are[7,8,14–16]

$$\rho_{(i)}\frac{\partial^2 u_{(i)}^{(1)}}{\partial_t^2}-\left[\kappa_{(i)}+4\mu_{(i)}/3\right]\nabla\left[\nabla\cdot u_{(i)}^{(1)}\right]+\mu_{(i)}\nabla\times\left[\nabla\times u_{(i)}^{(1)}\right]=0, \tag{1}$$

and

$$\rho_{(i)}\frac{\partial^2 u_{(i)}^{(2)}}{\partial_t^2}-\left[\kappa_{(i)}+4\mu_{(i)}/3\right]\nabla\left[\nabla\cdot u_{(i)}^{(2)}\right]+\mu_{(i)}\nabla\times\left[\nabla\times u_{(i)}^{(2)}\right]=F\left[u_{(i)}^{(1)}\right]. \tag{2}$$

In Eqs.(1) and (2), the superscripts (1) or (2) of the displacement vector $\boldsymbol{u}$ denotes the corresponding $\boldsymbol{u}$ is of the fundamental frequency or of the second-harmonic one, and the subscript $(i)=(1)$ or (2) of $\boldsymbol{u}$ indicate the corresponding $\boldsymbol{u}$ is associated with M_1 or with M_2, respectively. $\rho_{(i)}$, $\kappa_{(i)}$ and $\mu_{(i)}$ denote the density and the second-order elastic constants of M_1 or M_2, respectively. The term on the right-hand side of Eq.(2), i.e. $F\left[u_{(i)}^{(1)}\right]$, is a driving force produced by the fundamental waves, expressed as the quadratic terms.

The solutions of the eight partial bulk acoustic waves constituting a Lamb wave propagating with angular frequency ω in the waveguide are found from Eq.(1) by taking into account the Snell's law,[9] which can be written in the following from:,

$$\boldsymbol{u}_{(i)Ln}=u_{0(i)Ln}K_{(i)Ln}^0\exp[\mathrm{j}K_{(i)Ln}\cdot r-\mathrm{j}\omega t],$$

and

$$\boldsymbol{u}_{(i)Tn}=u_{0(i)Tn}(-1)^n\left(\hat{x}\times K_{(i)Tn}^0\right)\times\exp[\mathrm{j}K_{(i)Tn}\cdot r-\mathrm{j}\omega t] \tag{3}$$

with

$$\begin{aligned}&K_{(i)Pn}\cdot r=k_{(i)}z+(-1)^n\alpha_{(i)P}k_{(i)}y,\\&n=1,2,\quad P=T\quad\text{or}\quad L,\\&K_{(i)P}=\omega/c_{(i)P}=\left|K_{(i)P1}\right|=\left|K_{(i)P2}\right|,\\&k_{(i)}=K_{(i)P}\sin\theta_{(i)P},\\&\alpha_{(i)P}k_{(i)}=K_{(i)P}\cos\theta_{(i)P},\quad k_{(i)}=\omega/c,\\&\alpha_{(i)P}=\sqrt{\left(c^2/c_{(i)P}^2\right)-1},\end{aligned} \tag{4}$$

here $(i)=(1)$ or (2) denotes M_1 or M_2, respectively; $n=l$, 2; $K_{(i)Tn}^0$ and $K_{(i)Tn}^0$ are the unit vectors of the wave vectors $K_{(i)Ln}$ and $K_{(i)Tn}$; $\theta_{(i)P}$ is the angle between the wave vector $K_{(i)Pn}$ and the oy-axis (see Fig.2); $\hat{x}$ is an unit vector along the *ox*-axis; $c_{(i)P}(P=L$ or $T)$ denotes the longitudinal or the transverse acoustic velocity; $u_{0(i)Ln}$ and $u_{0(i)Tn}$ $(n=1,2)$ are the amplitudes of partial longitudinal and transverse waves; $K_{(i)L}=\omega/c_{(i)L}$ and $K_{(i)T}=\omega/c_{(i)T}$ are the magnitudes of $K_{(i)Ln}$ and

$K_{(i)Tn}$ respectively; $k_{(i)}$ is the oz-axis component of $K_{(i)Ln}$ or $K_{(i)Tn}$; $k_{(1)} = k_{(2)} = k$ because of the condition of phase matching among the partial bulk acoustic waves. c and $k = \omega/c$ are respectively called the phase velocity and the wave number of Lamb wave propagation, whose existence is postulated.

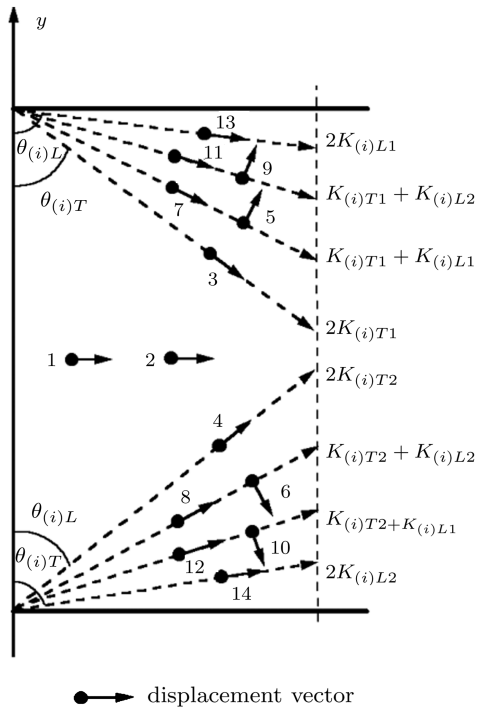


Fig.2. The drive second-harmonic fields of Lamb waves propagation, where numbers (1–14) represent 1-$u^{(\mathrm{DL})}_{(i)L1-L2}$, 2-$u^{(\mathrm{DL})}_{(i)T1-T2}$, 3-$u^{(\mathrm{DL})}_{(i)T1-T1}$, 4-$u^{(\mathrm{DL})}_{(i)T2-T2}$, 5-$u^{(\mathrm{DT})}_{(i)T1-L1}$, 6-$u^{(\mathrm{DT})}_{(i)T2-L2}$, 7-$u^{(\mathrm{DL})}_{(i)T1-L1}$, 8-$u^{(\mathrm{DL})}_{(i)T2-L2}$, 9-$u^{(\mathrm{DT})}_{(i)T1-L2}$,10-$u^{(\mathrm{DT})}_{(i)T2-L1}$, 11-$u^{(\mathrm{DL})}_{(i)T1-L2}$, 12-$u^{(\mathrm{DL})}_{(i)T2-L1}$, 13-$u^{(\mathrm{DL})}_{(i)L1-L1}$, 14-$u^{(\mathrm{DL})}_{(i)L2-L2}$.

Since the interface between M_1 and M_2 is an ideal one, the boundary condition requires that the tangential and normal stresses cancel out at the boundary $y = d_1$ and $y = -d_2$ and that they be continuous as well as the oy- and oz-axis displacement components of the partial bulk waves at the interface $y = 0$. Based on this boundary condition, we have the following matrix equation:

$$[M(\omega,k)]\begin{bmatrix}\cdots\\ u_{(i)Ln}\\ u_{(i)Tn}\\ \cdots\end{bmatrix} = 0, \quad i = 1,2, \quad n = 1,2. \tag{5}$$

The $[M(\omega,k)]$ is an 8×8 coefficient matrix which is shown in the appendix. For a nontrivial solution, $|M(\omega,k)| = 0$ must hold, which determines the Lamb wave dispersion relationship. Usually, we cannot express this dispersion relationship in an explicit form and we can only present the corresponding dispersion curve by numerical calculation.

We have the fundamental waves in the two materials as follows:

$$u^{(1)}_{(i)} = \sum_{n=1}^{2}\left[u_{(\mathrm{i})Ln} + u_{(\mathrm{i})Tn}\right], \quad i = 1,2.$$

According to Refs.[10–13], due to the self- and cross interactions of the partial bulk acoustic waves of Lamb wave propagation, the driving forces on M_1 and M_2 are composed of several terms and they are expressed as

$$\begin{aligned} F\left[u^{(1)}_{(i)}\right] = & \sum_{l,n=1}^{2}(-1)^l\left[\hat{x}\times(K^0_{(i)Tl}+K^0_{(i)Ln})\right]F^{(\mathrm{DT})}_{(i)Tl-Ln}\exp[\mathrm{j}(K_{(i)Tl}+K_{(i)Ln})\cdot r] \\ & + \sum_{n=1}^{2}K^0_{(i)Ln}F^{(\mathrm{DL})}_{(i)Ln-Ln}\exp[\mathrm{j}2K_{(i)Ln}\cdot r] + \sum_{n=1}^{2}K^0_{(i)Tn}F^{(\mathrm{DL})}_{(i)Tn-Tn}\exp[\mathrm{j}2K_{(i)Tn}\cdot r] \\ & + \sum_{l,n=1}^{2}(K^0_{(i)Tl}+K^0_{(i)Ln})F^{(\mathrm{DL})}_{(i)Tl-Ln}\exp[\mathrm{j}(K_{(i)Tl}+K_{(i)Ln})\cdot r] \\ & + \hat{\boldsymbol{z}}F^{(\mathrm{DL})}_{(i)L1-L2}\exp(\mathrm{j}2kz) + \hat{z}F^{(\mathrm{DL})}_{(i)T1-T2}\exp(\mathrm{j}2kz), \end{aligned} \tag{6}$$

where $\hat{\boldsymbol{z}}$ is a unit vector of the oz-axis; $\exp(-\mathrm{j}2\omega_t)$is omitted; The superscript (DT) or (DL) in $F^{(\mathrm{DT})}_{(i)Pl-Qn}$ or $F^{(\mathrm{DL})}_{(i)Pl-Qn}(P,\ Q = T,\ L)$ denotes the corresponding driving force component of the transverse wave or the longitudinal wave; and the subscript $(i)Pl - Qn(i = 1,2; P,Q = T,L)$ in $F^{(\mathrm{DT})}_{(i)Pl-Qn}$ or $F^{(\mathrm{DL})}_{(i)Pl-Qn}$ denotes the corresponding driving force component arising from the nonlinear interaction of the partial

bulk waves $u_{(i)Pl}$ and $u_{(i)Qn}$. The driving force on M_1 or M_2 shown in Eq.(6) has a similar form to that of Lamb-wave propagating in a solid plate,[7,8] and there are 14 driving force components on M_i ($i = 1$ or 2) (each with four transverse and ten longitudinal components). Combining Eqs.(2) and (6) yields the 14 driven second harmonics on M_i.

Figure 2 shows the driven second-harmonic fields in each layer of the two-layered solid plate. Because of the absence of dispersion in the two materials of M_1 and M_2, only the self-interaction of the partial bulk wave $u_{(i)L1}$ or $u_{(i)L2}$ can generate the driven second harmonic with a cumulative effect,[7,13] which is called driven cumulative second-harmonic (denoted by $u^{(\mathrm{DL})}_{(i)L1-L1}$ or $u^{(\mathrm{DL})}_{(i)L2-L2}$ respectively). We call the other driven second harmonic without the cumulative effect the driven plane second harmonic $\left(\text{denoted by } u^{(\mathrm{DP})}_{(i)}\right)$. Each driven cumulative second harmonic is expressed as[13]:

$$u^{(\mathrm{DL})}_{(i)Ln-Ln} = u^{(\mathrm{DL})}_{0(i)Ln-Ln}\left[\sin\theta_{(i)L}z + (-1)^n\cos\theta_{(i)L}y\right]K^0_{(i)Ln}\exp\left[\mathrm{j}2K_{(i)Ln}\cdot r\right],$$

and

$$u^{(\mathrm{DL})}_{0(i)Ln-Ln} = \frac{\left[3\kappa_{(i)} + 4\mu_{(i)} + 2A_{(i)} + 6B_{(i)} + 2C_{(i)}\right]}{4(\kappa_{(i)} + 4\mu_{(i)}/3)}\left(\frac{K_{(i)L}}{k}\right)^2(kd_i)^2\left(\frac{u_{0(i)Ln}}{d_i}\right)^2, \tag{7}$$

where $A_{(i)}$, $B_{(i)}$ and $C_{(i)}$ are third-order elastic constants of the materials of M_1 and M_2, d_i is the thickness of M_i, $i = 1$ or 2.

The bulk nonlinearities of materials of M_1 and M_2 bring on the second-harmonic generation as Lamb wave propagates in the composite plate. There is a second-order boundary condition that requires that the components of second-harmonic stress equal zero at $y = d_1$ and $-d_2$ and that the components of second-harmonic stress, as well as second-harmonic displacement components along the *oy* and *oz* axes, be continuous at the interface $y = 0$. Usually, the second-order boundary condition cannot be satisfied if we consider only the driven second harmonic in a two-layered plate. In order to satisfy the boundary condition, there must exist a freely propagating second harmonic in M_1 and M_2, separately.[7,13] In fact, the driven second harmonic produced by the components of $F\left[u^{(1)}_{(i)}\right]$ may be considered as a particular solution of Eq.(2), and the freely propagating second harmonic can be introduced as a general solution of Eq.(2) when its term on the right-hand side equals zero.

For a Lamb wave propagating in a two-layered plate, eight partial bulk waves (i.e. $u_{(i)T1}$, $u_{(i)L1}$, $u_{(i)T2}$ and $u_{(i)L2}$) are reflected at upper, lower and interface boundaries of the composite plates. In fact, the partial bulk waves $u_{(i)L1}$ and $u_{(i)T1}$ may be regarded as the reflected waves of $u_{(i)L2}$ and $u_{(i)T2}$, and vice versa. So, we may use the technique of analysing the nonlinear acoustic reflection at an interface for reference to determine the formal solution of the freely propagating second harmonic.[13] As the driven cumulative second harmonic $u^{(\mathrm{DL})}_{(i)L1-L1}$and $u^{(\mathrm{DL})}_{(i)L2-L2}$ retain the growth effects along the *oy*- and *oz*-axes, the freely propagating second harmonic to be introduced should include the cumulative terms along the *oy*- and *oz*-axes too. Taking into account the condition of phase matching between the driven and the freely propagating second harmonics, the formal solution of the freely propagating second harmonic satisfying Eq.(2) when $F\left[u^{(1)}_{(i)}\right] = 0$ is found to be[7,13]

$$\begin{aligned} u^{(\mathrm{F})}_{(i)Tn} &= u^{(\mathrm{FC})}_{(i)Tn} + u^{(\mathrm{FP})}_{(i)Tn} = \left\{\left[\cos\theta_{(i)T}z + (-1)^{n-1}\sin\theta_{(i)T}y\right]u^{(\mathrm{FC})}_{0(i)Tn} + u^{(\mathrm{FP})}_{0(i)T(n)}\right\} \\ &\quad\times(-1)^n(\hat{x}\times K^0_{(i)Tn})\exp[\mathrm{j}2K_{(i)Tn}\cdot r], \end{aligned} \tag{8}$$

and

$$\begin{aligned} u^{(\mathrm{F})}_{(i)Ln} &= u^{(\mathrm{FC})}_{(i)Ln} + u^{(\mathrm{FP})}_{(i)Ln} = \left\{\left[\cos\theta_{(i)L}y + (-1)^{n-1}\sin\theta_{(i)L}y\right]u^{(\mathrm{FC})}_{0(i)Ln} + u^{(\mathrm{FP})}_{0(i)Ln}\right\} \\ &\quad\times K^0_{(i)Ln}\exp[\mathrm{j}2K_{(i)Ln}\cdot r], \quad n = 1, 2, \end{aligned} \tag{9}$$

where $n = 1, 2$; $u^{(\mathrm{FC})}_{(i)Tn}$ and $u^{(\mathrm{FC})}_{(i)Ln}$ are called the freely propagating cumulative second-harmonic; $u^{(\mathrm{FP})}_{(i)Tn}$ and $u^{(\mathrm{FP})}_{(i)Ln}$ are called freely propagating plane second-harmonic. Figure 3 shows the freely propagating second-harmonic fields.

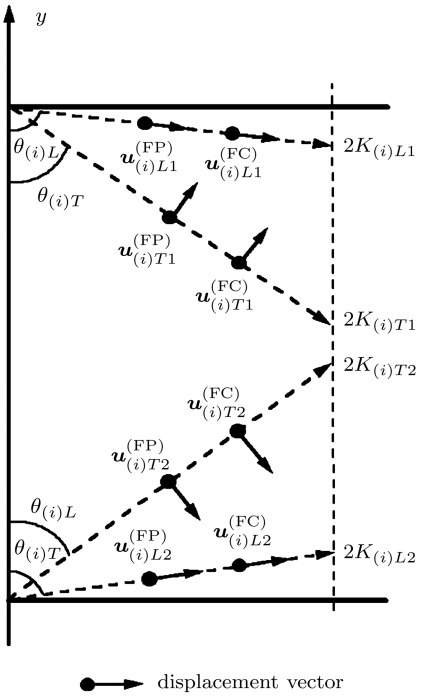


Fig.3. The freely propagating plane and cumulative second-harmonic fields.

In the i-th layer ($i = 1$ or 2), the sum of the driven and the freely propagating second-harmonics consist of the ultimate second-harmonics $\left(\text{denoted by } u^{(2)}_{(i)}\right)$. Thus we have

$$\begin{aligned} u^{(2)}_{(i)} &= u^{(2\mathrm{C})}_{(i)} + u^{(2\mathrm{P})}_{(i)} \\ u^{(2\mathrm{C})}_{(i)} &= \sum_{n=1}^{2} \left[u^{(\mathrm{DL})}_{(i)Ln-Ln} + u^{(\mathrm{FC})}_{(i)Ln} + u^{(\mathrm{FC})}_{(i)Tn} \right] \\ u^{(2\mathrm{P})}_{(i)} &= \sum_{n=1}^{2} \left[u^{(\mathrm{FP})}_{(i)Ln} + u^{(\mathrm{FP})}_{(i)Tn} \right] + u^{(\mathrm{DP})}_{(i)}. \end{aligned} \tag{10}$$

We called $u^{(2\mathrm{C})}_{(i)}$ and $u^{(2\mathrm{P})}_{(i)}$ the ultimate cumulative second-harmonic and the ultimate plane second-harmonic in solid layers respectively, we are interested in analysing the process of generation of $u^{(2\mathrm{C})}_{(i)}$ for its cumulative growth effect. A second-order boundary condition may be satisfied by taking into account the driven and the freely propagating second-harmonics simultaneously. Second-harmonic stresses in a two-layered plate include two terms. One term is produced by the ultimate second-harmonic through the linear Hook's law, and the other is caused by the fundamental waves due to the nonlinear Hook's law.[17] According to the second-order boundary condition, we have[7]

$$[M(2\omega, 2k)] \begin{bmatrix} \cdots \\ u^{(\mathrm{FC})}_{0(i)Ln} \cos\theta_{(i)L} + u^{(\mathrm{DL})}_{0(i)Ln-Ln} \sin\theta_{(i)L} \\ u^{(\mathrm{FC})}_{0(i)Tn} \cos\theta_{(i)T} \\ \cdots \end{bmatrix} z + [M(2\omega, 2k)] \begin{bmatrix} \cdots \\ u^{(\mathrm{FP})}_{0(i)Ln} \\ u^{(\mathrm{FP})}_{0(i)Tn} \\ \cdots \end{bmatrix} = \begin{bmatrix} \cdots \\ R_{(i)} \\ R'_{(i)} \\ \cdots \end{bmatrix} \tag{11}$$

with

$$\begin{bmatrix} \cdots \\ R_{(i)} \\ R'_{(i)} \\ \cdots \end{bmatrix} = \begin{bmatrix} u^{(2)}_y(0) \\ u^{(2)}_z(0) \\ T^{(2)}_{yz}(0) \\ T^{(2)}_{yy}(0) \\ T^{(2)}_{yz}(d_1) \\ T^{(2)}_{yy}(d_1) \\ T^{(2)}_{yz}(-d_2) \\ T^{(2)}_{yy}(-d_2) \end{bmatrix}, \tag{12}$$

where the coefficient matrix $[M(2\omega, 2k)]$ can be obtained from $[M(\omega, k)]$ by substituting $2kd$ for kd. The term on the right-hand side of Eq.(11) is an 8×1 matrix, in which $u^{(2)}_y(0)$ and $u^{(2)}_z(0)$ are the displacement components of the driven plane second-harmonic at $y = 0$; $T^{(2)}_{yz}(0)$, $T^{(2)}_{yy}(0)$, $T^{(2)}_{yz}(d_1)$, $T^{(2)}_{yy}(d_1)$, $T^{(2)}_{yz}(-d_2)$ and $T^{(2)}_{yy}(-d_2)$ are second-harmonic stress components arising from the fundamental waves and the driven second-harmonic (without the factor z). Equation (11) should be satisfied at any point along the oz-axis. Thus we have[7,13]

$$[M(2\omega,2k)]\begin{bmatrix}\cdots\\ u^{(\mathrm{FC})}_{0(i)Ln}\cos\theta_{(i)L}+u^{(\mathrm{DL})}_{0(i)Ln-Ln}\sin\theta_{(i)L}\\ u^{(\mathrm{FC})}_{0(i)Tn}\cos\theta_{(i)T}\\ \cdots\end{bmatrix}=\begin{bmatrix}\cdots\\0\\0\\\cdots\end{bmatrix}, \tag{13}$$

and

$$[M(2\omega,2k)]\begin{bmatrix}\cdots\\ u^{(\mathrm{FP})}_{0(i)Ln}\\ u^{(\mathrm{FP})}_{0(i)Tn}\\ \cdots\end{bmatrix}=\begin{bmatrix}\cdots\\R_{(i)}\\R'_{(i)}\\\cdots\end{bmatrix}. \tag{14}$$

Similar to the case of Lamb-wave propagating in a solid plate, $|M(\omega,_k)| = 0$ determines the dispersion curves of the ω Lamb-wave propagation and $|M(2\omega,2k)| = 0$ determines the dispersion curves of the 2ω Lamb-wave propagation. Generally, $|M(2\omega,2k)| = 0$ cannot be deduced from $|M(\omega,_k)| = 0$ due to the dispersion of Lamb-wave propagation. Thus Eq.(13) has a zero solution, that is, no cumulative second-harmonic is generated in a composite plate. Since the interest of our analysis is focused on the cumulative second-harmonic generation of Lamb waves propagation, we now discuss the existence conditions of cumulative second-harmonic generation and obtain the corresponding analytical solution.

2.1. Existence conditions of cumulative second-harmonic generation

We find from the foregoing that the cumulative second-harmonic generation of Lamb waves propagation can occur for the particular cases in which $|M(2\omega,2k)| = 0$ along with $|M(\omega,_k)| = 0$. These special cases are of interest and we will analyse them thoroughly. Because the dispersion relationship of Lamb waves propagation is very complicated and its explicit form cannot be presented, the existence conditions can be analysed only by quantitative evaluation. As an example, we use a numerical result to illustrate the existence conditions of cumulative second-harmonic generation of Lamb wave propagation. The materials of M_1 and M_2 are glass and iron, respectively. The corresponding parameters are $\rho(1) = 4463\,\mathrm{kg\cdot m^{-3}}$, $\kappa(1) = 41.1\times10^9\,\mathrm{Pa}$, $\mu(1) = 21.7\times10^9\,\mathrm{Pa}$, $\rho(2) = 7700\,\mathrm{kg\cdot m^{-3}}$, $\kappa(2) = 172\times10^9\,\mathrm{Pa}$, and $\mu(2) = 84\times10^9\,\mathrm{Pa}$.[18] In Fig.4 the solid curves represent the dispersion curves due to $|M(\omega,_k)| = 0$, and the dash curves represent the dispersion curves due to $|M(2\omega,2k)| = 0$. The intersection points of the dash and solid curves satisfy $|M(\omega,_k)| = 0$ and $|M(2\omega,2k)| = 0$ simultaneously. These points are presented by coordinates (fd_1, c). Figure 4 gives a series of intersection values, for example (fd_1, c)—(0.634, 5.961), (0.958, 5.732), (1.125, 6.32), (1.916, 5.833). At these particular points, the second-harmonic generation of Lamb wave propagation may retain the cumulative growth effect.

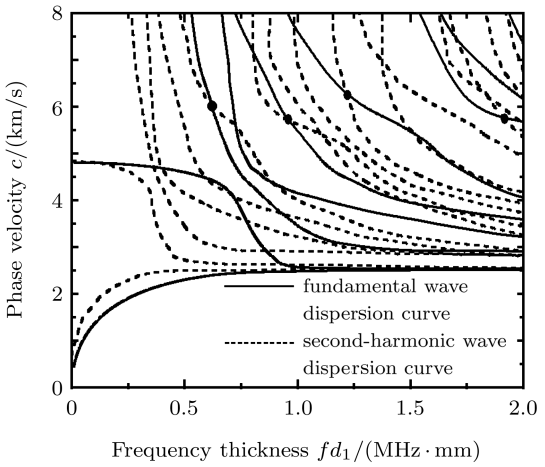


Fig.4. The dispersion curves of two-layered solid plates and the intersection point of cumulative second-harmonic generation.

2.2. Analytical solution of the cumulative second harmonic

There exists a nontrivial solution for Eq.(13) when $|M(2\omega,2k)| = 0$ along with $|M(\omega,_k)| = 0$. We can obtain the following matrix expression from Eq.(13)

$$\begin{bmatrix} \cdots \\ u^{(\mathrm{FC})}_{0(i)Ln}\cos\theta_{(i)L} + u^{(\mathrm{DL})}_{0(i)Ln-Ln}\sin\theta_{(i)L} \\ u^{(\mathrm{FC})}_{0(i)Tn}\cos\theta_{(i)T} \\ \cdots \end{bmatrix} = \begin{bmatrix} \cdots \\ E^{(2\omega)}_{(i)Ln} \\ E^{(2\omega)}_{(i)Tn} \\ \cdots \end{bmatrix} u^{(\mathrm{FC})}_{0(2)T2}\cos\theta_{(2)T}. \tag{15}$$

The seven constants $E^{(2\omega)}_{(i)Ln}$ and $E^{(2\omega)}_{(i)Tn}$ $\left(i = 1, 2;\ n = 1, 2;\ E^{(2\omega)}_{(2)T2} = 1\right)$ are easily determined from any seven equations of Eq.(13). Change the form of Eq.(15) into the following form:

$$\begin{bmatrix} \cdots \\ u^{(\mathrm{FC})}_{0(i)Ln} \\ u^{(\mathrm{FC})}_{0(i)Tn} \\ \cdots \end{bmatrix} = \begin{bmatrix} \cdots \\ E^{(2\omega)}_{(i)Ln}u^{(\mathrm{FC})}_{0(2)T2}\cos\theta_{(2)T}/\cos\theta_{(i)L} - u^{(\mathrm{DL})}_{0(i)Ln-Ln}\tan\theta_{(i)L} \\ E^{(2\omega)}_{(i)Tn}u^{(\mathrm{FC})}_{0(2)T2}\cos\theta_{(2)T}/\cos\theta_{(i)T} \\ \cdots \end{bmatrix}. \tag{16}$$

The oz-axis components of the ultimate cumulative second-harmonic, denoted by $u^{(2\mathrm{C})}_{(i)z}$, are given in the following form (neglecting the factor $\exp(2\mathrm{j}kz - 2\mathrm{j}\omega t)$)

$$\begin{aligned} u^{(2\mathrm{C})}_{(i)z} = &\sum_{n=1}^{2} \begin{bmatrix} \sin^2\theta_{(i)L}u^{(\mathrm{DL})}_{0(i)Ln-Ln}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)L}ky] \\ +\sin\theta_{(i)L}\cos\theta_{(i)L}u^{(\mathrm{FC})}_{0(i)Ln}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)L}ky] \\ +\cos^2\theta_{(i)T}u^{(\mathrm{FC})}_{0(i)Tn}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)T}ky] \end{bmatrix} z \\ &+\sum_{n=1}^{2}(-1)^{n-1} \begin{bmatrix} -\sin\theta_{(i)L}\cos\theta_{(i)L}u^{(\mathrm{DL})}_{0(i)Ln-Ln}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)L}ky] \\ +\sin^2\theta_{(i)L}u^{(\mathrm{FC})}_{0(i)Ln}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)L}ky] \\ +\sin\theta_{(i)T}\cos\theta_{(i)T}u^{(\mathrm{FC})}_{0(i)Tn}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)T}ky] \end{bmatrix} y. \end{aligned} \tag{17}$$

The amplitude of cumulative second-harmonic is zero at the position of the excitation source.[13] For simplicity, we assume the excitation source to be a line source located at the position $(y,\ z) = (\xi, 0)$ of M_1 or M_2. According to the initial condition, from Eq.(17), we have

$$\sum_{n=1}^{2}(-1)^{n-1} \begin{bmatrix} -\sin\theta_{(i)L}\cos\theta_{(i)L}u^{(\mathrm{DL})}_{0(i)Ln-Ln}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)L}k\xi] \\ +\sin^2\theta_{(i)L}u^{(\mathrm{FC})}_{0(i)Ln}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)L}k\xi] \\ +\sin\theta_{(i)T}\cos\theta_{(i)T}u^{(\mathrm{FC})}_{0(i)Tn}\exp[(-1)^n 2\mathrm{j}\alpha_{(i)T}k\xi] \end{bmatrix} = 0. \tag{18}$$

Combining Eq.(16) and Eq.(18) leads to

$$u^{(\mathrm{FC})}_{0(2)T2} = \frac{\sum\limits_{n=1}^{2}(-1)^{n-1}\tan\theta_{(i)L}u^{(\mathrm{DL})}_{0(i)Ln-Ln}\exp\left[(-1)^n 2\mathrm{j}\alpha_{(i)L}k\xi\right]}{\cos\theta_{(2)T}\sum\limits_{n=1}^{2}(-1)^{n-1}\left\{\begin{array}{l} \dfrac{\sin^2\theta_{(i)L}}{\cos\theta_{(i)L}}E^{(2\omega)}_{(i)Ln}\exp\left[(-1)^n 2\mathrm{j}\alpha_{(i)L}k\xi\right] \\ +\sin\theta_{(i)T}E^{(2\omega)}_{(i)Tn}\exp\left[(-1)^n 2\mathrm{j}\alpha_{(i)T}k\xi\right] \end{array}\right\}}. \tag{19}$$

In Eq.(19), the subscript (i) is dependent on whether the excitation source is in M_1 or M_2. Now, the freely propagating cumulative second-harmonics $u^{(\mathrm{FC})}_{(i)Ln}$ and $u^{(\mathrm{FC})}_{(i)Tn}$ can be easily determined by substituting Eq.(19) into Eq.(16). So, the ultimate cumulative second-harmonic in the ith layer is

$$\begin{aligned} u^{(2\mathrm{C})}_{(i)} = &\sum_{n=1}^{2} \begin{bmatrix} u^{(\mathrm{DL})}_{0(i)Ln-Ln}[\sin\theta_{(i)L}z + (-1)^n\cos\theta_{(i)L}y] \\ +u^{(\mathrm{FC})}_{0(i)Ln}[\cos\theta_{(i)L}z + (-1)^{n-1}\sin\theta_{(i)L}y] \end{bmatrix} K^0_{(i)Ln}\exp\left[2\mathrm{j}K_{(i)Ln}\cdot r\right] \\ &+\sum_{n=1}^{2} u^{(\mathrm{FC})}_{0(i)Tn}[\cos\theta_{(i)T}z + (-1)^{n-1}\sin\theta_{(i)T}y](-1)^n\left[\hat{x}\times K^0_{(i)Tn}\right]\exp\left[2\mathrm{j}K_{(i)Tn}\cdot r\right], \end{aligned} \tag{20}$$

which is obtained from Eqs.(7), (16) and (19). $u^{(\mathrm{DL})}_{0(i)Ln-Ln}$ is given by Eq.(8). We can find from Eq.(14) that the cumulative second-harmonic fields of Lamb wave propagation are dependent on the position of the excitation source ξ. Equation (20) is the solution that we need because it satisfies the boundaries and initial conditions of excitation.

In general, Eq.(14) does not have a solution with physical significance, for its right-hand side does not equal zero, that is, the freely propagating plane second harmonic cannot be determined in the present analysis. This is a shortcoming of the present analysis. After thoroughly studying the process of analysing the cumulative second-harmonic generation, we can find that this shortcoming cannot be avoided due to the assumption (i.e. the higher harmonic energy is much less than the fundamental one and there is no coupling between the fundamental waves and the higher harmonics). However, to some extent, the solution of the cumulative second harmonic of Lamb-wave propagation shown in Eq.(20) may be used to describe the effect of growth along the propagation distance. On the other hand, after the second harmonic propagates some distance, the cumulative second-harmonic generation plays a dominant role due to its growth effect. Thus it is less important that the freely propagating plane second harmonic cannot be determined.

3. Quantitative analyses

A quantitative computation is used to illustrate the foregoing analysis results. The relation between Lamb wave propagation velocity c and frequency thickness fd_1 are shown in Fig.4. The *oy*- and *oz*-axis components of the cumulative second harmonic are given by

$$u^{(2\mathrm{C})}_{(i)y} = \sum_{n=1}^{2} \left\{ \begin{array}{l} \left[\begin{array}{l} u^{(\mathrm{DL})}_{0(i)Ln-Ln}[\sin\theta_{(i)L}z + (-1)^n \cos\theta_{(i)L}y] \\ +u^{(\mathrm{FC})}_{0(i)Ln}[\cos\theta_{(i)L}z + (-1)^{n-1}\sin\theta_{(i)L}y] \end{array} \right] (-1)^n \cos\theta_{(i)L} \exp\left[2\mathrm{j}K_{(i)Ln}\cdot r\right] \\ +u^{(\mathrm{FC})}_{0(i)Tn}\left[\cos\theta_{(i)T}z + (-1)^{n-1}\sin\theta_{(i)T}y\right](-1)^{n-1}\sin\theta_{(i)T}\exp\left[2\mathrm{j}K_{(i)Tn}\cdot r\right], \end{array} \right. \tag{21}$$

and

$$u^{(2\mathrm{C})}_{(i)z} = \sum_{n=1}^{2} \left\{ \begin{array}{l} \left[\begin{array}{l} u^{(\mathrm{DL})}_{0(i)Ln-Ln}[\sin\theta_{(i)L}z + (-1)^n \cos\theta_{(i)L}y] \\ +u^{(\mathrm{FC})}_{0(i)Ln}[\cos\theta_{(i)L}z + (-1)^{n-1}\sin\theta_{(i)L}y] \end{array} \right] \sin\theta_{(i)L} \exp\left[2\mathrm{j}K_{(i)Ln}\cdot r\right] \\ +u^{(\mathrm{FC})}_{0(i)Tn}\left[\cos\theta_{(i)T}z + (-1)^{n-1}\sin\theta_{(i)T}y\right]\cos\theta_{(i)T}\exp\left[2\mathrm{j}K_{(i)Tn}\cdot r\right], \end{array} \right. \tag{22}$$

where $n = 1, 2$. We still assume that the materials of $\mathrm{M}_1$and M_2 are glass and iron, respectively. The corresponding third-order elastic constants are $A_{(1)} = 420 \times 10^9\,\mathrm{Pa}$, $B_{(1)} = -120 \times 10^9\,\mathrm{Pa}$, $C_{(1)} = 130 \times 10^9\,\mathrm{Pa}$, $A_{(2)} = 1100 \times 10^9\,\mathrm{Pa}$, $B_{(2)} = -1580 \times 10^9\,\mathrm{Pa}$, and $C_{(2)} = 1230 \times 10^9\,\mathrm{Pa}$. We can give the ultimate cumulative second-harmonic fields in the two-layer plate by combining Eqs.(7), (16), (19), (21) and (22).

Figure 5 shows cumulative second-harmonic fields where the line source is located as $(y, z) = (d_1, 0)$, the other is located at $(y, z) = (-d_2, 0)$ (real parts of $u^{(2\mathrm{C})}_{(i)y}$and $u^{(2\mathrm{C})}_{(i)z}$, and corresponding to $u^2_{(2)T2}/d_1$). The results illustrate that cumulative second-harmonic field patterns are dependent on the position of the excitation source.

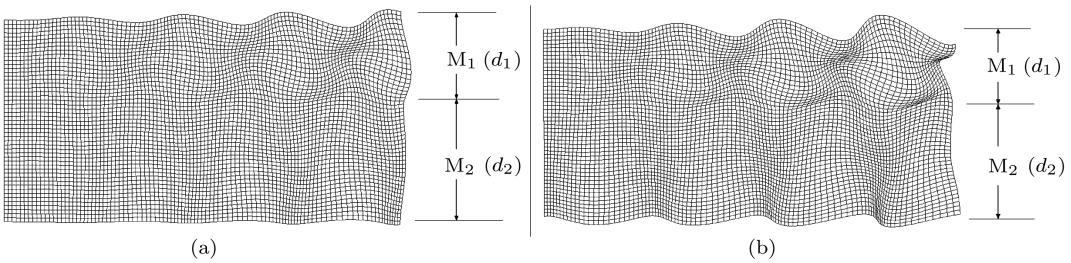


Fig.5. Cumulative second-harmonic field patterns of Lamb wave propagation, with $fd_1 = 1.125\,(\mathrm{MHz}\cdot\mathrm{mm})$, $c = 6.32\,(\mathrm{km}\cdot\mathrm{s}^{-1})$, and $(y, z) = (d_1, 0)$ (a); $fd_1 = 1.125\,(\mathrm{MHz}\cdot\mathrm{mm})$, $c = 6.32\,(\mathrm{km}\cdot\mathrm{s}^{-1})$, and $(y, z) = (-d_2, 0)$ (b).

4. Conclusions

We have demonstrated that cumulative second-harmonic generation of Lamb wave propagating in a two-layer solid plate may be described in terms of the second-order perturbation and the technique of analysing the nonlinear reflection of acoustic waves at an interface. It is shown that there are some particular cases in which the second-harmonic of Lamb wave propagation does have a cumulative effect. The corresponding existence conditions of cumulative second-harmonic generation are given for an idiographic waveguide made of glass and iron. These conditions may be of significance in practical applications, for example, we can select an appropriate thickness of the solid plate and excitation frequency of Lamb wave propagation (the intersections in Fig.4), at the moment, the corresponding second harmonic can grow linearly with the propagation distance. Based on the initial condition of excitation, the cumulative second-harmonic solution is presented in this paper. This solution is applicable to the numerical analysis of cumulative second-harmonic field patterns.

Numerical results reveal the distortion of cumulative second-harmonic field patterns, as well as the dependence of the corresponding field patterns on the position of the excitation source. Although there are some shortcomings since the formula of the cumulative second harmonic is complicated and less straightforward and the freely propagating plane second harmonic cannot be fully determined, the present analysis yields a physical insight into the process of cumulative second-harmonic generation of Lamb-wave propagating in a two-layered solid plate.

Appendix

The coefficient matrix of Eq.(5), $[M(\omega,k)]$, is given by $[M(\omega,k)] = \begin{pmatrix} [M]_{11} & [M]_{12} \\ [M]_{21} & [M]_{22} \end{pmatrix}$, where the forms of the sub-matrix $[M]_{ij}$ $(i,j=1,2)$ are as follows:

$$[M]_{11} = \begin{bmatrix} -R_{(1)}L_{(1)}(-D_1) & -S_{(1)}T_{(1)}(-D_1) & R_{(1)}L_{(1)}(D_1) & S_{(1)}T_{(1)}(D_1) \\ X_{(1)}L_{(1)}(-D_1) & Y_{(1)}T_{(1)}(-D_1) & X_{(1)}L_{(1)}(D_1) & Y_{(1)}T_{(1)}(D_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$[M]_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -R_{(2)}L_{(2)}(-D_2) & -S_{(2)}T_{(2)}(-D_2) & R_{(2)}L_{(2)}(D_2) & S_{(2)}T_{(2)}(D_2) \\ X_{(2)}L_{(2)}(-D_2) & Y_{(2)}T_{(2)}(-D_2) & X_{(2)}L_{(2)}(D_2) & Y_{(2)}T_{(2)}(D_2) \end{bmatrix};$$

$$[M]_{21} = \begin{bmatrix} -R_{(1)} & -S_{(1)} & R_{(1)} & S_{(1)} \\ X_{(1)} & Y_{(1)} & X_{(1)} & Y_{(1)} \\ -\cos\theta_{(1)L} & \sin\theta_{(1)T} & \cos\theta_{(1)L} & -\sin\theta_{(1)T} \\ \sin\theta_{(1)L} & \cos\theta_{(1)T} & \sin\theta_{(1)L} & \cos\theta_{(1)T} \end{bmatrix};$$

$$[M]_{22} = \begin{bmatrix} R_{(2)} & S_{(2)} & -R_{(2)} & -S_{(2)} \\ -X_{(2)} & -Y_{(2)} & -X_{(2)} & -Y_{(2)} \\ \cos\theta_{(2)L} & -\sin\theta_{(2)T} & -\cos\theta_{(2)L} & \sin\theta_{(2)T} \\ -\sin\theta_{(2)L} & -\cos\theta_{(2)T} & -\sin\theta_{(2)L} & -\cos\theta_{(2)T} \end{bmatrix}$$

where $L_{(i)}(\pm D_i) = \exp\left[\pm \mathrm{j}\alpha_{(i)L} k D_i\right]$, $T_{(i)}(\pm D_i) = \exp\left[\pm \mathrm{j}\alpha_{(i)T} k D_i\right]$, $R_{(i)} = 2\mu_{(i)} \cos\theta_{(i)L}$, $S_{(i)} = \mu_{(i)}[\alpha^2_{(i)T} - 1] \sin\theta_{(i)T}$, $X_{(i)} = [\left(\kappa_{(i)} + 4\mu_{(i)}/3\right) \alpha^2_{(i)L} + \kappa_{(i)} - 2\mu_{(i)}/3] \sin\theta_{(i)L}$, $Y_{(i)} = -2\mu_{(i)} \cos\theta_{(i)T}$, and $D_1 = d_1$, $D_2 = -d_2$.

References

[1] Rose J L 2002 *J. Pressure Vessel Technol.* **124** 273

[2] Declercq N F, Degrieck J and Leroy O 2004 *Ultrasonics* **42** 173

[3] Kundu T, Maji A, Ghosh T and Maslov K 1998 *Ultrasonics* **35** 573

[4] Jhang K Y and Kim K C 1999 *Ultrasonics* **37** 39

[5] Solodov I Y 1998 *Ultrasonics* **36** 282

[6] Deng M X 2006 *Ultrasonics* **44** Suppl. 1 E1157

[7] Deng M X 1999 *J. Appl. Phys.* **85** 3051

[8] Deng M X 2000 *J. Phys.* D-*Appl. Phys.* **33** 207

[9] Auld B A 1973 *Acoustic Fields and Waves in Solids* (New York: John Wiley) p54

[10] Deng M X 1996 *Acta Acoust.* **21** 429 (in Chinese)

[11] Deng M X 1997 *Acta Acoust.* **22** 182 (in Chinese)

[12] Qian Z W 1995 *Chin. Phys.* **4** 670

[13] Zhou S and Shui Y 1992 *J. Appl. Phys.* **72** 5070

[14] Hamilton M F and Blackstock D T 1998 *Nonlinear Acoustics* (New York: Academic) p86

[15] Jones G L and Kobett D R 1963 *J. Acoust. Soc. Am.* **35** 5

[16] Du H W, Peng H, Jiang Z H and Feng H Q 2007 *Acta Phys. Sin.* **56** 6496

[17] Shui Y and Solodov I Y 1988 *J. Appl. Phys.* **64** 6155

[18] Hearmon R F S 1979 *Landolt–Bornstein Numerical Data and Functional Relationship in Science and Technology* (Berlin: Springer) Group III p235