Vol 17 No 11, November 2008
1674-1056/2008/17(11)/4204-03

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Exact analytical solution for quantum spins mixing in spin-1 Bose–Einstein condensates*

Chen Ai-Xi(陈爱喜)†, Qiu Wan-Ying(邱万英), and Wang Zhi-Ping(王志平)

Department of Applied Physics, East China Jiaotong University, Nanchang 330013, China

(Received 27 September 2007; revised manuscript received 15 January 2008)

This paper solves exactly a set of fully quantized coupled equations describing the quantum dynamics of quantum spins mixing in spin-1 Bose–Einstein condensates by deriving the exact explicit analytical expressions for the evolution of creation and annihilation operators.

Keywords: Bose–Einstein condensates, spin mixing
PACC: 4250

Recently there are considerable interests in multiwave mixing in cold atoms and Bose–Einstein condensates.[1−12] One of the great challenges in quantized multiwave mixing is to solve exactly a set of fully quantized coupled equations in order to better describe the quantum dynamics including the evolution of quantum fluctuations, entanglements, and correlations as well as revealing the possible Rabi oscillations, collapse and revivals.[12]

To the best of our knowledge, no one has been able to obtain the exact analytical results for the set of coupled operator equations for the quantum spins mixing in spin-1 Bose–Einstein condensates because it is a highly nonlinear and fully quantized set of coupled equations. Here we shall accomplish such very difficult task.

In this paper, we consider a dilute gas of trapped bosonic atom with hyperfine spin $f = 1$. Due to the nonlinear interactions between condensate atoms, the internal dynamics of the spin-mixing process will evolve with time. By the second quantized theory, the Hamiltonian describing quantum spins mixing in spin-1 Bose–Einstein condensates[1] can be expressed as

$$\begin{aligned}\hat{H} =&\hbar g(\hat{a}_1^\dagger\hat{a}_1^\dagger\hat{a}_1\hat{a}_1 + \hat{a}_{-1}^\dagger\hat{a}_{-1}^\dagger\hat{a}_{-1}\hat{a}_{-1} - 2\hat{n}_1\hat{n}_{-1}\\ &+ 2\hat{n}_1\hat{n}_0 + 2\hat{n}_{-1}\hat{n}_0 + 2\hat{a}_0^\dagger\hat{a}_0^\dagger\hat{a}_1\hat{a}_{-1}\\ &+ 2\hat{a}_1^\dagger\hat{a}_{-1}^\dagger\hat{a}_0\hat{a}_0),\end{aligned} \tag{1}$$

where g is a coupling parameter, $\hat{n}_j = \hat{a}_j^\dagger\hat{a}_j$, $\hat{a}_j$ $(j = 1, 0, -1)$ are the annihilation operators associated with the condensate modes of different spin states and they satisfy the usual commutation relations for bosons $[\hat{a}_j, \hat{a}_k^\dagger] = \delta_{jk}$ and $[\hat{a}_j, \hat{a}_k] = 0$. Furthermore, this $\hat{H}$ describes a non-symmetric part, the symmetric part of the Hamiltonian is a constant operator for a fixed number of atoms and is not considered.

Introducing an angular momentum of the three components $L_{x,y,z}$ as follows:[1,13,14]

$$\hat{L}_- = \hat{L}_x - \mathrm{i}\hat{L}_y = \sqrt{2}(\hat{a}_1^\dagger\hat{a}_0 + \hat{a}_0^\dagger\hat{a}_{-1}), \tag{2a}$$

$$\hat{L}_+ = \hat{L}_x + \mathrm{i}\hat{L}_y = \sqrt{2}(\hat{a}_0^\dagger\hat{a}_1 + \hat{a}_{-1}^\dagger\hat{a}_0), \tag{2b}$$

$$\hat{L}_z = \hat{a}_{-1}^\dagger\hat{a}_{-1} - \hat{a}_1^\dagger\hat{a}_1, \tag{2c}$$

where $\hat{L}_z$, the lowering operator $\hat{L}_-$ and raising operator $\hat{L}_+$ obey angular momentum commutation relations $[\hat{L}_+, \hat{L}_-] = 2\hat{L}_z$ and $[\hat{L}_z, \hat{L}_\pm] = \pm\hat{L}_\pm$, The model Hamiltonian (1) can be put into the form[1]

$$\hat{H} = \hbar g(\hat{L}^2 - 2\hat{N}), \tag{3}$$

where $\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$, and $\hat{N} = \hat{n}_1 + \hat{n}_0 + \hat{n}_{-1}$ denotes the total atom number in all three condensate modes of different spin states. Obviously, the operators $\hat{N}$, $\hat{L}^2$ and $\hat{L}_{z,\pm}$ ($\hat{L}_z$, $\hat{L}_+$, $\hat{L}_-$) are conserved quantities, i.e., $\hat{N}(t) = \hat{N}(0)$, $\hat{L}^2(t) = \hat{L}^2(0)$ and $\hat{L}_{z,\pm}(t) = \hat{L}_{z,\pm}(0)$ because they commute with the Hamiltonian. There exist the explicit expressions related all the eigenstates $|l, m\rangle$ of the angular momentum (2) to the Fock states[14] and the corresponding algebraic approach has been developed.[14−17]

*Project supported by Natural Science Foundation of Jiangxi, China (Grant Nos 0612006 and 2007GZW0819), the Scientific Research Foundation of Jiangxi Provincial Department of Education (Grant No [2007]191), and Funds from East China Jiaotong University.

†E-mail: aixichen@ecjtu.jx.cn

Using the Heisenberg equation of motion $\mathrm{i}\hbar\,\mathrm{d}\hat{A}/\mathrm{d}t = [\hat{A}, \hat{H}]$, we readily obtain the following equations:

$$\mathrm{i}\frac{\mathrm{d}\hat{a}_1}{\mathrm{d}\tau} = -2\hat{L}_z\hat{a}_1 + \sqrt{2}\hat{L}_+\hat{a}_0, \tag{4a}$$

$$\mathrm{i}\frac{\mathrm{d}\hat{a}_0}{\mathrm{d}\tau} = \sqrt{2}\hat{L}_-\hat{a}_1 + \sqrt{2}\hat{L}_+\hat{a}_-, \tag{4b}$$

$$\mathrm{i}\frac{\mathrm{d}\hat{a}_{-1}}{\mathrm{d}\tau} = 2\hat{L}_z\hat{a}_{-1} + \sqrt{2}\hat{L}_-\hat{a}_0, \tag{4c}$$

where $\tau = gt$.

Our purpose is to solve this set of operator-type differential equations to obtain the explicit analytical expressions showing how the annihilation operators vary with respect to the time $\tau = gt$. To achieve this goal, we follow the previous strategy[12] to put them into the more concise form with the corresponding formal solution,

$$\mathrm{i}\frac{\mathrm{d}\hat{\alpha}}{\mathrm{d}\tau} = \hat{K}\hat{\alpha} \Rightarrow \hat{\alpha}(\tau) = \mathrm{e}^{-\mathrm{i}\hat{K}\tau}\hat{\alpha}(0), \tag{5}$$

where $\hat{\alpha}(\tau) = (\hat{a}_1(\tau), \hat{a}_0(\tau), \hat{a}_{-1}(\tau))^{\mathrm{T}}$ and $\hat{K}$ is a time-independent 3×3 operator-type matrix of the form

$$\hat{\boldsymbol{K}} = \sqrt{2}\begin{pmatrix} -\sqrt{2}\,\hat{L}_z & \hat{L}_+ & 0 \\ \hat{L}_- & 0 & \hat{L}_+ \\ 0 & \hat{L}_- & \sqrt{2}\,\hat{L}_z \end{pmatrix}, \tag{6}$$

where $\hat{L}_z = \hat{L}_z(0)$ and $\hat{L}_\pm = \hat{L}_\pm(0)$.

Consequently, to solve the set of operator-type differential equations (4) has turned into finding the explicit expression of the operator $\exp(-\mathrm{i}\hat{\boldsymbol{K}}\tau)$. The extreme difficulty in doing so lies in the fact that the elements of the matrix $\hat{\boldsymbol{K}}$ are operators and they do not commute with each other due to $[\hat{L}_+, \hat{L}_-] = 2\hat{L}_z$ and $[\hat{L}_z, \hat{L}_\pm] = \pm\hat{L}_\pm$, and hence one cannot adopt the usual powerful techniques for c-number matrices.

We therefore utilize the previous approach[12] to diagonalize the operator-type matrix $\hat{\boldsymbol{K}}$. Defining $\hat{\Omega} = \sqrt{4\hat{L}^2+1}$ with $\hat{L}^2 = \hat{L}^2(0) = \hat{L}^2(t)$ and using the properties $[\hat{L}_z, \hat{L}_\pm] = \pm\hat{L}_\pm \Rightarrow \hat{L}_z\hat{L}_\pm = \hat{L}_\pm(\hat{L}_z \pm 1) \Rightarrow F(\hat{L}_z)\hat{L}_\pm = \hat{L}_\pm F(\hat{L}_z \pm 1)$, $\hat{L}_-\hat{L}_+ = \hat{L}^2 - \hat{L}_z^2 - \hat{L}_z = (\hat{\Omega}+1+2\hat{L}_z)(\hat{\Omega}-1-2\hat{L}_z)/4$, $\hat{L}_+\hat{L}_- = \hat{L}^2 - \hat{L}_z^2 + \hat{L}_z = (\hat{\Omega}+1-2\hat{L}_z)(\hat{\Omega}-1+2\hat{L}_z)/4$, we can from $\hat{\boldsymbol{K}}\hat{\psi} = \lambda\hat{\psi}$ with $\hat{\psi}(\tau) = (\alpha, \beta, \gamma)^{\mathrm{T}}$ to show that $\hat{\boldsymbol{K}}\hat{\boldsymbol{W}} = \hat{\boldsymbol{W}}\hat{\boldsymbol{D}}$ or $\hat{\boldsymbol{K}} = \hat{\boldsymbol{W}}\hat{\boldsymbol{D}}\hat{\boldsymbol{W}}^{-1}$ with the diagonal matrix $\hat{\boldsymbol{D}} = \mathrm{diag}(-2, -1+\hat{\Omega}, -1-\hat{\Omega})$ and hence

$$\mathrm{e}^{-\mathrm{i}\hat{\boldsymbol{K}}\tau} = \hat{\boldsymbol{W}}\begin{pmatrix} \mathrm{e}^{2i\tau} & 0 & 0 \\ 0 & \mathrm{e}^{\mathrm{i}\tau-\mathrm{i}\tau\hat{\Omega}} & 0 \\ 0 & 0 & \mathrm{e}^{\mathrm{i}\tau+\mathrm{i}\tau\hat{\Omega}} \end{pmatrix}\hat{\boldsymbol{W}}^{-1}, \tag{7}$$

where the operator-type matrix $\hat{\boldsymbol{W}}$ is

$$\hat{\boldsymbol{W}} = \begin{pmatrix} \hat{L}_+ & \sqrt{2}\,\hat{L}_+(1+\hat{\Omega}-2\hat{L}_z) & \sqrt{2}\,\hat{L}_+(1-\hat{\Omega}-2\hat{L}_z) \\ \sqrt{2}\,\hat{L}_z & (1+\hat{\Omega})^2 - 4\hat{L}_z^2 & (1-\hat{\Omega})^2 - 4\hat{L}_z^2 \\ -\hat{L}_- & \sqrt{2}\,\hat{L}_-(1+\hat{\Omega}+2\hat{L}_z) & \sqrt{2}\,\hat{L}_-(1-\hat{\Omega}+2\hat{L}_z) \end{pmatrix}. \tag{8}$$

All the operators in Eq.(8) are the corresponding values at $t = 0$. Besides, we can consider the conserved operators $\hat{\Omega}$ and $\hat{L}^2$ in Eqs.(7) and (8) as c-numbers because they commutes with all the other operators in Eqs.(6–8). However it should be noted that the operators $\hat{L}^2$ and $\hat{\Omega} = \sqrt{4\hat{L}^2+1}$ do not commute with the operators $\hat{a}_j$ and $\hat{a}_j^\dagger$, $j = 0, \pm$.

Then after some tedious but straightforward manipulations, we derive from Eqs.(5), (7) and (8) the exact explicit analytical expressions for the evolution of creation and annihilation operators as follows:

$$\hat{a}_1(\tau)\mathrm{e}^{-\mathrm{i}\tau} = f(\tau, \hat{L}_z)\hat{a}_1(0) + g(\tau, \hat{L}_z)\hat{a}_0(0) + h(\tau, \hat{L}_z)\hat{a}_{-1}(0), \tag{9a}$$

$$\hat{a}_0(\tau)\mathrm{e}^{-\mathrm{i}\tau} = s(\tau, \hat{L}_z)\hat{a}_1(0) + v\hat{a}_0(0) + s(\tau, -\hat{L}_z)\hat{a}_{-1}(0), \tag{9b}$$

$$\hat{a}_{-1}(\tau)\mathrm{e}^{-\mathrm{i}\tau} = h(\tau, -\hat{L}_z)\hat{a}_1(0) + g(\tau, -\hat{L}_z)\hat{a}_0(0) + f(\tau, -\hat{L}_z)\hat{a}_{-1}(0), \tag{9c}$$

with

$$f(\tau, \hat{L}_z) = \frac{\cos(\hat{\Omega}\tau) + \mathrm{e}^{\mathrm{i}\tau}}{2} + \frac{4\hat{\Omega}[\cos(\hat{\Omega}\tau) - \mathrm{e}^{\mathrm{i}\tau}](1+\hat{L}_z)\hat{L}_z - \mathrm{i}[\hat{\Omega}^2 + 3 + 4(4 - 3\hat{L}_z - \hat{\Omega}^2)\hat{L}_z]\sin(\hat{\Omega}\tau)}{2\hat{\Omega}(\hat{\Omega}^2 - 5)}, \tag{10a}$$

$$g(\tau, \hat{L}_z) = \frac{2\sqrt{2}\hat{L}_z[\mathrm{e}^{\mathrm{i}\tau} - \cos(\hat{\Omega}\tau)]}{\hat{\Omega}^2 - 5} - \mathrm{i}\frac{\sqrt{2}(6\hat{L}_z + \hat{\Omega}^2 - 9)}{\hat{\Omega}(\hat{\Omega}^2 - 5)}, \tag{10b}$$

$$h(\tau,\hat{L}_z) = \frac{[\hat{\Omega}^2 - 4\hat{L}_z(\hat{L}_z-1)-5][\cos(\hat{\Omega}\tau)-\mathrm{e}^{\mathrm{i}\tau}]}{2(\hat{\Omega}^2-5)} - \mathrm{i}\frac{\hat{\Omega}^2+4\hat{L}_z(3\hat{L}_z-5)+3}{2(\hat{\Omega}^2-5)\hat{\Omega}}\sin(\hat{\Omega}\tau), \tag{10c}$$

$$s(\tau,\hat{L}_z) = \frac{[-\hat{\Omega}^2+4\hat{L}_z(\hat{L}_z+1)+5]\hat{L}_z[\cos(\hat{\Omega}\tau)-\mathrm{e}^{\mathrm{i}\tau}]}{\sqrt{2}(\hat{\Omega}^2-5)} - \mathrm{i}\frac{(\hat{\Omega}^2-1)^2-4\hat{L}_z^2(1+6\hat{L}_z+\hat{\Omega}^2)+2(3+\hat{\Omega}^2)\hat{L}_z}{2\sqrt{2}(\hat{\Omega}^2-5)\hat{\Omega}}\sin(\hat{\Omega}\tau), \tag{10d}$$

$$v(\tau,\hat{L}_z) = \cos(\hat{\Omega}\tau) - \frac{4\hat{L}_z^2[\cos(\hat{\Omega}\tau)-\mathrm{e}^{\mathrm{i}\tau}]}{\hat{\Omega}^2-5} + \mathrm{i}\frac{\hat{\Omega}^2-12\hat{L}_z^2+3}{(\hat{\Omega}^2-5)\hat{\Omega}}\sin(\hat{\Omega}\tau). \tag{10e}$$

Equations (9) and (10) are the central results of the present paper and they represent the exact analytical solution to a set of four highly nonlinear and fully quantized coupled equations (4). They completely describe the quantum dynamics of the quantum spins mixing in spin-1 Bose–Einstein condensates including the evolution of quantum fluctuations, entanglements, average atom numbers ($\langle\hat{a}_k^\dagger\hat{a}_k\rangle$, $k=0,\pm1$) and correlations such as $\langle\hat{a}_i\hat{a}_j\rangle$ and $\langle\hat{a}_i^\dagger\hat{a}_j\rangle$ ($i\neq j$).

In summary, we have derived and solved exactly a set of fully quantized coupled equations describing the quantum dynamics including the evolution of quantum fluctuations, entanglements and correlations for the quantum spins mixing in spin-1 Bose–Einstein condensates. The exact analytical solution could also be used to demonstrate the possible Rabi oscillations, collapse and revivals.[18−21]

Acknowledgments

We would like to thank Professor Wu Ying for enlightening discussions.

References

[1] Law C K, Pu H and Bigelow N P1998 *Phys. Rev. Lett.* **81** 5257
[2] Pu H, Law C K, Raghavan S, Eberly J H and Bigelow N P 1999 *Phys. Rev.* A **60** 1463
[3] Koashi M and Ueda M 2000 *Phys. Rev. Lett.* **84** 1066
Huang G X 2004 *Chin. Phys.* **13** 1866
[4] Ohtsuka S and Kurihara S 2003 *Phys. Rev.* A **68** 013601
Xie B H and Jing H 2002 *Chin. Phys.* **11** 115
[5] Wu Y, Yang X and Xiao Y 2001 *Phys. Rev. Lett.* **86** 2200
[6] Goldstein E V and Meystre P 1999 *Phys. Rev.* A **59** 1509
Wei W 2005 *Chin. Phys.* **14** 2407
[7] Wu Y, Yang X, Sun C P, Zhou X J and Wang Y Q 2000 *Phys. Rev.* A **61** 043604
[8] Villain P, Ohberg P, Santos L, Sanpera A and Lewenstein M 2001 *Phys. Rev.* A **64** 023606
[9] Heurich J, Pu H, Moore M G and Meystre P 2001 *Phys. Rev.* A **63** 033605
[10] Wu Y and Yang X 2006 *J. Opt. Soc. Am.* B **23** 2888
[11] Wu Y and Yang X 2006 *Opt. Lett.* **31** 519
[12] Wu Y and Yang X 2005 *Opt. Lett.* **30** 311
[13] Wang L, Puri R R and Eberly J H 1992 *Phys. Rev.* A **46** 7192
[14] Wu Y 1996 *Phys. Rev.* A **54** 4534
[15] Yang X, Wu Y and Li Y J 1997 *Phys. Rev.* A **55** 4545
[16] Wu X and Yang X 2001 *Phys. Rev.* A **63** 043816
[17] Wu Y, Yang X and Sun C 2000 *Phys. Rev.* A **62** 063603
[18] Yao Y S, Zou J and Shao B 2003 *Chin. Phys.* **12** 649
[19] Yang R S and Yang J H 2008 *Chin. Phys.* B **17** 1189
[20] Li F, Shu W X, Luo H L and Ren Z Z 2007 *Chin. Phys.* **16** 650
[21] Zhou L, Kong L B and Zhan M S 2008 *Chin. Phys.* B **17** 1601