Vol 17 No 11, November 2008
1674-1056/2008/17(11)/4153-05

Chinese Physics B

© 2008 Chin. Phys. Soc.
and IOP Publishing Ltd

Classical integrability of strings in γ-deformed backgrounds*

Xie Xiao-Ning(解小宁)[a)c)†] and Yue Rui-Hong(岳瑞宏)[b)c)]

[a)] *SKLLQG, Institute of Earth Environment, Chinese Academy of Sciences, Xi'an 710075, China*
[b)] *Department of Physics, Ningbo University, Ningbo 315211, China*
[c)] *Institute of Modern Physics, Northwest University, Xi'an 710069, China*

(Received 9 August 2007; revised manuscript received 2 March 2008)

This paper considers classical strings propagating in γ-deformed $AdS_3 \times S^3$ backgrounds generated by certain shift T-dualities accompanied (TsT) transformations on S^3 and AdS_3, respectively. It finds that the $U(1)$ currents of strings with the twisted boundary conditions are equal to those in γ-deformed backgrounds generated by TsT transformations on both S^3 and AdS_3. Applying the TsT transformations, it derives the local Lax connections and the monodromy matrices in γ-deformed backgrounds with the spectral parameter which ensure the classical integrability of the string theories.

Keywords: TsT transformation, Lax connection, monodromy matrix, integrability
PACC: 1110L, 1117

1. Introduction

Recently, Lunin and Maldacena[1] found the gravity dual to the β-deformations of $\mathcal{N}$=4 super Yang–Mills (SYM) theory.[2] This deformation of the string background can be obtained by applying two T-dualities accompanied by certain shift parametrized by β (TsT transformation). For real values of $\beta \equiv \gamma$, Frolov obtained the strings in the γ-deformed background by TsT transformation.[3] Furthermore, Rashkov *et al* studied the multi-shift Ts...sT transformation applied to the backgrounds with at least two $U(1)$ isometries.[4]

In recent years, integrabilities of nonlinear equations have received wide attention.[5−10] Especially, integrabilities in string theories are expected to play an important role in establishing strings on the anti-de Sitter space/conformal field theory (AdS/CFT) correspondence.[11−17] The classical integrability and non-local charges in the type IIB string theory on $AdS_5 \times S^5$ were suggested in the early work[18] by regarding the S^5 part as a nonlinear σ model. For such a non-symmetric coset space $\dfrac{PSU(2,2|4)}{SO(1,4) \times SO(5)}$, Bena–Polchinski–Roiban[19] found an infinite set of classically conserved currents for the Green–Schwarz (GS) superstring on $AdS_5 \times S^5$, such that it may be exactly solvable. For this new γ-deformed backgrounds, the integrabilities are still useful for searching new less supersymmetric examples of the AdS/CFT correspondence. Frolov[3] obtained the Lax operator for the bosonic part of superstring in the γ-deformed background generated by TsT transformations on S^5 which proves the integrability at classical level, and the Lax operator in the TsT transformed AdS_5 was constructed in Ref.[20]. Moreover, the Lax operator of GS superstring included the fermionic sector was given in γ-deformed backgrounds by redefinition of the fermionic variables.[21] In Ref.[22], the Lax pair for the pure spinor strings was determined in the TsT-deformed backgrounds.

The bosonic part of the worldsheet action for a string on $AdS_3 \times S^3$ is the sigma model action on the group manifold $SU(1,1) \times SU(2)$.[23] (In Ref.[24], since the group manifold $SU(1,1)$ is locally isomorphic to $SL(2,R)$, the manifold $AdS_3 \times S^3$ can be understood as the group manifold $SL(2,R) \times SU(2)$.) In the present paper, the string sigma model action on $AdS_3 \times S^3$ is described by the group manifold $SU(1,1) \times SU(2)$. Moreover, according to TsT transformations on both S^3 and AdS_3, we obtain the Lax connections and the monodromy matrices of the strings in γ-deformed $AdS_3 \times S^3$ backgrounds.

*Project supported by the National Natural Science Foundation of China (Grant Nos 90403019 and 10575080).
†Corresponding author. E-mail: xnxie@126.com

This paper is organized as follows. In Section 2, we use TsT transformations on S^3 and AdS_3, respectively, and then derive the string actions in γ-deformed $AdS_3 \times S^3$ backgrounds. In Section 3, the Lax connections and the monodromy matrices in γ-deformed backgrounds generated by TsT transformations on S^3 and AdS_3 are both acquired from the strings on $AdS_3 \times S^3$.

2. The γ-deformed $AdS_3 \times S^3$ action

2.1. The γ-deformed action generated by TsT transformations on S^3

In this section, we consider the string sigma model on S^3, using a T-duality on one circle of S^3, a shift of a second angle variable, followed by another T-duality, then we will derive the strings in the γ-deformed background.

Since the TsT-transformation involves only variables of S^3, it is sufficient to consider that the S^3 part of the string action is a sigma model by unitary $SU(2)$ matrices of the form

$$\boldsymbol{g} = \begin{pmatrix} X_1 & X_2 \\ -X_2^* & X_1^* \end{pmatrix}, \qquad |X_1|^2 + |X_2|^2 = 1, \quad (1)$$

where

$$X_1 = r_1 \mathrm{e}^{\mathrm{i}\phi_1''}, \quad X_2 = r_2 \mathrm{e}^{\prime \mathrm{i}\phi_2''}, \quad \text{and} \quad r_1^2 + r_2^2 = 1. \quad (2)$$

In terms of the parametrization of $\boldsymbol{g}$, the string action on S^3 takes the form

$$S'' = -\frac{\sqrt{\lambda}}{2} \int \mathrm{d}\tau \frac{\mathrm{d}\sigma}{2\pi} [\gamma^{\alpha\beta} (\partial_\alpha r_i \partial_\beta r_i + g_{ij} \partial_\alpha \phi_i'' \partial_\beta \phi_j'') + \Lambda (r_1^2 + r_2^2 - 1)], \quad (3)$$

where $\sqrt{\lambda} = R^2/\alpha'$, R is the radius of S^3, Λ is a Lagrange multiplier, $i = 1, 2$, and $\gamma^{\alpha\beta} = \mathrm{diag}(-1, 1)$. The metric g_{ij} and the anti-symmetric 2-form field b_{ij} are

$$g_{11} = r_1^2, \qquad g_{22} = r_2^2,$$
$$g_{12} = g_{21} = 0, \qquad b_{ij} = 0. \quad (4)$$

The action is invariant under the $SU(2)$ group, and the two $U(1)$ isometry transformations are realized as the angle variables ϕ_i''.

Firstly, we make the T-duality transformation[3,4] on the circle parametrized by ϕ_1''. We obtain the action for the T-duality theory

$$S' = -\frac{\sqrt{\lambda}}{2} \int \mathrm{d}\tau \frac{\mathrm{d}\sigma}{2\pi} [\gamma^{\alpha\beta} (\partial_\alpha r_i \partial_\beta r_i + g'_{ij} \partial_\alpha \phi_i' \partial_\beta \phi_j') + \Lambda (r_1^2 + r_2^2 - 1)], \quad (5)$$

where

$$g'_{11} = \frac{1}{r_1^2}, \qquad g'_{22} = r_2^2,$$
$$g'_{12} = g'_{21} = 0, \qquad b'_{ij} = 0. \quad (6)$$

The T-duality variables ϕ_i' are related to ϕ_i'' as follows

$$\partial_\alpha \phi_1'' = \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_1' g'_{11},$$
$$\phi_2'' = \phi_2'. \quad (7)$$

The relations in Eq.(7) are satisfied only on-shell, it means that their consistency conditions lead to the equations of motion for ϕ_i'' and ϕ_1'.

Secondly, we make the following shift of the angle variable ϕ_2':

$$\phi_2' \to \phi_2' + \hat{\gamma} \phi_1', \quad (8)$$

where $\hat{\gamma}$ is any constant. After the shift the T-transformed metric g' and the field b' acquire the following form, $g'_{ij} \to G'_{ij}$, $b'_{ij} \to B'_{ij}$:

$$G'_{11} = g'_{11} + \hat{\gamma}^2 g'_{22} = \frac{G^{-1}}{r_1^2},$$
$$G^{-1} = 1 + \hat{\gamma}^2 r_1^2 r_2^2,$$
$$G'_{12} = \hat{\gamma} g'_{22} = \hat{\gamma} r_2^2,$$
$$G'_{22} = g'_{22} = r_2^2, \quad B'_{ij} = 0, \quad (9)$$

and the variables ϕ_i' transforms into

$$\partial_\alpha \phi_1'' = \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_1' g'_{11},$$
$$\phi_2'' = \phi_2' + \hat{\gamma} \phi_1'. \quad (10)$$

The final step is to make again the T-duality transformation on the circle parametrized by ϕ_1'. This leads to the string action in the γ-deformed background

$$S = -\frac{\sqrt{\lambda}}{2} \int \mathrm{d}\tau \frac{\mathrm{d}\sigma}{2\pi} [\gamma^{\alpha\beta} (\partial_\alpha r_i \partial_\beta r_i + G_{ij} \partial_\alpha \phi_i \partial_\beta \phi_j) - \epsilon^{\alpha\beta} B_{ij}\, \partial_\alpha \phi_i \partial_\beta \phi_j + \Lambda (r_1^2 + r_2^2 - 1)], \quad (11)$$

where

$$G_{ij} = G g_{ij}, \quad B_{12} = -B_{21} = \hat{\gamma} G r_1^2 r_2^2. \quad (12)$$

The variables ϕ_i' are related to the T-duality variables ϕ_i as follows:

$$\partial_\alpha \phi_1' = \gamma_{\alpha\beta} \epsilon^{\beta\gamma} \partial_\gamma \phi_1 G_{11} - \partial_\alpha \phi_2 B_{12},$$
$$\phi_2' = \phi_2. \quad (13)$$

We have the string action in the γ-deformed background about S^3

$$\begin{aligned} S = &-\frac{\sqrt{\lambda}}{2}\int \mathrm{d}\tau\frac{\mathrm{d}\sigma}{2\pi}[\gamma^{\alpha\beta}(\partial_\alpha r_i\partial_\beta r_i + Gr_i^2\partial_\alpha\phi_i\partial_\beta\phi_i) \\ &- 2\hat{\gamma}Gr_1^2r_2^2\epsilon^{\alpha\beta}\partial_\alpha\phi_1\partial_\beta\phi_2 \\ &+ \Lambda(r_1^2+r_2^2-1)]. \end{aligned} \tag{14}$$

Equations (7), (l0) and (l3) allow us to determine the following simple relations between the angle variables ϕ_i'' and TsT-transformed variables ϕ_i:

$$\begin{aligned} &\partial_\alpha\phi_1'' = G(\partial_\alpha\phi_1 - \hat{\gamma}r_2^2\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi_2), \\ &\partial_\alpha\phi_2'' = G(\partial_\alpha\phi_2 + \hat{\gamma}r_1^2\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi_1), \\ &G^{-1} = 1+\hat{\gamma}^2r_1^2r_2^2. \end{aligned} \tag{15}$$

To clarify the meaning of the relations in Eq.(l5) we compute the conserved $U(1)$ isometry currents for the string theories on the backgrounds under consideration

$$\begin{aligned} &J_i''^{\alpha} = -\sqrt{\lambda}r_i^2\gamma^{\alpha\beta}\partial_\beta\phi_i'', \\ &J_1^{\alpha} = -\sqrt{\lambda}r_1^2\gamma^{\alpha\beta}G(\partial_\beta\phi_1 - \hat{\gamma}r_2^2\gamma_{\beta\rho}\epsilon^{\rho\gamma}\partial_\gamma\phi_2), \\ &J_2^{\alpha} = -\sqrt{\lambda}r_2^2\gamma^{\alpha\beta}G(\partial_\beta\phi_2 + \hat{\gamma}r_1^2\gamma_{\beta\rho}\epsilon^{\rho\gamma}\partial_\gamma\phi_1). \end{aligned} \tag{16}$$

Comparing the relations in Eq.(l5) with the expressions for the currents in Eq.(16), we see that Eq.(l5) is just a statement that the $U(1)$ currents of strings on S^3 are equal to those in the γ-deformed background:

$$J_i''^{\alpha} = J_i^{\alpha}. \tag{17}$$

Moreover, we can label canonical momenta as $p_i = J_i^0$, and the associated charges $J_i = \int\frac{\mathrm{d}\sigma}{2\pi}p_i$, so the twisted boundary conditions for the $U(1)$ variables ϕ'' on S^3 take the form[3]

$$\begin{aligned} &\phi_1''(2\pi) - \phi_1''(0) = 2\pi(m_1+\gamma J_2), \\ &\phi_2''(2\pi) - \phi_2''(0) = 2\pi(m_2-\gamma J_1), \end{aligned} \tag{18}$$

where $\gamma = \frac{\hat{\gamma}}{\sqrt{\lambda}}$, and m_i denotes integer winding numbers defined by

$$\phi_i(2\pi) - \phi_i(0) = 2\pi m_i. \tag{19}$$

2.2. The γ-deformed action generated by TsT transformations on AdS_3

In this section, we consider the string sigma model action on AdS_3, and find that the AdS_3 sector takes the form by unitary $SU(1,1)$ matrices,

$$\boldsymbol{g} = \begin{pmatrix} X_1 & X_2 \\ X_2^* & X_1^* \end{pmatrix} \tag{20}$$

with

$$\boldsymbol{g}^\dagger\, s\, \boldsymbol{g} = s, \quad s \equiv \mathrm{diag}(-1,1), \quad -|X_1|^2 + |X_2|^2 = -1, \tag{21}$$

where

$$X_1 = r_1\mathrm{e}^{\mathrm{i}\phi_1''}, \quad X_2 = r_2\mathrm{e}^{\mathrm{i}\phi_2''}, \quad \text{and} \quad r_1^2 - r_2^2 = 1. \tag{22}$$

Hence, the string action on AdS_3 takes the form

$$\begin{aligned} S'' = &-\frac{\sqrt{\lambda}}{2}\int \mathrm{d}\tau\frac{\mathrm{d}\sigma}{2\pi}[\gamma^{\alpha\beta}(\partial_\alpha r_1\partial_\beta r_1 - \partial_\alpha r_2\partial_\beta r_2 \\ &+ g_{ij}\partial_\alpha\phi_i''\partial_\beta\phi_j'') + \Lambda(r_1^2 - r_2^2 - 1)], \end{aligned} \tag{23}$$

where the metric g_{ij} and the anti-symmetric 2-form field b_{ij} are

$$\begin{aligned} &g_{11} = r_1^2, \qquad g_{22} = -r_2^2, \\ &g_{12} = g_{21} = 0, \qquad b_{ij} = 0. \end{aligned} \tag{24}$$

The action is invariant under the $SU(1,1)$ group, and the two $U(1)$ isometry transformations are realized as the angle variables ϕ_i''.

Similar to the TsT transformation on S^3, we make the TsT transformation on AdS_3. The ϕ_2'' direction is spacelike, so we invoke a TsT transformation which acts as a T-duality along the ϕ_2'' direction, a shift in the ϕ_1' direction $\phi_1' \to \phi_1' + \hat{\gamma}\phi_2'$, followed by a second T-duality in the new ϕ_2' direction.

After the TsT transformation, we derive the string action in γ-deformed background about the AdS_3 sector,

$$\begin{aligned} S = &-\frac{\sqrt{\lambda}}{2}\int \mathrm{d}\tau\frac{\mathrm{d}\sigma}{2\pi}[\gamma^{\alpha\beta}(\partial_\alpha r_1\partial_\beta r_1 - \partial_\alpha r_2\partial_\beta r_2 \\ &- Gr_1^2\partial_\alpha\phi_1\partial_\beta\phi_1 + Gr_2^2\partial_\alpha\phi_2\partial_\beta\phi_2) \\ &+ 2\hat{\gamma}Gr_1^2r_2^2\epsilon^{\alpha\beta}\partial_\alpha\phi_1\partial_\beta\phi_2 \\ &+ \Lambda(r_1^2 - r_2^2 - 1)] \end{aligned} \tag{25}$$

with the relation between ϕ_i'' and ϕ_i

$$\begin{aligned} &\partial_\alpha\phi_1'' = -G(\partial_\alpha\phi_1 - \hat{\gamma}r_2^2\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi_2), \\ &\partial_\alpha\phi_2'' = -G(\partial_\alpha\phi_2 - \hat{\gamma}r_1^2\gamma_{\alpha\beta}\epsilon^{\beta\gamma}\partial_\gamma\phi_1), \\ &G^{-1} = -1+\hat{\gamma}^2r_1^2r_2^2. \end{aligned} \tag{26}$$

We compute the conserved $U(1)$ isometry currents for the string theories in the backgrounds

$$\begin{aligned} &J_1''^{\alpha} = -\sqrt{\lambda}r_1^2\gamma^{\alpha\beta}\partial_\beta\phi_1'', \\ &J_2''^{\alpha} = \sqrt{\lambda}r_2^2\gamma^{\alpha\beta}\partial_\beta\phi_2'', \\ &J_1^{\alpha} = -\sqrt{\lambda}r_1^2\gamma^{\alpha\beta}(-G)(\partial_\beta\phi_1 - \hat{\gamma}r_2^2\gamma_{\beta\rho}\epsilon^{\rho\gamma}\partial_\gamma\phi_2), \end{aligned}$$

$$J_2^\alpha = \sqrt{\lambda} r_2^2 \gamma^{\alpha\beta}(-G)(\partial_\beta \phi_2 - \hat{\gamma} r_1^2 \gamma_{\beta\rho}\epsilon^{\rho\gamma}\partial_\gamma \phi_1). \quad (27)$$

According to the relations in Eq.(26) and the currents in Eq.(27), we easily find that the $U(1)$ currents of strings on AdS_3 are equal to those in the γ-deformed background:

$$J_i''^\alpha = J_i^\alpha. \quad (28)$$

The twisted boundary conditions for the $U(1)$ variables ϕ'' on AdS_3 take the form

$$\begin{aligned} \phi_1''(2\pi) - \phi_1''(0) =& 2\pi(m_1 - \gamma J_2)\ , \\ \phi_2''(2\pi) - \phi_2''(0) =& 2\pi(m_2 + \gamma J_1), \end{aligned} \quad (29)$$

where m_i denotes integer winding numbers defined by

$$\phi_i(2\pi) - \phi_i(0) = 2\pi m_i. \quad (30)$$

It is emphasized that the twisted boundary conditions (29) on AdS_3 are different from those in Eq.(18) on S^3.

So we have derived the γ-deformed actions generated by TsT transformations on S^3 and AdS_3. What is more, it is found that the $U(1)$ currents of strings with the twisted boundary conditions are equal to those in γ-deformed backgrounds generated by TsT transformation on both S^3 and AdS_3. In next section we will use these relations in Eqs.(l5) and (26) to find the Lax connections for the string theory in γ-deformed backgrounds.

3. Classical integrability in γ-deformed backgrounds

In this section, from the known Lax connection for the string theory on $AdS_3 \times S^3$, we will derive the Lax connections and the monodromy matrices for the string theory in γ-deformed backgrounds.

The sigma model on S^3 (or AdS_3) follows from the usual action for the principal chiral field:

$$S = \int \mathrm{d}z_+ \mathrm{d}z_- \mathrm{tr}(J_+ J_-), \quad (31)$$

where we use $z_\pm = \frac{1}{2}(\tau \pm \sigma)$, and $\partial_\pm = \partial_\tau \pm \partial_\sigma$, the right currents $J_\pm = g^{-1}\partial_\pm g$. The Euler–Lagrange equation can be written as

$$\partial_+ J_- + \partial_- J_+ = 0. \quad (32)$$

$J_\pm$ also satisfies the Maurer–Cartan equation

$$\partial_+ J_- - \partial_- J_+ + [J_+, J_-] = 0. \quad (33)$$

Now the Lax connection $A_i(l)$ with the spectral parameter l is constructed,

$$\begin{aligned} A_+(l) = a_+(l)J_+ = \frac{-l}{1-l}J_+, \\ A_-(l) = a_-(l)J_- = \frac{l}{1+l}J_-. \end{aligned} \quad (34)$$

It is obvious that the Lax connection $A_\pm(l)$ in Eq.(34) satisfies the compatibility condition for the integrable system (or the zero curvature condition)

$$\partial_+ A_- - \partial_- A_+ + [A_+, A_-] = 0, \quad (35)$$

or

$$[D_+, D_-] = 0, \quad D_\pm = \partial_\pm + A_\pm. \quad (36)$$

The Lax connection in Eq.(34) for the sigma model on S^3 (or AdS_3) cannot be used to derive a local Lax connection for the string in the γ-deformed background, because it is not invariant under the $U(1)$ isometry transformations.

About the sector S^3, for the right currents $J_\pm$, the dependence on the angle variables ϕ_i'' can be easily found if one notices that the matrix $\boldsymbol{g}$ (1) can be represented in the following factorized form,[3]

$$\boldsymbol{g}(r_i, \boldsymbol{\phi}_i'') = \boldsymbol{M}'(\phi_i'')\hat{\boldsymbol{g}}(r_i)\,\boldsymbol{M}(\phi_i''), \quad X_i = r_i \mathrm{e}^{\mathrm{i}\phi_i''}, \quad (37)$$

where

$$\begin{aligned} \boldsymbol{M} &= \begin{pmatrix} 0 & \mathrm{e}^{\frac{\mathrm{i}}{2}(\phi_2'' - \phi_1'')} \\ \mathrm{e}^{-\frac{\mathrm{i}}{2}(\phi_2'' - \phi_1'')} & 0 \end{pmatrix}, \\ \boldsymbol{M}' &= \begin{pmatrix} 0 & \mathrm{e}^{\frac{\mathrm{i}}{2}(\phi_1'' + \phi_2'')} \\ \mathrm{e}^{-\frac{\mathrm{i}}{2}(\phi_1'' + \phi_2'')} & 0 \end{pmatrix}, \\ \hat{\boldsymbol{g}} &= \begin{pmatrix} r_1 & -r_2 \\ r_2 & r_1 \end{pmatrix}, \qquad \hat{\boldsymbol{g}}^{-1} = \hat{\boldsymbol{g}}^{\mathrm{T}}, \\ \boldsymbol{M}^{-1} &= \boldsymbol{M}, \qquad \boldsymbol{M}'^{-1} = \boldsymbol{M}', \end{aligned} \quad (38)$$

we derive the following formula[3]

$$J_\pm(r_i, \phi''^i) = \boldsymbol{M}^{-1}\hat{J}_\pm(r_i, \partial\phi_i'')\boldsymbol{M}, \quad (39)$$

where

$$\begin{aligned} \hat{J}_\pm =& \frac{\mathrm{i}}{2}(\partial_\pm \phi_2'' - \partial_\pm \phi_1'')\sigma_3 + \hat{\boldsymbol{g}}^{-1}\partial_\pm \hat{\boldsymbol{g}} \\ & - \frac{\mathrm{i}}{2}(\partial_\pm \phi_1'' + \partial_\pm \phi_2'')\hat{\boldsymbol{g}}^{-1}\sigma_3 \hat{\boldsymbol{g}}. \end{aligned} \quad (40)$$

Gauging away the dependence of the Lax connection $A_\pm$ on the matrix $\boldsymbol{M}$, we obtain the Lax connection that depends only on $\partial_\alpha \phi_i''$,

$$\begin{aligned} D_\pm \to \boldsymbol{M} D_\pm \boldsymbol{M}^{-1} =& \partial_\pm + A'_\pm, \\ A'_\pm(l) =& \boldsymbol{M} A_\pm \boldsymbol{M}^{-1} + \boldsymbol{M}\partial_\pm \boldsymbol{M}^{-1} \\ =& a_\pm(l)\hat{J}_\pm + \frac{\mathrm{i}}{2}(\partial_\pm \phi_2'' - \partial_\pm \phi_1'')\sigma_3. \end{aligned} \quad (41)$$

The local Lax connection $A'_{\pm}(l)$ in the γ-deformed background about S^3 is now obtained by expressing the angle variables $\partial_\alpha \phi''_i$ of S^3 in terms of the angle variables $\partial_\alpha \phi_i$ of the γ-deformed background by using the explicit relations in Eq.(l5).

For the sector AdS_3, the matrix g in Eq.(20) can also be represented in the following factorized form:

$$\boldsymbol{g}(r_i, \phi''_i) = \boldsymbol{M}'(\phi''_i)\hat{g}(r_i)\,\boldsymbol{M}(\phi''_i), \quad X_i = r_i \mathrm{e}^{\mathrm{i}\phi''_i}, \tag{42}$$

where $\boldsymbol{M}, \boldsymbol{M}'$ have the same form as those in Eq.(38), but

$$\hat{\boldsymbol{g}} = \begin{pmatrix} r_1 & r_2 \\ r_2 & r_1 \end{pmatrix},$$
$$\hat{\boldsymbol{g}}^{-1} = \begin{pmatrix} r_1 & -r_2 \\ -r_2 & r_1 \end{pmatrix}. \tag{43}$$

Hence $J_{\pm}$, $\hat{J}_{\pm}$ and $A'_{\pm}$ have the same form with those in the Eqs.(39) (40) and (41). So the local Lax connection in the γ-deformed background about AdS_3 is now obtained by expressing the angle variables $\partial_\alpha \phi''_i$ in terms of the angle variables $\partial_\alpha \phi_i$ by the explicit relations in Eq.(26).

Therefore, one constructs the monodromy matrix $\boldsymbol{T}(l)$ by using the given Lax connection in Eq.(41),

$$\boldsymbol{T}(l) = \mathcal{P}\mathrm{exp}\Big\{ -\int_0^{2\pi} A'_\sigma(l)\mathrm{d}\sigma \Big\}$$
$$= \mathcal{P}\mathrm{exp}\Big\{ \int_0^{2\pi} \frac{1}{2}\left(A'_-(l) - A'_+(l)\right)\mathrm{d}\sigma \Big\}. \tag{44}$$

The Lax connection and the monodromy matrix of the string theory would naturally lead to an infinite number of non-local conserved charges by the standard method[19] which are intimately connected to the classical integrability of the theories. This fact is a characteristic property that the string theories in the γ-deformed backgrounds generated by TsT transformations on both S^3 and AdS_3 are completely integrable at least at classical level.

In summary, we have discussed TsT transformations on S^3 and AdS_3, and generated the string actions in γ-deformed backgrounds. Then we have found that the $U(1)$ currents of strings with the twisted boundary conditions are equal to those in γ-deformed backgrounds. Finally, we have derived the local Lax connections and the monodromy matrices in γ-deformed backgrounds which imply the classical integrability of the string theories.

Acknowledgments

We would like to thank Professor Shi Kang-Jie for many stimulating discussions. Xie is also grateful for Yang A-Ping's encouragement.

References

[1] Lunin O and Maldacena J 2005 *J. High Energy Phys.* **05**(05) 033
[2] Leigh R G and Strassler M J 1995 *Nucl. Phys.* B **447** 95
[3] Frolov S 2005 *J. High Energy Phys.* **05**(05) 069
[4] Rashkov R C, Viswanathan K S and Yang Y 2005 *Phys. Rev.* D **72** 106008
[5] Tian X D and Yue R H 2005 *Acta Phys. Sin.* **54** 1485 (in Chinese)
[6] Ma H C and Lou S Y 2005 *Chin. Phys.* **14** 1495
[7] Wang Y S and Yan X W 2006 *Acta Phys. Sin.* **55** 2885 (in Chinese)
[8] Li B and Wang Y S 2007 *Acta Phys. Sin.* **56** 1260 (in Chinese)
[9] Ouyang S G, Guo Q, Wu L J and Lan S 2007 *Chin. Phys.* **16** 2331
[10] Chen X H, Xia T C and Zhu L C 2007 *Chin. Phys.* **16** 2493
[11] Xiong C H 2005 *Acta Phys. Sin.* **54** 47 (in Chinese)
[12] Wang Z Y, Wang X H, Shi K J and Yue R H 2007 *Acta Phys. Sin.* **56** 755 (in Chinese)
[13] Chen B, He Y L, Zhang P and Song X C 2005 *Phys. Rev.* D **71** 086007
[14] Xie X N and Yue R H 2007 *Commun. Theor. Phys.* **48** 680
Zhang L X, Wu S, Xie X N, Xiong C H, Yue R H, and Shi K J 2007 *Chin. Phys. Lett.* **24** 3092
Xie X N, Yue R H, Shi K J, Wu S and Zhang L X 2008 *Front. Phys. China* **3** 69
[15] Hou B Y, Wang X H, Xiong C H and Yue R H 2006 *Commun. Theor. Phys.* **45** 301
Wang X H, Wang Z Y, Cai X L, Song P, Hou B Y and Shi K J 2006 *Commun. Theor. Phys.* **45** 663
[16] Alday L F 2003 *J. High Energy Phys.* **03**(12) 033
Arutyunov G and Frolov S 2004 *J. High Energy Phys.* **05**(02) 059
Alday L F, Arutyunov G and Tseytlin A A 2005 *J. High Energy Phys.* **05**(07) 002
[17] Young C A S 2006 *Phys. Lett.* B **632** 559
[18] Mandal G, Suryanarayana N V and Wadia S R 2002 *Phys. Lett.* B **543** 81
[19] Bena I, Polchinski J and Roiban R 2004 *Phys. Rev.* D **69** 046002
[20] McLoughlina T and Swanson I 2006 *J. High Energy Phys.* **06**(08) 084
[21] Alday L F, Arutyunov G and Frolov S 2006 *J. High Energy Phys.* **06**(06) 018
[22] Grassi P A and Kluson J 2006 arXiv:hep-th/0611151
[23] Berkovits N 1999 *J. High Energy Phys.* **99**(03) 018
[24] Maharana J 2005 *Mod. Phys. Lett.* A **20** 2317