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Consensus problems of multi-agent systems with noise perturbation*

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In this paper, by using the stability theory of stochastic differential equations, the average-consensus problem with noise perturbation is investigated. It is analytically proved that the consensus could be achieved with a probability of one. Furthermore, numerical examples are taken to illustrate the effectiveness of the theoretical result.

Keywords: consensus problems, multi-agent systems, white noise
PACC: 0565, 0540

1. Introduction

In recent years the consensus problems have aroused an increasing interest due to their applications in many areas such as biology, physics and engineering. The studies of multiple agents focus on an understanding of the general mechanisms and interconnection rules of cooperative phenomena as well as their potential applications in various engineering problems.[1−6]

In Ref.[1], Vicsek *et al* proposed a simple but interesting discrete-time model which can be viewed as a particular case of computer model mimicking animal aggregation. In Ref.[7], Jadbabaie *et al* provided a theoretical explanation for the consensus behaviour of the Vicsek model. In the past few years, consensus problems have developed very fast and several research topics have been addressed, such as agreement over random networks,[8] networks with communication time-delays,[4,6] consensus with leaders[6] and consensus in discrete-time multi-agent systems.[9] For more details, see the relevant review paper[10] and references cited therein.

Noise, commonly regarded as a random and persistent disturbance obscuring or reducing the clarity of a signal, is of omnipresence in nature and in man-made systems.[11−14] And there are also many synchronous phenomena in noisy environments, such as the blinking synchronization of a swarm of fireflies through random mutual communications and the chaos synchronization inducted by noise coupling.[15−17] In Ref.[18], the numerical simulation results show that flocking motions can be maintained for weak noise.

In reality, the motion of agent group is inevitably subjected to noise and disturbance in environments. Hence, it becomes interesting to ask such questions: can consensus be achieved in a noisy environment? Besides the numerical evidence, are there any analytical conditions that can be rigorously established?

In this paper, we investigate an average-consensus problem in the environment with white noise. Our contribution is to consider the effect of noise on interconnection communication. In noisy environments, the communication between two agents is disturbed, i.e. the noise can affect an agent's sensing of other agent's value, so we assume that the communication links are fixed and the link weights are stochastically perturbed by white noise. The average-consensus problem is more challenging than other consensus problems. The analysis becomes more difficult if the noise perturbation is involved. Based on the stability theory of stochastic differential equations, we present an analytical proof of some sufficient conditions, which ensure consensus in a noisy environment with a probability of one.

2. Problem statement

Algebraic graph theory is a natural framework for analysing consensus problems. If each agent is

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regarded as a node of a graph, then their coupling topology is conveniently described by a simple graph. Let $\mathcal{G} = (\mathcal{V}, \varepsilon, \mathcal{A})$ be a weighted digraph of order n with a set of nodes $\mathcal{V} = \{1, 2, \ldots, n\}$, a set of edges $\varepsilon \subseteq \mathcal{V} \times \mathcal{V}$, and a weighted adjacency matrix $\mathcal{A} = (a_{ij}) \in R^{n\times n}$ with nonnegative elements. The node indices belong to a finite set $\mathcal{I} = \{\infty, \in, \ldots, \backslash\}$. An edge of $\mathcal{G}$ is denoted by (i, j), which starts from node i and ends at node j. Then element a_{ij} associated with the edge of the digraph is positive, i.e. $a_{ij} > 0 \Leftrightarrow (i, j) \in E$. Moreover, we assume $a_{ii} = 0$ for all $i \in \mathcal{I}$. A set of neighbours of node i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \varepsilon\}$. A path in a digraph is a sequence $i_0, i_1, \ldots, i_f$ of different nodes such that (i_{j-1}, i_j) is an edge for $j = 1, 2, \ldots, f, f \in Z^+$. Digraph $\mathcal{G}$ is strongly connected if there exists a path between any two distinct nodes. Moreover, if $\sum_{j\in N_i} a_{ij} = \sum_{j\in N_i} a_{ji}$ for all $i = 1, 2, \ldots, n$, then the digraph $\mathcal{G}$ is called being balanced. A diagonal matrix $\boldsymbol{D} = \mathrm{diag}(d_1, \ldots, d_n) \in R^{n\times n}$ is a degree matrix, with diagonal elements $d_i = \sum_{j\in N_i} a_{ij}$ for all $i = 1, 2, \ldots, n$. Then the Laplacian matrix of the weighted digraph $\mathcal{G}$ is defined as $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A} \in R^{n\times n}$.

Remark 1 The Laplacian matrix $\boldsymbol{L}$ of the digraph $\mathcal{G}$ is a zero row-sum matrix, so $\boldsymbol{L}$ has a zero eigenvalue with eigenvector $\mathbf{1} = (1, \ldots, 1) \in \boldsymbol{R}^n$. In addition, the digraph $\mathcal{G}$ is a strongly connected diagraph if and only if $\boldsymbol{L}$ has a simple zero eigenvalue, and the digraph $\mathcal{G}$ is balanced if and only if $\mathbf{1}^{\mathrm{T}}\boldsymbol{L} = 0$.

Let $x_i \in R$ denote the value of node i, we refer to $\mathcal{G}_x = (\mathcal{G}, x)$ with $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathrm{T}}$ as a network with value $x \in R^n$ and topology $\mathcal{G}$. Node i and node j are said to be in agreement with each other in a network if and only if $x_i = x_j$. The nodes of a network are said to reach a consensus if and only if $x_i = x_j$ for all $i, j \in \mathcal{I}, i \neq j$.

Suppose that each node of a group is a dynamic agent with dynamics

$$x_i = f(x_i, u_i), i \in \mathcal{I}, \tag{1}$$

where u_i is a coupling input. For any initial state x_0, if $x(t)$ converges to some equilibrium point x^* such that $x_i^* = x_j^*$ for all $i, j \in \mathcal{I}$ as $t \to \infty$, then system (1) is said to have solved a consensus problem, especially, if $x_i^* = \mathrm{Ave}(x_0) = \dfrac{1}{n}\sum_{i=1}^{n} x_i(0)$, system (1) is said to have solved the average-consensus problem.

In Ref.[4], Olfati-Saber and Murray proposed a typical continuous-time model

$$\dot{x}_i = \sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i), i \in \mathcal{I}. \tag{2}$$

The authors proved that system (2) could solve the average-consensus problem provided that the digraph $\mathcal{G}$ was a strongly connected and balanced diagraph. Based on this model, a number of conditions for asymptotic consensus under a variety of assumptions on interconnection communication have recently been reported. Our contribution here in this paper is to extend these recent contributions to considering the effect of noise on interconnection communication.

As mentioned above, noise is unavoidable in the real world. The motion of the agent group is subject to environmental noise. In order to investigate the average-consensus problem with noise perturbation, we stochastically perturb system (2) into the following stochastic differential equation

$$\begin{aligned} \mathrm{d}x_i = &\sum_{j\in\mathcal{N}_i} a_{ij}(x_j - x_i)\mathrm{d}t \\ &+ \sum_{j\in\mathcal{N}_i} \sigma_{ij}(x_j - x_i)\mathrm{d}w, i \in \mathcal{I}, \end{aligned} \tag{3}$$

where w is a Brownian motion defined in a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with a natural filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Accordingly, $\dot{w}$ is white noise. $\boldsymbol{A}_\sigma = [\sigma_{ij}] \in \mathbb{R}^{n\times n}$ is the matrix representing the intensity of noise. To highlight the presence of noise, it is natural to assume that $\sigma_{ij} > 0$ if $j \in \mathcal{N}_i; \sigma_{ij} = 0$ if $j \notin \mathcal{N}_i; j \neq i$, and $\sigma_{ii} = \sum\limits_{j\in\mathcal{N}_i} \sigma_{ij}$. Let $\boldsymbol{D}_\sigma = \mathrm{diag}\{\sigma_{11}, \ldots, \sigma_{nn}\}$ and $\boldsymbol{G} = \boldsymbol{D}_\sigma - \boldsymbol{A}_\sigma$.

Remark 2 In noisy environments, the noise affects an agent's sensing of its neighbouring agent's value, i.e. the communication between two agents is disturbed, so we assume that the parameter a_{ij} is stochastically perturbed in the form of $a_{ij} \to a_{ij} + \sigma_{ij}\dot{w}$ for all $a_{ij} > 0$.

In what follows, it is shown that system (3) could almost surely solve the average-consensus problem provided that certain conditions on matrices $\boldsymbol{L}$ and $\boldsymbol{G}$ are satisfied. As is well known, the term "almost surely" means that an event occurs with a probability of one.

3. Convergence analysis

Before the discussion, we introduce some results for stochastic differential equations and matrix analysis.

Consider the following n-dimensional stochastic system:

$$\mathrm{d}x(t) = f(t, x(t))\mathrm{d}t + g(t, x(t))\mathrm{d}w(t), \tag{4}$$

where $f(t,x) : [0,+\infty] \times R^n \to R^n$ and $g(t,x) : [0,+\infty] \times R^n \to M^{n\times m}$ are continuous and satisfy the local Lipschitz condition and linear growth condition with $f(t,0)=0$, $g(t,0)=0$. $w(t)$ is an m-dimensional Wiener process.

Then, for each $V(t,x) \in C^{2,1}(R\times R^n)$, define the differential operator $\mathcal{L}$ that is related to system (4) by

$$\begin{aligned}\mathcal{L}V =&\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f\\&+\frac{1}{2}\mathrm{trace}\left[g^T\cdot\frac{\partial^2 V}{\partial^2 x}\cdot g\right],\end{aligned}\tag{5}$$

where $\partial V/\partial x = (\partial V/\partial x_1,\ldots,\partial V/\partial x_n)$, $\partial^2 V/\partial x^2 = (\partial^2 V/\partial x_i\partial x_j)_{n\times n}$.

The following result gives conditions on stability of the trivial solution of system (4), of which the details can be found in Ref.[19].

Lemma 1 Suppose that there exist a non-negative function $V(t,x) \in C^{2,1}([t_0,+\infty]\times R^n, R^+)$, and three positive numbers, i.e. p,θ,λ, such that for all $x\neq 0, t\geq t_0$,

$$\theta\,|x|^p \leq V(t,x),$$

$$\mathcal{L}V(t,x) \leq -\lambda V(t,x)$$

hold, then for any $x_0 \in R^n$ the trivial solution of system (4) is almost surely exponentially asymptotically stable, i.e.

$$\lim_{t\to\infty}\frac{1}{t}\lg\|x(t,t_0,x_0)\| \leq -\frac{\lambda}{p}\quad a.s.$$

The following lemma is obtained from Ref.[20].

Lemma 2 Let $\boldsymbol{A}\in M_n$ be a real and symmetric matrix and denote the eigenvalues of A as

$$\lambda_{\min}=\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_n=\lambda_{\max},$$

then

$$\lambda_1\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}\leq\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}\leq\lambda_n\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}$$

for all $x\in R^n$. Furthermore, if u_1 is the right eigenvector of $\boldsymbol{A}$ associated with λ_1
then

$$\min_{x\neq 0, x\perp u_1}\frac{\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{x}}{\boldsymbol{x}^{\mathrm{T}}\boldsymbol{x}}=\lambda_2.$$

Theorem Consider a network of integrators with topology $\mathcal{G}=(\mathcal{V},\varepsilon,\mathcal{A})$ that is a strongly connected and balanced digraph, then system (3) will solve the average-consensus problem with a probability of one if the following conditions are satisfied:
i) $\mathbf{1}^{\mathrm{T}}\boldsymbol{G}=0$;
ii) $\lambda_{\max}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})<\lambda_2(\boldsymbol{L}+\boldsymbol{L}^{\mathrm{T}})$,
where $\lambda_{\max}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})$ is the largest eigenvalue of matrix $\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}$, and $\lambda_2(\boldsymbol{L}+\boldsymbol{L}^{\mathrm{T}})$ is the second smallest eigenvalue of matrix $\boldsymbol{L}+\boldsymbol{L}^{\mathrm{T}}$.

Proof With matrices $\boldsymbol{L}$ and $\boldsymbol{G}$, system (5) can be written in a matrix form as follows:

$$\mathrm{d}\boldsymbol{x}=-\boldsymbol{L}\boldsymbol{x}\mathrm{d}t-\boldsymbol{G}\boldsymbol{x}\mathrm{d}w,\tag{6}$$

where $\boldsymbol{x}=(x_1,\ldots,x_n)^{\mathrm{T}}$, $\mathrm{d}\boldsymbol{x}=(\mathrm{d}x_1,\ldots,\mathrm{d}x_n)^{\mathrm{T}}$.

Let $\alpha=\mathrm{Ave}(\boldsymbol{x})$, with $\mathbf{1}^{\mathrm{T}}\boldsymbol{L}=\mathbf{1}^{\mathrm{T}}\boldsymbol{G}=0$ we have

$$\dot{\alpha}=\frac{1}{n}\sum_{i=1}^{n}\dot{x}_i=-\frac{1}{n}(\mathbf{1}^{\mathrm{T}}\boldsymbol{L}\boldsymbol{x}+\mathbf{1}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{x}\dot{\omega})=0.\tag{7}$$

So $\alpha=\mathrm{Ave}(\boldsymbol{x})$ is an invariant quantity. This allows the decomposition of $\boldsymbol{x}$ in the form $\boldsymbol{x}=\boldsymbol{e}+\alpha\mathbf{1}$, where $\boldsymbol{e}$ is a disagreement vector. With $L\mathbf{1}=G\mathbf{1}=0$, we can rewrite system (6) as a disagreement system given by

$$\mathrm{d}\boldsymbol{e}=-\boldsymbol{L}\boldsymbol{e}\mathrm{d}t-\boldsymbol{G}\boldsymbol{e}\mathrm{d}w,\tag{8}$$

Clearly, $e(t,0)=0$ is a trivial solution of the disagreement system (8). Moreover, the asymptotic consensus could be achieved with a probability of one, provided that the trivial solution of system (8) is almost surely globally asymptotically stable, i.e. for any initial value $e_0\in R^n$, $\|\boldsymbol{e}(t,t_0,e_0)\|\to 0$ almost surely holds as $t\to\infty$. Here, $\|\cdot\|$ denotes the Euclidean norm.

To prove that the trivial solution of system (8) is almost surely globally asymptotically stable, take a positive-define function $V(\boldsymbol{e})=\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$. Thus, the operator $\mathcal{L}$ defined in expression (5) is exerted onto the disagreement system (8), giving

$$\begin{aligned}\mathcal{L}V=&-\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L})\boldsymbol{e}\\&+\mathrm{trace}(\boldsymbol{e}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{e})\\&-\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L})\boldsymbol{e}+\boldsymbol{e}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{e}.\end{aligned}\tag{9}$$

To further estimate $\mathcal{L}V$, we first prove that $\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L})\boldsymbol{e}=\lambda_2(\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L})\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}$ holds. Note that $\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L}$ is a symmetric and zero row-sum matrix, based on the Gersgorin disc theorem[20] all the eigenvalues of matrix $\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L}$ are real and non-negative. So all the eigenvalues of $\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L}$ can be ordered as

$$0=\lambda_{\min}=\lambda_1\leq\lambda_2\leq\ldots\leq\lambda_n=\lambda_{\max}.\tag{10}$$

Since the digraph $\mathcal{G}$ is balanced, then from the theorem 7 in Ref.[4] $(\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L})/2$ is a valid Laplacian matrix for a graph $\hat{\mathcal{G}}$. In addition, the digraph $\mathcal{G}$ is strongly connected, then the graph $\hat{\mathcal{G}}$ is also strongly connected. So the matrix $\boldsymbol{L}^{\mathrm{T}}+\boldsymbol{L}$ is irreducible, and

has a single zero eigenvalue with eigenvector $\mathbf{1}$. Moreover, we have

$$\mathbf{1}^{\mathrm{T}}\boldsymbol{e} = \sum_{i=1}^{n} \boldsymbol{x}_i - n\alpha = 0. \tag{11}$$

Hence, $\boldsymbol{e}$ is orthogonal to $\mathbf{1}$ and belongs to the $(n-1)$-dimensional subspace called the disagreement eigenspace of $\boldsymbol{L}^{\mathrm{T}} + \boldsymbol{L}$. Then with Lemma 2 we have

$$\boldsymbol{e}^{\mathrm{T}}(\boldsymbol{L}^{\mathrm{T}} + \boldsymbol{L})\boldsymbol{e} = \lambda_2(\boldsymbol{L}^{\mathrm{T}} + \boldsymbol{L})\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}, \tag{12}$$

$$\boldsymbol{e}^{\mathrm{T}}\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\boldsymbol{e} < \lambda_{\max}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}. \tag{13}$$

Therefore, with expressions (9), (12) and (13), we have

$$\begin{aligned}\mathcal{L}V &\leq -\lambda_2(\boldsymbol{L}^{\mathrm{T}} + \boldsymbol{L})\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} + \lambda_{\max}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G})\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e} \\ &= -(\lambda_2(\boldsymbol{L}^{\mathrm{T}} + \boldsymbol{L}) - \lambda_{\max}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}))\boldsymbol{e}^{\mathrm{T}}\boldsymbol{e}. \end{aligned} \tag{14}$$

If the condition $\lambda_{\max}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}) < \lambda_2(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}})$ holds, applying Lemma 1, one arrives at the disagreement vector $\boldsymbol{e}$ almost surely globally asymptotically converging to $\mathbf{0}$. The proof is now completed here.

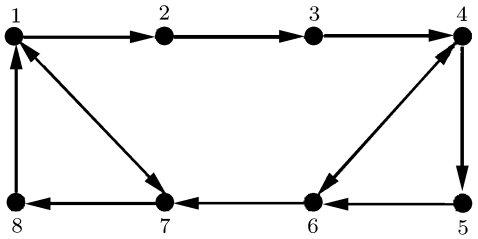


Fig.1. An example of a strongly connected and balanced digraph with 8 nodes.

Numerical simulations will be performed to illustrate the theoretical results. Figure 1 shows the network with n=8 nodes and 0–1 weights. Clearly, the graph $\mathcal{G}$ is strongly connected and balanced. For simplicity, we take the noise intensity matrix $\boldsymbol{G} = \sigma_0\boldsymbol{L}$ in the simulation, and the initial state $\boldsymbol{x}_0 = (1, -1, 2, -2, 3, -3, 8, 0)^{\mathrm{T}}$. It is not hard to obtain:

$$\begin{aligned}&\mathrm{Ave}(\boldsymbol{x}(0)) = 1,\\ &\lambda_2(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}}) = 0.6971,\\ &\lambda_{\max}(\boldsymbol{L}^{\mathrm{T}}\boldsymbol{L}) = 12.0924,\\ &\lambda_{\max}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}) = 12.0924\sigma_0^2.\end{aligned}$$

When $\sigma_0 < 0.24$, then the condition $\lambda_{\max}(\boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}) < \lambda_2(\boldsymbol{L} + \boldsymbol{L}^{\mathrm{T}})$ holds. If we take $\sigma_0 = 0.2$ in the simulation, then Fig.2 will show the state trajectories of the deterministic system (2) and the stochastic system (3).

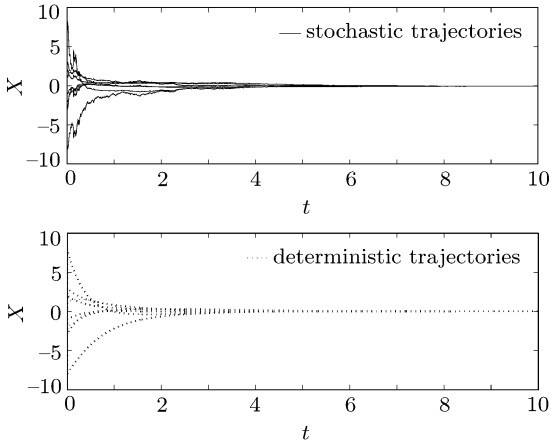


Fig.2. The trajectories of the deterministic system (2) and the stochastic system (3) with $\sigma_0 = 0.2$.

The group disagreement δ (i.e. $\|\boldsymbol{e}\|$) is shown in Fig.3. Clearly, the group disagreement is monotonically decreasing. One can observe that an average-consensus is reached asymptotically both in system (2) and in system (3). But it takes much time to reach the average-consensus in a noisy environment.

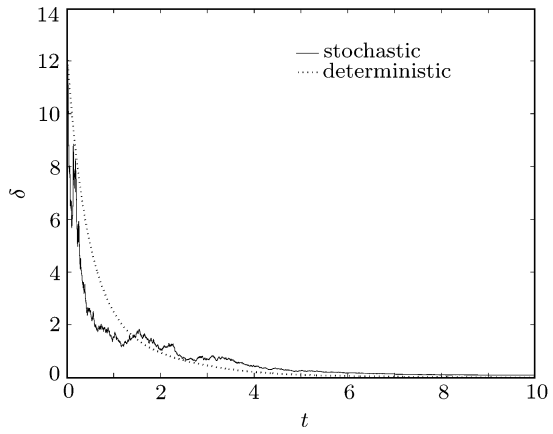


Fig.3. The group disagreement of the deterministic system (2) and the stochastic system (3) with $\sigma_0 = 0.2$.

With $\sigma_0 = 2$ chosen, Fig.4 shows that the consensus fails simply due to the violation of noise intensities. It is worth mentioning that the hypotheses in Theorem are sufficient conditions for a successful consensus but not necessary. So, the violation of them does not necessarily lead the consensus to fail. But the simulation result in Fig.4 shows that the strong noise can prevent the consensus from occurring.

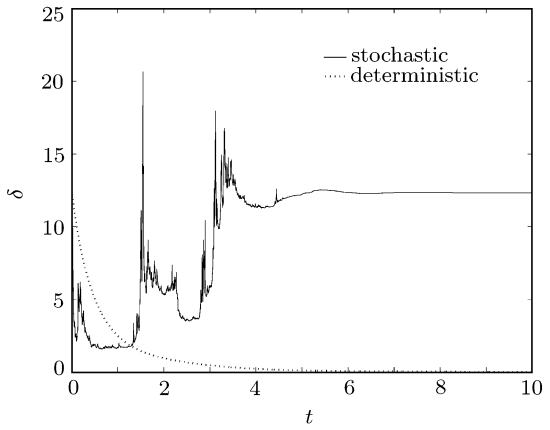


Fig.4. The group disagreement of the deterministic system (2) and the stochastic system (3) with $\sigma_0 = 2$.

4. Conclusion

In this paper, the average-consensus problem with noise perturbation is investigated. Based on the stability theory of stochastic differential equations, sufficient conditions ensuring a successful consensus with a probability of one have been rigorously established. Numerical examples are taken to illustrate the effectiveness of the theoretical result. Compared with the result of consensus problems without noise perturbation, it takes much time to reach the consensus in a noisy environment.

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