Vol 17 No 11, November 2008
1674-1056/2008/17(11)/4114-09

Chinese Physics B

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Periodic solutions and flip bifurcation in a linear impulsive system*

Jiang Gui-Rong(蒋贵荣)[a)b)] and Yang Qi-Gui (杨启贵)[a)†]

[a)] *School of Mathematical Sciences, South China University of Technology, Guangzhou* 510641, *China*
[b)] *School of Mathematics and Computing Science, Guilin University of Electronic Technology, Guilin* 541004, *China*

(Received 20 May 2008; revised manuscript received 9 June 2008)

In this paper, the dynamical behaviour of a linear impulsive system is discussed both theoretically and numerically. The existence and the stability of period-one solution are discussed by using a discrete map. The conditions of existence for flip bifurcation are derived by using the centre manifold theorem and bifurcation theorem. The bifurcation analysis shows that chaotic solutions appear via a cascade of period-doubling in some interval of parameters. Moreover, the periodic solutions, the bifurcation diagram, and the chaotic attractor, which show their consistence with the theoretical analyses, are given in an example.

Keywords: linear impulsive equation, periodic solution, flip bifurcation
PACC: 0547

1. Introduction

Impulsive differential equations (IDEs) are suitable for the mathematical simulation of evolutionary processes in which the parameters undergo relatively long periods of smooth variation followed by a short-term rapid change in their values. The study of IDEs is mainly concerned with the properties of their solutions, such as, existence, uniqueness, stability, boundedness, oscillation, etc.[1,2] In recent years the potential applications of IDEs in various scientific fields, such as impulsive control,[3] impulsive synchronization,[4] etc have been extensively studied.

Finding the explicit and exact periodic solutions plays an important role in studying the dynamics of IDEs (see Refs.[5, 6] and references cited there for example). However, little is known about the bifurcation theory of IDEs. Lakmeche and Arino[7] converted a problem of periodic solution into a fixed-point problem and discussed the bifurcation of non-trivial periodic solution. Akhmet[8] obtained the conditions of existence for Hopf bifurcation. Gan *et al*[9] discussed the bifurcation and chaos in a ratio-dependent predator-prey system by using the normal form theory. Jiang and Lu[10,11] discussed fold and flip bifurcations theoretically and numerically by using the Poincaré map in several autonomous systems with impulses. Among the papers about IDEs in the literature, many authors discussed numerically the complex dynamics of impulsive differential systems, such as period-doubling bifurcation, chaos, crisis, etc. (see Ref.[12] for example). Therefore, the theoretical analysing of the dynamical behaviours of impulsive differential systems is a challenging task.

In this paper, a linear system, whose dynamics is very simple, is used to build a linear impulsive system with impulses at fixed time. Here we construct a discrete map and present analytical results about the complex dynamical behaviour. The purpose of this paper is to find the conditions of existence for periodic solutions and bifurcations theoretically by using the centre manifold theorem and the bifurcation theorem. The organization of the rest of this paper is as follows. A linear impulsive system is introduced and a discrete map is constructed in Section 2. In Section 3, the existence and the stability of the period-one solutions are given. In Section 4, the flip bifurcation analysis is given in detail. The detailed numerical results are presented in Section 5 to verify the theoretical analysis, and the conclusion is presented in Section 6 finally.

*Project supported by the National Natural Science Foundation of China (Grant Nos 10572011, 100461002, and 10661005) and the Natural Science Foundation of Guangxi Province, China (Grant Nos 0575092 and 0832244).
†Corresponding author. E-mail: qgyang@scut.edu.cn

2. The linear impulsive system

Linear differential equation is widely used to describe various dynamical systems. For example, to analyse the vibration of a mass m on a spring with a dashpot, the second-order linear differential equation $m\ddot{x} + c\dot{x} + kx = F(t)$ is used, where c is the damping constant of the dashpot, k is the spring constant, and $F(t)$ is a given external force. This equation transforms via the substitution $\dot{x} = y$ into a two-dimensional linear system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\dfrac{k}{m}x - \dfrac{c}{m}y + F(t). \end{cases}$$

In view of the external force $F(t)$, the continuous control is used here. However, in many cases impulsive control is more efficient than continuous control.[3,13] By using impulsive control inputs, the speed of the mass m changes impulsively. It follows from the above linear system that

$$\left.\begin{array}{l} \dot{x} = y, \\ \dot{y} = -\dfrac{k}{m}x - \dfrac{c}{m}y, \end{array}\right\} \quad t \neq nT,$$
$$\left.\begin{array}{l} \triangle x = 0, \\ \triangle y = h(x,\ y), \end{array}\right\} \quad t = nT. \tag{1}$$

Now we consider the following linear system with impulse control at periodic fixed time:

$$\left.\begin{array}{l} \dot{x} = ax + by, \\ \dot{y} = cx - ay, \end{array}\right\} \quad t \neq nT,$$
$$\left.\begin{array}{l} \triangle x = -(1-p)x + h, \\ \triangle y = qy^2 - y, \end{array}\right\} \quad t = nT, \tag{2}$$

where $q > 0$, $0 < p < 1$, $\triangle x(t) = x(t^+) - x(t)$, $x(t^+) = \lim_{\tau \to 0^+} x(t+\tau)$, $\triangle y(t) = y(t^+) - y(t)$, and $y(t^+) = \lim_{\tau \to 0^+} y(t+\tau)$. More details about the existence and the uniqueness of solutions of impulsive systems can be found in Ref.[1].

It is well known that the dynamics of the following linear system:

$$\begin{cases} \dot{x} = ax + by, \\ \dot{y} = cx - ay \end{cases}$$

is very simple. For $a^2 + bc > 0$, the characteristic equation of the equilibrium point $O(0,0)$ is

$$\lambda^2 - a^2 - bc = 0,$$

which has a positive root and a negative root, and then $O(0,0)$ is a saddle. For simplicity, we set

$$a > 0, \quad b = 0, \quad \text{and} \quad c < 0 \tag{3}$$

in this paper and the solution from the initial point $(x_0,\ y_0)$ is

$$\begin{aligned} x(t) &= x_0 \exp(at), \\ y(t) &= \frac{cx_0}{2a}\exp(at) + \frac{y_0 - cx_0}{2a}\exp(-at). \end{aligned} \tag{4}$$

The phase portrait of system (2) is shown in Fig.1, where the trajectory originating from the initial point $A_k(x_k,\ y_k)$ reaches the point $B_k(\bar{x}_k,\ \bar{y}_k)$, next jumps to the point $A_{k+1}(x_{k+1},\ y_{k+1})$ due to the effect of impulse.

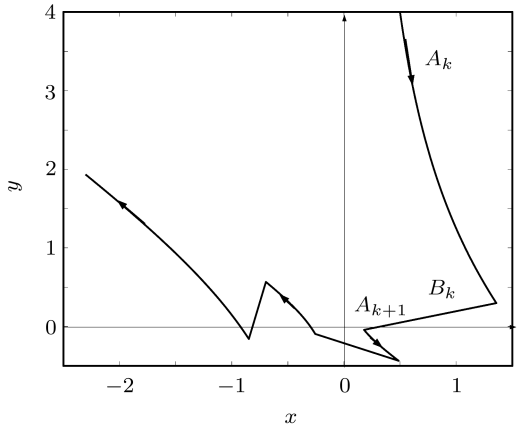


Fig.1. The phase portrait of system (2).

From system (4) it follows that

$$\begin{aligned} \bar{x}_k &= x_k \exp(aT), \\ \bar{y}_k &= \frac{cx_k}{2a}\exp(aT) + \frac{y_k - cx_k}{2a}\exp(-aT). \end{aligned} \tag{5}$$

Meanwhile, considering the effect of impulse in systems (2) and (4), one obtains

$$\begin{aligned} x_{k+1} &= p\bar{x}_k + h, \\ y_{k+1} &= q\bar{y}_k^2. \end{aligned} \tag{6}$$

It follows from systems (5) and (6) we obtain the discrete map

$$\begin{aligned} x_{k+1} &= p\exp(aT)x_k + h, \\ y_{k+1} &= q\Big(\frac{c}{2a}\left(\exp(aT) - \exp(-aT)\right)x_k \\ &\quad + \exp(-aT)y_k\Big)^2. \end{aligned} \tag{7}$$

For each fixed point of map (7) there exists an associated periodic solution of system (2), and vice versa. So in the following, on the assumption that T is a control parameter, we discuss the dynamics of system (2) in virtue of map (7).

3. Existence and stability of periodic solutions

In this section we consider the existence and the stability of period-one solution of system (2). Suppose that the fixed point of (7) is $(X,\ Y)$, then $X = \dfrac{h}{1-p\exp(aT)}$, and

$$q\exp(-2aT)Y^2 + \left(\frac{qch(1-\exp(-2aT))}{a(1-p\exp(aT))} - 1\right)Y + q\left(\frac{ch\,(\exp(aT)-\exp(-aT))}{2a(1-p\exp(aT))}\right)^2 = 0. \tag{8}$$

It follows from Eq.(8) that map (7) has no fixed point for $1 - \dfrac{2qch(1-\exp(-2aT))}{a(1-p\exp(aT))} < 0$, one fixed point for $1 - \dfrac{2qch(1-\exp(-2aT))}{a(1-p\exp(aT))} = 0$, and two fixed points for $1 - \dfrac{2qch(1-\exp(-2aT))}{a(1-p\exp(aT))} > 0$.

Now consider the following function:

$$H(T) = 1 - \frac{2qch(1-\exp(-2aT))}{a(1-p\exp(aT))},\ T \in (0,\ +\infty). \tag{9}$$

There exists a unique $T^* \in (0,\ +\infty)$ such that $H(T^*) = 0$. In fact, in view of $a > 0,\ h > 0,\ q > 0,\ c < 0$ and $T \in (0,\ +\infty)$, we have

$$H(0) = 1 - \frac{2qch(1-1)}{a(1-p)} = 1 > 0,$$
$$\lim_{T\to\frac{1}{a}\ln\left(\frac{1}{p}\right)^-} H(T) = +\infty,$$
$$\lim_{T\to\frac{1}{a}\ln\left(\frac{1}{p}\right)^+} H(T) = -\infty,$$

and

$$\lim_{T\to+\infty} H(T) = 1. \tag{10}$$

Setting $\bar{H}(T) = p\exp(3aT) - 3p\exp(aT) + 2$, then

$$\bar{H}(0) = p - 3p + 2 = 2(1-p) > 0,$$

$\bar{H}'(T) = 3ap(\exp(3aT) - \exp(aT)) > 0$ for $a > 0$ and $T \in (0,\ +\infty)$, which leads to $\bar{H}(T) > 0$ for $T \in (0,\ +\infty)$, hence for $T \in \left(0,\ \dfrac{1}{a}\ln\left(\dfrac{1}{p}\right)\right) \cup \left(\dfrac{1}{a}\ln\left(\dfrac{1}{p}\right),\ +\infty\right)$,

$$H'(T) = \frac{-2qch\exp(-2aT)}{(1-p\exp(aT))^2},$$

and

$$p\exp(3aT) - 3p\exp(aT) + 2 > 0. \tag{11}$$

Considering expressions (9), (10) and (11), there exists an unique $T^* \in \left(\dfrac{1}{a}\ln\left(\dfrac{1}{p}\right),\ +\infty\right)$ such that $H(T^*) = 0$.

For $\dfrac{1}{a}\ln\left(\dfrac{1}{p}\right) < T < T^*$, $H(T) < 0$ and map (7) has no fixed point, and hence system (2) has no period-one solution; For $T = T^*$, $H(T) = 0$ and map (7) has one fixed point, and hence system (2) has a unique period-one solution; For $0 < T < \dfrac{1}{a}\ln\left(\dfrac{1}{p}\right)$ or $T > T^*$, $H(T) > 0$ and map (7) has two fixed points, and hence system (2) has two period-one solutions.

Now we consider the stability of the period-one solutions in the cases of $T \in \left(0,\ \dfrac{1}{a}\ln\left(\dfrac{1}{p}\right)\right) \cup (T^*, +\infty)$ and $T = T^*$.

In the case of $T = T^*$,

$$1 - \frac{2qch(1-\exp(-2aT^*))}{a(1-p\exp(aT^*))} = 0,$$

and the unique fixed point $A(x_k,\ y_k)$ of map (7) is

$$\left(\frac{h}{1-p\exp(aT^*)},\ \frac{1-\dfrac{qch(1-\exp(-2aT^*))}{a(1-p\exp(aT^*))}}{2q\exp(-2aT^*)}\right) = \left(\frac{h}{1-p\exp(aT^*)},\ \frac{1}{4q\exp(-2aT^*)}\right).$$

The characteristic polynomial is given by

$$\begin{vmatrix} \lambda - p\exp(aT^*) & 0 \\ a_{11} & \lambda - 1 \end{vmatrix},$$

where $a_{11} = -2q\Big(\dfrac{c}{2a}(\exp(aT^*)-\exp(-aT^*))x_k + (1-aT^*)y_k\Big)\dfrac{c}{2a}(\exp(aT^*)-\exp(-aT^*))$, and the eigenvalues are

$$\lambda_{1A} = p\exp(aT^*),\ \lambda_{2A} = 1.$$

For $T^* > \dfrac{1}{a}\ln\left(\dfrac{1}{p}\right)$, $\lambda_{1A} = p\exp(aT^*) > 1$ and the fixed point A is unstable. Hence the unique period-one solution corresponding to the fixed point A is unstable.

In the case of $T \in \left(0,\ \dfrac{1}{a}\ln\left(\dfrac{1}{p}\right)\right) \cup (T^*, +\infty)$, the two fixed points of map (7) are

$$B\left(\frac{h}{1-p\exp(aT)},\ \frac{\left(1+\sqrt{H(T)}\right)^2}{4q\exp(-2aT)}\right),$$

and

$$C\left(\frac{h}{1-p\exp(aT)},\ \frac{\left(1-\sqrt{H(T)}\right)^2}{4q\exp(-2aT)}\right). \tag{12}$$

Similar to the case of $T = T^*$, it is easy to calculate the eigenvalues of the fixed points B and C as

$$\lambda_{1B} = p\exp(aT),$$
$$\lambda_{2B} = 1 + \sqrt{H(T)},$$

$$\lambda_{1C} = p\exp(aT),$$

and

$$\lambda_{2C} = 1 - \sqrt{H(T)}. \tag{13}$$

For $T > T^*$ and $T > \frac{1}{a}\ln\left(\frac{1}{p}\right)$, then $\lambda_{1B} = p\exp(aT) > 1$ and $\lambda_{1C} = p\exp(aT) > 1$. So the fixed points B and C are unstable. Therefore the associated two period-one solutions are unstable for $T > T^*$.

For $0 < T < \frac{1}{a}\ln\left(\frac{1}{p}\right)$, the eigenvalues $\lambda_{1B} \in (0,\ 1)$ and

$$\lambda_{2B} = 1 + \sqrt{H(T)} > 1,$$

which means that the period-one solution corresponding to the fixed point B is unstable. We also have $\lambda_{1C} \in (0,\ 1)$ and $\lambda_{2C} = 1 - \sqrt{H(T)}$.

It follows from expressions (10) and (11) that there exits an unique $T_0 \in \left(0,\ \frac{1}{a}\ln\left(\frac{1}{p}\right)\right)$ such that

$$H(T_0) = 4. \tag{14}$$

Then

$$-1 < \lambda_{2C} < 1 \ \text{ for } \ 0 < T < T_0,$$

and

$$\lambda_{2C} < -1 \ \text{ for } \ T_0 < T < \frac{1}{a}\ln\left(\frac{1}{p}\right), \tag{15}$$

from which it follows that the period-one solution corresponding to the fixed point C is stable for $0 < T < T_0$ and unstable for $T_0 < T < \frac{1}{a}\ln\left(\frac{1}{p}\right)$.

To summarize the above discussion, the following results about the existence and the stability of period-one solutions of system (2) are obtained.

Proposition 3.1

(i) System (2) has no periodic solution for $\frac{1}{a}\ln\left(\frac{1}{p}\right) < T < T^*$, where T^* is the solution of the equation

$$1 - \frac{2qch(1-\exp(-2aT))}{a(1-p\exp(aT))} = 0,\ T \in (0,\ +\infty); \tag{16}$$

(ii) System (2) has a unique unstable period-one solution for $T = T^*$;

(iii) System (2) has two period-one solutions for $T \in \left(0,\ \frac{1}{a}\ln\left(\frac{1}{p}\right)\right) \cup (T^*, +\infty)$. Furthermore, if $T \in \left(T_0,\ \frac{1}{a}\ln\left(\frac{1}{p}\right)\right) \cup (T^*, +\infty)$, where T_0 is the solution of the equation

$$1 - \frac{2qch(1-\exp(-2aT))}{a(1-p\exp(aT))} = 4,\ T \in (0,\ +\infty), \tag{17}$$

then these two period-one solutions are unstable. If $T \in (0,\ T_0)$, then one period-one solution is stable and one is unstable.

4. Flip bifurcation

There are three ways for a fixed point P of a discrete map $f:\ \mathbb{R}^n \to \mathbb{R}^n$ to be nonhyperbolic, that is, $\mathrm{D}f(P)$ has an eigenvalue $+1$, an eigenvalue -1 or a pair of complex eigenvalues λ and $\bar{\lambda}$ with $|\ \lambda\ | = 1$.

Lemma 4.1

Let $f_\mu:\ \mathbb{R} \to \mathbb{R}$ be a one-parameter family of maps such that f_{μ_0} has a fixed point x_0 with an eigenvalue of -1, and assume the following conditions:

(F_1) $\left(\frac{\partial f}{\partial \mu}\frac{\partial^2 f}{\partial x^2} + 2\frac{\partial^2 f}{\partial x \partial \mu}\right) \neq 0$ at $(x_0,\ \mu_0)$;

(F_2) $\bar{a} = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}\right)^2 + \frac{1}{3}\left(\frac{\partial^3 f}{\partial x^3}\right) \neq 0$ at $(x_0,\ \mu_0)$,

then there will be a smooth curve of fixed points of f_μ passing through $(x_0,\ \mu_0)$, whose stability changes at $(x_0,\ \mu_0)$. There is also a smooth curve γ passing through $(x_0,\ \mu_0)$ so that $\gamma - (x_0,\ \mu_0)$ is a union of hyperbolic period-2 orbits.

The proof of this lemma is can be found in Ref.[14]. In (F_2) the sign of $\bar{a}$ determines the stability and the direction of bifurcation of the orbits of period-2. If $\bar{a}$ is positive, the orbits are stable; if $\bar{a}$ is negative they are unstable. As mentioned above, an eigenvalue of -1 is associated with a flip bifurcation, which is also referred to as a period-doubling bifurcation. Now we discuss the flip bifurcation of system (2) by using Lemma 4.1.

Proposition 4.1

Assume that the following condition holds:

(F_3) $A_4\left(A_7 - \frac{A_1 apx_0\exp(aT_0)}{1+p\ \exp(aT_0)}\right) + \Big(A_3 apx_0 \exp(aT_0) + A_6 + \frac{2A_1 apx_0\exp(aT_0)}{1+p\exp(aT_0)}\Big) \neq 0$, where T_0

is the solution of the equation $3ap\exp(3aT_0)-(3a+2chq)\exp(2aT_0)+2chq=0$,

$$x_0=\frac{h}{1-p\exp(aT_0)},\quad y_0=\frac{\exp(2aT_0)}{4q},$$
$$A_1=\frac{2c^2q^2x_0y_0}{a^2}-\frac{c^2qx_0}{a^2}+\frac{c^2x_0}{8a^2y_0}+\frac{cqy_0}{a}-\frac{c}{4a},$$
$$A_2=\frac{c^2q(\exp(aT_0)-\exp(-aT_0))^2}{4a},$$
$$A_3=\frac{cq(1-\exp(-2aT_0))}{a},$$
$$A_4=q\exp(-2aT_0),$$
$$A_5=-\frac{3}{2}c\exp(2aT_0)-c,$$
$$A_6=2cqx_0\exp(-2aT_0),$$

and

$$A_7=\frac{8c^2q^2x_0^2y_0}{a}-\frac{2c^2qx_0^2}{a}+2cqx_0y_0-2ay_0,$$

then a flip bifurcation will occur at $T=T_0$ in system (2). For some cases where $\epsilon>0$, system (2) has a positive stable period-two solution for $T\in(T_0,T_0+\epsilon)$.

Proof It follows from Section 3 that one of the eigenvalues of the fixed point C is $\lambda_{2C}=-1$ at $T=T_0$, and hence T_0 is a candidate of flip bifurcation. Map (7) may be rewritten as

$$F_{\mathrm{T}}:\begin{pmatrix}x\\y\end{pmatrix}\mapsto\begin{pmatrix}p\exp(aT)x+h\\ q\left(\dfrac{c}{2a}(\exp(aT)-\exp(-aT))x+\exp(-aT)y\right)^2\end{pmatrix}. \tag{18}$$

Expression (14) yields

$$3ap\exp(3aT_0)-(3a+2chq)\exp(2aT_0)+2chq=0, \tag{19}$$

where $T_0\in\left(0,\ \dfrac{1}{a}\ln\left(\dfrac{1}{p}\right)\right)$. In this case, the fixed point $C(x_0,\ y_0)$ is

$$\left(\frac{h}{1-p\exp(aT_0)},\ \frac{\exp(2aT_0)}{4q}\right). \tag{20}$$

Let $u=x-x_0$, $v=y-y_0$ and $\tilde{T}=T-T_0$, we transform the fixed point $C(x_0,\ y_0)$ of map (18) to the origin, then map (18) becomes

$$\begin{pmatrix}u\\v\end{pmatrix}\mapsto\begin{pmatrix}p\exp(aT_0)\exp(a\tilde{T})u+p\exp(aT_0)\exp(a\tilde{T})x_0+h-x_0\\ q\exp(-2a(T_0+\tilde{T}))\Big(\dfrac{c}{2a}(\exp(2a(T_0+\tilde{T}))-1)(u+x_0)+v+y_0\Big)^2-y_0\end{pmatrix}. \tag{21}$$

By using $x_0=p\exp(aT_0)+h$,

$$y_0=q\exp(-2aT_0)\left(\frac{c}{2a}(\exp(2aT)-1)x_0+y_0\right)^2,$$

and $\exp(a\tilde{T})=1+a\tilde{T}+O(|\tilde{T}|^2)$, map (21) becomes

$$\begin{pmatrix}u\\v\end{pmatrix}\mapsto\begin{pmatrix}p\exp(aT_0)&0\\A_1&-1\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}+\begin{pmatrix}f_1(u,\ v,\ \tilde{T})\\ g_1(u,\ v,\ \tilde{T})\end{pmatrix}, \tag{22}$$

where

$$f_1(u,\ v,\ \tilde{T})=ap\exp(aT_0)\tilde{T}u+apx_0\exp(aT_0)\tilde{T}+O(|\tilde{T}|^2+|u|^2+|v|^2),$$

and

$$\begin{aligned}&g_1(u,\ v,\ \tilde{T})\\ &=A_2u^2+A_3uv+A_4v^2+A_5u\tilde{T}+A_6v\tilde{T}+A_7\tilde{T}\\ &\quad+O(|\tilde{T}|^2+|u|^2+|v|^2).\end{aligned} \tag{23}$$

Let

$$J=\begin{pmatrix}0&\dfrac{1}{A_1}\\ 1&\dfrac{1}{1+p\exp(aT_0)}\end{pmatrix}$$

and use the translation $\begin{pmatrix}u\\v\end{pmatrix}=J\begin{pmatrix}\tilde{x}\\\tilde{y}\end{pmatrix}$, then map (22) will become

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}\mapsto\begin{pmatrix}-1 & 0\\ 0 & p\exp(aT_0)\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\end{pmatrix}+\begin{pmatrix}f(\tilde{x},\ \tilde{y},\ \tilde{T})\\ g(\tilde{x},\ \tilde{y},\ \tilde{T})\end{pmatrix}, \tag{24}$$

where

$$f(\tilde{x},\ \tilde{y},\tilde{T})=g_1\left(\frac{1}{A_1}\tilde{y},\tilde{x}+\frac{1}{1+p\exp(aT_0)}\tilde{y},\ \tilde{T}\right)-\frac{A_1}{1+p\exp(aT_0)}f_1\left(\frac{1}{A_1}\tilde{y},\tilde{x}+\frac{1}{1+p\exp(aT_0)}\tilde{y},\ \tilde{T}\right),$$

and

$$g(\tilde{x},\ \tilde{y},\ \tilde{T})=A_1f_1\left(\frac{1}{A_1}\tilde{y},\ \tilde{x}+\frac{1}{1+p\exp(aT_0)}\tilde{y},\ \tilde{T}\right).$$

Now the centre manifold theorem is used to determine the nature of the bifurcation of the fixed point $(\tilde{x},\ \tilde{y})=(0,\ 0)$ at $\tilde{T}=0$. There exists a centre manifold for map (24) which can be represented as follows:

$$\begin{aligned}W^c(0)\ =\ &\{(\tilde{x},\ \tilde{y},\ \tilde{T})\in R^3|\tilde{y}=h(\tilde{x},\ \tilde{T}),\\ &h(0,\ 0)=0,\ \mathrm{D}h(0,\ 0)=0\}.\end{aligned}$$

Letting

$$\tilde{y}\ =\ h\left(\tilde{x},\ \tilde{T}\right)=B_0\tilde{T}+B_1\tilde{T}\tilde{x}+B_2\tilde{x}^2$$

$$+O\left(|\tilde{T}|^2+|\tilde{x}|^3\right)$$

and substituting it into map (24) yield

$$\begin{aligned}&h(-\tilde{x}+f(\tilde{x},\ h(\tilde{x},\ \tilde{T}),\ \tilde{T}),\tilde{T}))\\ &-p\exp(aT_0)h(\tilde{x},\ \tilde{T})-g\tilde{x},\ h(\tilde{x},\ \tilde{T}),\ \tilde{T})=0.\end{aligned} \tag{25}$$

Equating terms of like powers to zero gives

$$B_0=A_1apx_0\exp(aT_0),\ B_1=0,\ \text{ and }\ B_2=0.$$

Therefore, the map restricted to the centre manifold is given by

$$\begin{aligned}\bar{f}:\tilde{x}\ 7\mapsto\ &-\tilde{x}+\left(A_7-\frac{A_1apx_0\exp(aT_0)}{1+p\ exp(aT_0)}\right)\tilde{T}+\left(A_3apx_0\exp(aT_0)+A_6+\frac{2A_1apx_0\exp(aT_0)}{1+p\exp(aT_0)}\right)\tilde{X}\tilde{T}\\ &+A_4\tilde{X}^2+O\left(|\tilde{T}|^2+|\tilde{x}|^3\right).\end{aligned}$$

In view of (F_3), we have

$$\begin{aligned}&\left(\frac{\partial\bar{f}}{\partial\bar{T}}\frac{\partial^2\bar{f}}{\partial\bar{x}^2}+2\frac{\partial^2\bar{f}}{\partial\bar{x}\partial\bar{T}}\right)\\ &=\ 2A_4\left(A_7-\frac{A_1apx_0\exp(aT_0)}{1+p\exp(aT_0)}\right)+2\left(A_3apx_0\exp(aT_0)+A_6+\frac{2A_1apx_0\exp(aT_0)}{1+p\exp(aT_0)}\right)\neq 0\end{aligned}$$

and

$$\bar{a}=\frac{1}{2}\left(\frac{\partial^2\bar{f}}{\partial\bar{x}^2}\right)^2+\frac{1}{3}\left(\frac{\partial^3\bar{f}}{\partial\bar{x}^3}\right)=2A_4^2>0$$

at $\left(\tilde{x},\ \tilde{T}\right)=(0,\ 0)$. Then conditions (F_1) and (F_2) hold. So a flip bifurcation occurs in view of Lemma 4.1. A positive period-two solution bifurcates from the positive period-one solution at $\bar{T}=0$, that is, $T=T_0$. Since $\bar{a}>0$ in (F_2), the positive period-two solution is stable. It also means that for some cases where $\epsilon>0$, system (2) has a positive stable period-two solution for $q\in(T_0,\ T_0+\epsilon)$. At the same time, the positive period-one solution loses its stability. Thus we complete the proof of Theorem 4.1.

5. Numerical example

In this section, viewing T as parameter, we give the numerical results about the dynamics of considered impulsive differential systems through the follow-

ing example:

$$\left\{\begin{array}{l} \left.\begin{array}{l} \dot{x} = x, \\ \dot{y} = -2x - y, \end{array}\right\} t \neq nT, \\ \left.\begin{array}{l} \triangle x = -(1-0.5)x + 0.5, \\ \triangle y = 1.5y^2 - y, \end{array}\right\} t = nT. \end{array}\right. \tag{26}$$

In our case, $a = 1$, $b = 0$, $c = -2$, $p = 0.5$, $q = 1.5$, $h = 0.5$, and $\dfrac{1}{a}\ln\left(\dfrac{1}{p}\right) = \ln(2)$.

It follows from $H(T^*) = 0$ that

$$1 + \frac{3(1 - \exp(-2T))}{(1 - 0.5\exp(T))} = 0,\ T \in (0,\ +\infty), \tag{27}$$

which yields $T^* \approx 2.067365$. Then the fixed point $A\left(\dfrac{h}{1 - p\exp(aT^*)},\ \dfrac{1}{4q\exp(-2aT^*)}\right)$ is $(-0.1694,\ 10.4122)$ and the eigenvalues are $\lambda_1 = 1 - p\exp(aT^*) = -2.9520 < -1$, and $\lambda_2 = 1$. The unique period-one solution corresponding to this fixed point is unstable, see Fig.2.

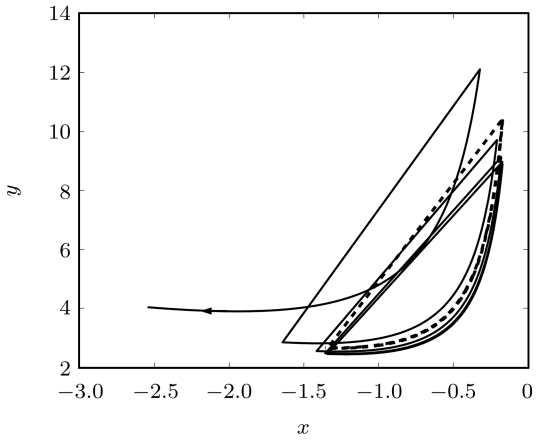


Fig.2. Stability of period-one solution of system (26) with $T = 2.067365$.

Setting $T = \ln(10)$, $T \in (T^*,\ +\infty)$ and the fixed points are $B(-0.125,\ 37.8732)$ and $C(-0.125,\ 4.0435)$. Therefore system (26) with $T = \ln(10)$ has two period-one solutions, see Fig.3. It can be seen from Fig.3 that the solution from the initial point $(-0.125,\ 30.8732)$ does not tend to these two periodic solutions and then these periodic solutions are unstable.

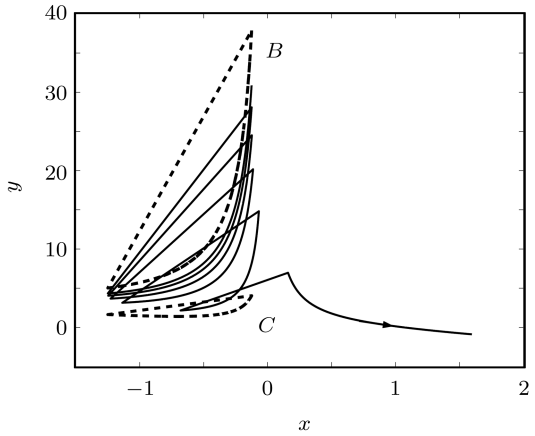


Fig.3. Stability of period-one solution of system (26) with $T = 2.067365$.

From expression (14), we have

$$1 + \frac{3\,(1 - \exp(-2T))}{1 - 0.5\exp(T)} = 4,\ T \in (0,\ +\infty), \tag{28}$$

which yields $T_0 = \dfrac{1}{3}\ln(2)$. Setting $T = \dfrac{1}{4}\ln(2)$, $T \in (0,\ T_0)$ and the fixed points are $B(1.2334,\ 1.8213)$ and $C(1.2334,\ 0.1433)$. Then system (26) with $T = \dfrac{1}{4}\ln(2)$ has two period-one solutions, see Fig.4. It can be seen that the solution from the initial point $(1.2334,\ 1.8)$ tends to the period-one solution corresponding to the fixed point C. Hence system (26) with $T = \dfrac{1}{4}\ln(2)$ has one stable period-one solution and one unstable period-one solution.

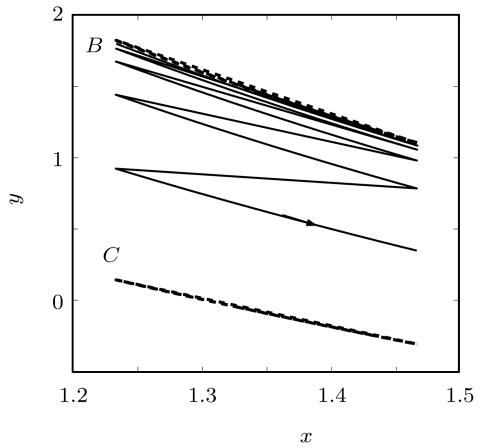


Fig.4. Stability of period-one solution of system (26) with $T = 2.067365$.

For $T = T_0 = \dfrac{1}{3}\ln(2)$, the fixed point $C(x_0,\ y_0)$ is $\left(\dfrac{1}{2 - \sqrt[3]{2}},\ \dfrac{\sqrt[3]{4}}{6}\right)$ in view of proposition 4.1. It follows from expression (23) that $A_1 = -1.7867$, $A_2 =$

0.3260, $A_3 = -1.1101$, $A_4 = 0.9449$, $A_5 = 6.7622$, $A_6 = -5.1072$, and $A_7 = 10.1993$. Then

$$2A_4\left(A_7 - \frac{A_1 apx_0 \exp(aT_0)}{1 + p\exp(aT_0)}\right) + 2\left(A_3 apx_0 \exp(aT_0) + A_6 + \frac{2A_1 apx_0 \exp(aT_0)}{1 + p\exp(aT_0)}\right) = 5.2023 \neq 0.$$

Condition (F_3) holds and then a flip bifurcation occurs at $T = T_0 = \frac{1}{3}\ln(2) \approx 0.2310$. Figure 5 shows the bifurcation diagram of stable periodic solutions of system (26). It can be seen from the bifurcation diagram in Fig.5 that there exists a stable period-two solution in system (26) for $T \in \left(\frac{1}{3}\ln(2),\ 0.345\right)$ and the route to chaos is period-doubling bifurcation. A period-two solution with $T = 0.3$ is shown in Fig.6 and a chaotic solution of system (26) with $T = 0.432$ is given in Fig.7.

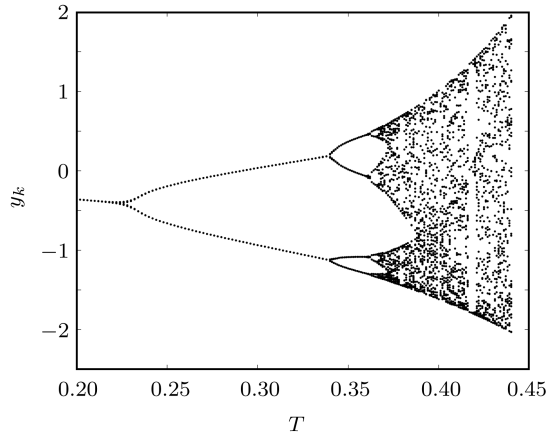


Fig.5. The bifurcation diagram of system (26) with respect to T.

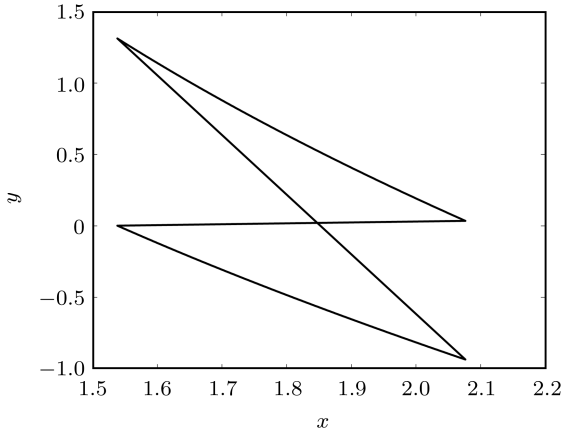


Fig.6. The stable period-2 solution of system (26) with $T = 0.3$.

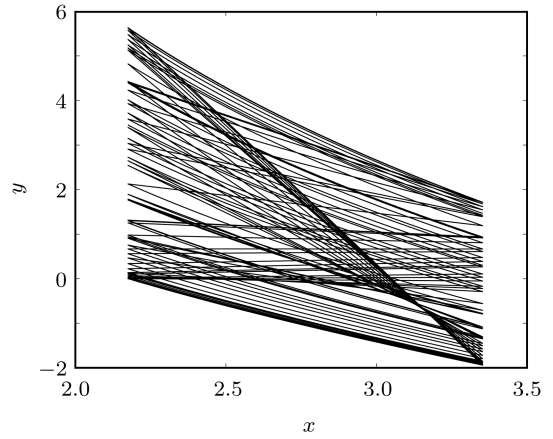


Fig.7. A chaotic solution of system (26) with $T = 0.432$.

6. Conclusion

The dynamics of a linear impulsive system is studied. It is seen that the impulsive system (2) is very rich in dynamics, although the corresponding system without impulses is simple. By using a discrete map, we obtain the conditions of existence and stability of the period-one solutions. The conditions of existence for flip bifurcation are derived by using the centre manifold theorem and the bifurcation theorem. Viewing T as a parameter, the numerical results reveal the complicated dynamical behaviour of system (2), which verifies the theoretical analysis. These results about linear impulsive systems are of significance for the dynamic analysis of nonlinear impulsive systems and we will adopt more real economics, ecology and environment models for the further study.

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