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Synchronization of spiral waves in a two-layer coupled inhomogeneous excitable system*

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We studied synchronization behaviours of spiral waves in a two-layer coupled inhomogeneous excitable system. It was found that phase synchronization can be observed under weak coupling strength. By increasing the coupling strength, the synchronization is broken down. With the further increase of the coupling strength, complete synchronization and phase synchronization occur again. We also found that the inhomogeneity in excitable systems is helpful to the synchronization.

Keywords: spiral waves, synchronization, inhomogeneous excitable media
PACC: 0545, 8700, 8730C

1. Introduction

Spiral waves are observed in a variety of physical, chemical, and biological systems. It is interesting to find that the appearance of spiral wave in the cardiac tissues may give information about one kind of heart disease (life-threatening cardiac arrhythmias) and the instability of spiral waves in the cardiac tissue indicates the rapid death of the heart. Up to now, many schemes about suppression and transition of spiral waves have been studied extensively.[1−5] Meanwhile, those studies have led to a detailed understanding of spiral wave dynamics.

In nature, the coupled systems are ubiquitous. For example, the heart-muscle tissue can be taken as composing of a number of slices of heart-muscle tissues. The human neural system is composed of a large number of nerve cells with synaptic connections to each other. Therefore, the studying of synchronization of coupled systems has theoretical importance and application perspectives.[6−12] Complete synchronization (CS) and phase synchronization (PS) have been found in systems exhibiting spatiotemporal chaos.[7,9−17] The previous works used to focus on exploring the qualitative behaviours of synchronization in coupled homogeneous excitable media.

In this paper, we study the synchronization of spiral waves in a two-layer coupled inhomogeneous excitable system. We find that phase synchronization and complete synchronization occurred at certain coupling strength threshold values and the inhomogeneity in excitable systems is helpful to the synchronization.

In the next section we describe the model of the two-layer coupled inhomogeneous excitable system and the initial conditions in our numerical simulations. In Section 3 we quantitatively study the CS of spiral waves and the relationship between the CS and the coupling strength. In Section 4, we study the PS of spiral waves. The conclusion and a short discussion are presented in the last section.

2. Model

For the single layer system, oxidation of CO absorbed on a Pt(110) single-crystal surface[18] is described by

$$\frac{\partial u}{\partial t} = \varepsilon^{-1} u(1-u)\left(u - \frac{v+b}{a}\right) + \nabla^2 u, \tag{1a}$$

$$\frac{\partial v}{\partial t} = f(u) - v, \tag{1b}$$

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And the function $f(u)$ is often defined with

$$f(u) = \begin{cases} 0 & 0 \le u < 1/3, \\ 1 - 6.75u(u-1)^2 & 1/3 \le \mathrm{u} \le 1, \\ 1 & u > 1, \end{cases} \quad (1c)$$

where the activator variable u is the coverage of the adsorbed CO, while the inhibitor variable v describes a structural change. The decisive parameters are b and ε, and the excitability of the system is defined by the inverse of ε. Equation (1) exhibits a saddle-node bifurcation at $b = 0$, which leads to an oscillatory process for negative b, and is excitable for small positive b and $a < 1 + b$.

For the two-layer system, the model is described by

$$\begin{aligned} \frac{\partial u_1}{\partial t} =& \varepsilon^{-1}\left[u_1(1-u_1)\left(u_1 - \frac{v_1+b}{a}\right) + c(u_2 - u_1)\right] \\ &+ \nabla^2 u_1, \quad (2a) \\ \frac{\partial v_1}{\partial t} =& f(u_1) - v_1, \\ \frac{\partial u_2}{\partial t} =& \varepsilon^{-1}\left[u_2(1-u_2)\left(u_2 - \frac{v_2+b}{a}\right) + c(u_1 - u_2)\right] \\ &+ \nabla^2 u_2, \quad (2b) \\ \frac{\partial v_2}{\partial t} =& f(u_2) - v_2, \end{aligned}$$

where subscripts 1 and 2 denote the variables of the first and the second layer, respectively. For depicting a complex inhomogeneous media, we suppose that the parameters exhibit normal distribution with the values of parameter a, b and ε, ranging from 0.3 to 1.1, 0 to 0.2 and 0.02 to 0.06, respectively. To compare the different influences on the synchronization of homogeneous media and inhomogeneous media, we set the values of parameters a, b and ε at 0.7, 0.1, and 0.04, respectively, for the homogeneous media, which are equal to their mean values of normal distributions in the inhomogeneous media. In addition, we also suppose that the two layers are identical (i.e., with the same parameter values) in our numerical simulations. Equation (2) is integrated by an explicit scheme with fixed time step $\Delta t = 0.5$, spatial step $\Delta x = \Delta y = 0.2$, and grid number 100×100. No-flux boundary condition is applied in the simulations.

3. Complete synchronization

Complete synchronization is the simplest and strongest form among the diverse synchronization behaviours. In the case of CS, each site or element keeps pace with the others in coupled systems (i.e., each element having the same dynamics and parameter set). We let the initial conditions for Eq.(2) be a spiral wave in the first layer and the rest state in the second layer. In order to obtain this initial condition, we set the coupling length to zero before $t = 20$ and only give the initial values to the first layer. For a strong coupling strength, the second layer is excited by the first layer through the action of coupling, and the two-layer coupled system is synchronized after a transient period, and the evolution of activator in the two layers are shown in Fig.1.

With the increasing of coupling strength, the two-layer coupled system is synchronized quickly and comes to a rest state in the end (see Fig.2).

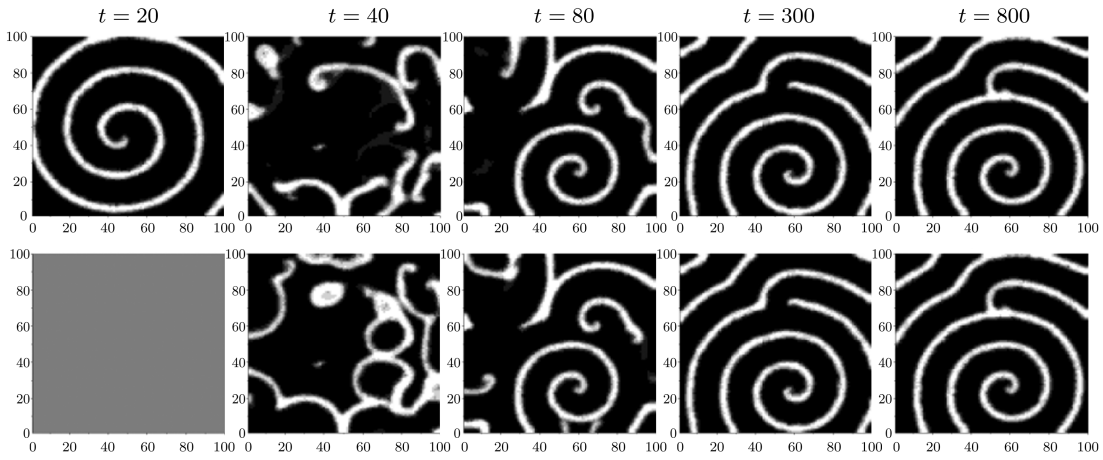


Fig.1. Sequence of snapshots of spiral waves in the two layers. Parameter value: $c = 0.022$, inhomogeneous media.

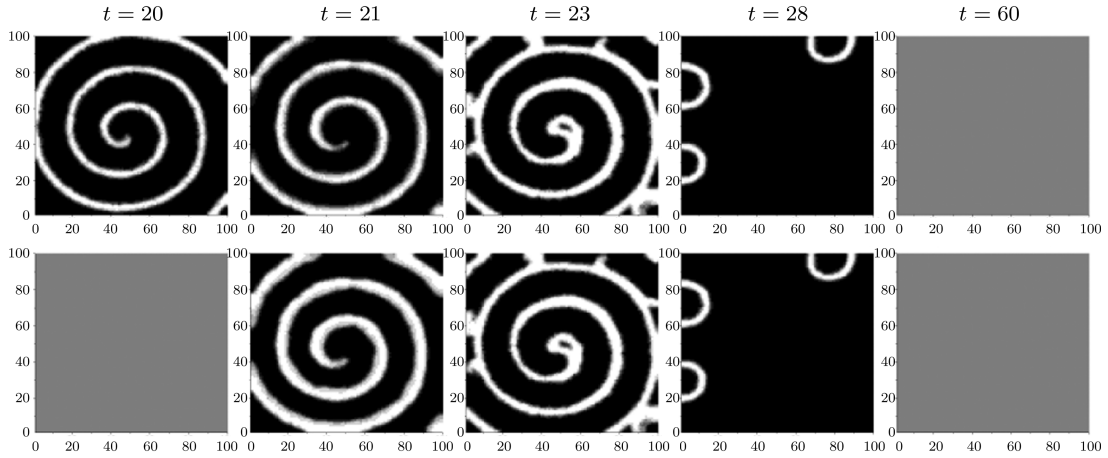


Fig.2. Sequence of snapshots of spiral waves in the two layers. Parameter value: $c = 0.08$, inhomogeneous media.

Comparing Fig.1 with Fig.2, it is found that a stronger coupling coefficient will take a shorter transient period to reach CS.

To quantitatively describe the synchronization of the two layers, we have calculated an instantaneous arithmetic mean difference of u

$$\mathrm{Diff}(t) = N^{-2} \sum_{i,j=1}^{N} \left| u_{1_{i,j}}(t) - u_{2_{i,j}}(t) \right|, \qquad (3)$$

and its average

$$\mathrm{AverDiff} = T^{-1} \sum_{t=0}^{T} \left(N^{-2} \sum_{i,j=1}^{N} \left| u_{1_{i,j}}(t) - u_{2_{i,j}}(t) \right| \right), \qquad (4)$$

where T denotes the duration of time, $i, j = 1, 2, \ldots, N$. Diff $= 0$ or the value of AverDiff is close to zero indicates that the patterns in the two layers are completely synchronized. Figure 3 shows the arithmetic mean of difference of the activator variable u and its average against the coupling strength c.

Figure 3(a) shows the two layers completely synchronized when $c = 0.0206$ for the inhomogeneous media and $c = 0.0224$ for the homogeneous media. The values of a, b and ε in the homogeneous media are equal to their means of normal distribution in the inhomogeneous media. It indicates that the coupled inhomogeneous media is easier to synchronize than the coupled homogeneous media.

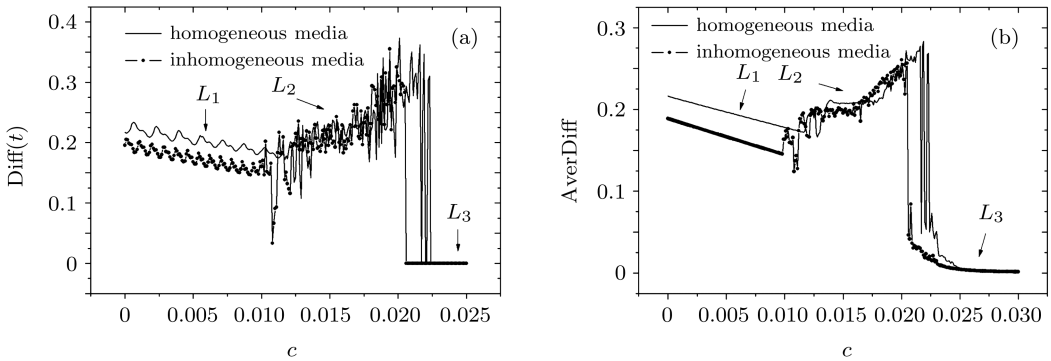


Fig.3. The arithmetic mean of difference of u and its average against the coupling strength c. The parameter c changes from 0 to 0.025, $t = 500$ time units in (a), c from 0 to 0.03, $T = 500$ time units in (b). The solid line corresponds to homogeneous media; the line with dots corresponds to inhomogeneous media, the parameters are $a = 0.7$, $b = 0.1$, and $\varepsilon = 0.04$ for the homogeneous media.

The curves in Fig.3 can be divided into three parts L_1, L_2 and L_3. Part L_1 corresponds to too low coupling strength of c from 0 to 0.01, approximately, which cannot excite the elements in the second layer. Figure 4 shows that when taking off the term of coupling at $t = 100$, the second layer enters the rest state quickly.

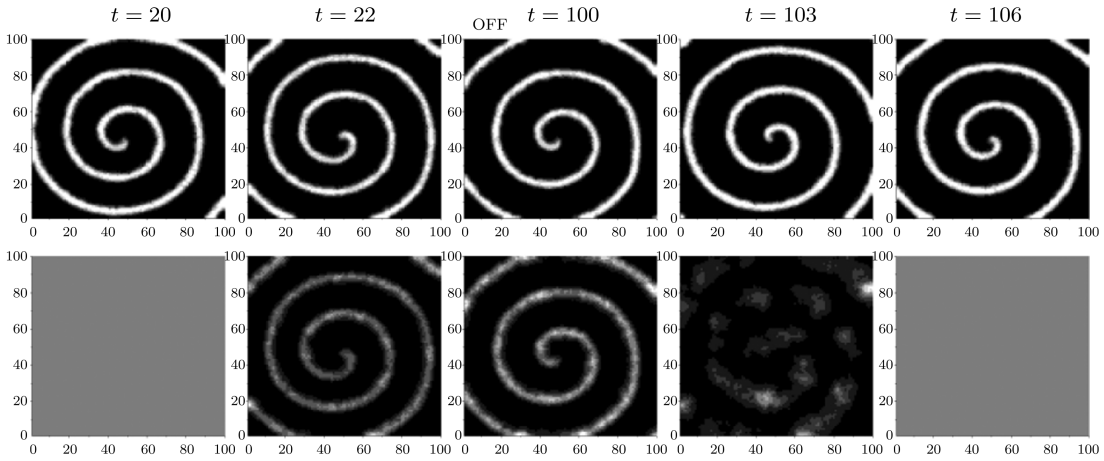


Fig.4. Sequence of snapshots of spiral waves in the two layers. Parameter value: $c = 0.008$, inhomogeneous media. At first, there exists a weak coupling between the two layers. However, the system is subsequently switched to no coupling at $t = 100$ (indicated by "OFF" on the plot).

Part L_2 in Fig.3 corresponds to no synchronization between the two layers but the elements in the second layer are excited. Figure 5 illustrates that, the elements in the second layer have been excited, and they perturb the elements in the first layer in turn. Consequently, the feedback destroys the spiral waves in the first layer. In addition, for some coupling coefficients (close to the curve L_1) the feedback is not strong enough to destroy the spiral waves in the first layer, but only affects the behaviour of the tip of the spiral waves. Here, we shall not dwell on this issue. Part L_3 corresponds to strong couple strength of c from 0.02 to 0.03, approximately, and corresponds to complete synchronization between the two layers.

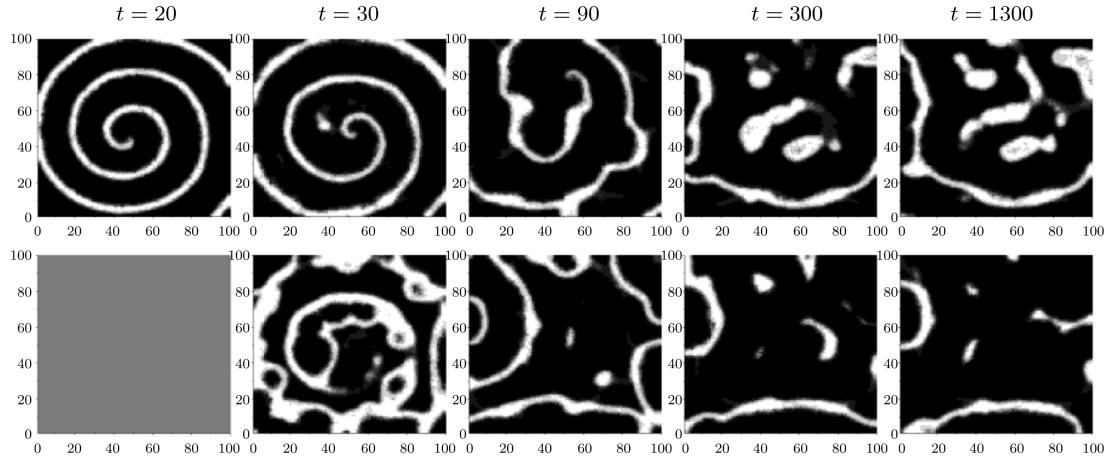


Fig.5. Sequence of snapshots of spiral wave in the two layers. Parameter value: $c = 0.018$, inhomogeneous media.

4. Phase synchronization

Phase synchronization means the entrainment of phases of chaotic oscillators, whereas their amplitudes remain chaotic and generally not correlated. As mentioned in Section 3, the elements in the second layer cannot be excited by weak coupling strength. However, the elements in the second layer are perturbed by the first layer and forms a spiral wave similar to the spiral wave in the first layer (see Fig.6). It is found that the phases are synchronized while the amplitudes are uncorrelated.

In order to quantitatively describe the phase synchronization, it is convenient to introduce an instantaneous phase $\phi(t)$,[19,20] called the marked-events phase. The marked-events method requires that an event which occurs once per period at a well-defined

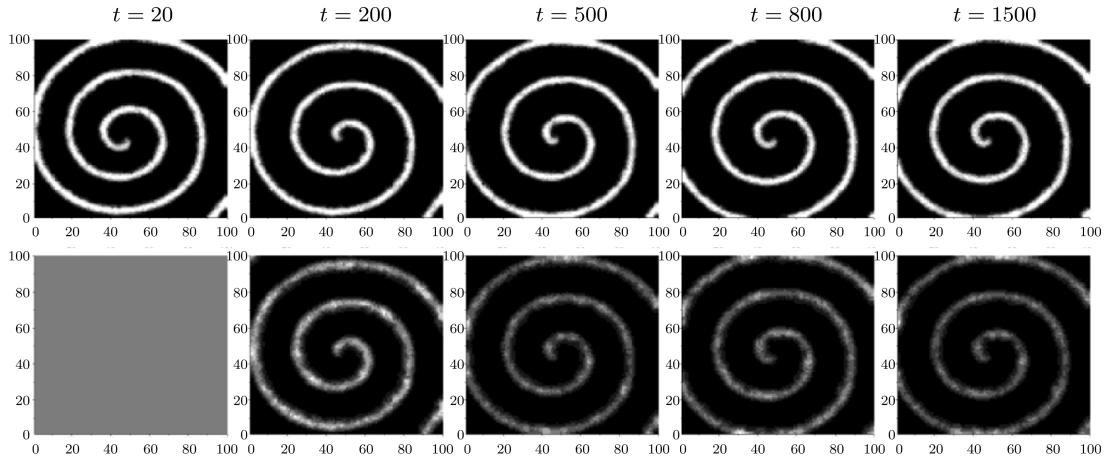


Fig.6. Sequence of snapshots of spiral wave in the two layers. Parameter value: $c = 0.008$, inhomogeneous media.

time instant is detected with the highest possible precision. It is then possible to define an instantaneous phase by linear interpolation, i.e,

$$\phi(t) = \frac{t - t_k}{t_{k+1} - t_k} 2\pi + k2\pi, \quad (t_k \le t \le t_{k+1}), \tag{5}$$

where t_k is the sequence of time when the selected events has been detected. The instantaneous phase difference between two elements is then given by

$$\psi(t) = \phi_1(t) - \phi_2(t), \tag{6}$$

where the subscripts 1 and 2 denote any two elements in the grid. The mean firing rate $\omega_{i,j}$ can be defined as:[11]

$$\omega_{i,j} = \lim_{T\to\infty} \left(2\pi \frac{M_{i,j}(T)}{T} \right), \tag{7}$$

where $M_{i,j}(T)$ is the number of peak values of activator variable u of the grid element (i, j) during the time interval T. Phase synchronization between two elements implies that difference of the firing rate equals to zero. Therefore, for describing the phase synchronization between two layers, we further define an average of mean firing rate difference between two layers

$$\Delta(\omega) = N^{-2} \sum_{i,j=1}^{N} \left| \omega_{1_{i,j}} - \omega_{2_{i,j}} \right|, \tag{8}$$

where the number 1 and 2 denote variables of the first and the second layer, respectively, subscripts i, j denote the grid element (i, j). For convenience, we choose N points located on a line in each layer and the two lines lie in corresponding positions in the two layers. We select the centre line in each layer. Figure 7 shows the average of mean firing rate difference against the coupling strength c after an evolution of 800 time units.

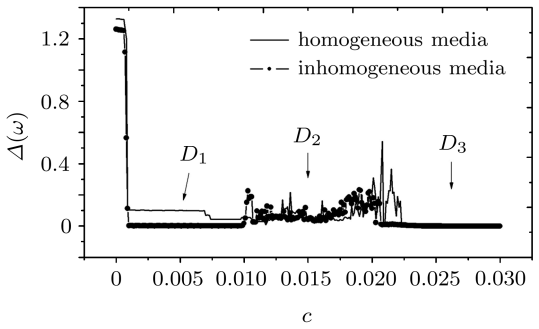


Fig.7. The average of mean firing rate difference against the coupling strength c. Parameter value: $T = 800$ time units, $c = 0.0, \ldots, 0.03$. The solid line corresponds to homogeneous media; the line with dots corresponds to inhomogeneous media, the parameters are $a = 0.7$, $b = 0.1$, and $\varepsilon = 0.04$ for the homogeneous media.

Similar as in Fig.3, Fig.7 is divided into three parts, D_1, D_2, and D_3. Part D_1 corresponds to weak coupling strength c from 0 to 0.01, approximately. This part also corresponds to the frequency synchronization while the amplitudes are totally uncorrelated. Furthermore, in part D_1, we observe that the coupled inhomogeneous media frequency synchronized better than the coupled homogeneous media. Part D_2 corresponds to neither complete synchronization nor frequency synchronization. Part D_3 corresponds to strong coupling strength $c > 0.02$, where we observe both frequency synchronization and complete synchronization.

Now we want to examine the phase synchronization in the frequency synchronization parameter region. An average of phase difference between two layers is defined by

$$\Delta\phi(t) = N^{-2} \sum_{i,j=1}^{N} \left| \phi_{1_{i,j}}(t) - \phi_{2_{i,j}}(t) \right|, \tag{9}$$

where the number 1 and 2 denote variables of the first and the second layer, respectively, subscripts i, j denote the grid element (i, j). In the same way, we select N elements in the centre line from each layer to compute Eq.(9).

Figure 8 shows that the average of phase difference equals to 2.5 with the evolution of time. It indicated that phase synchronization occurred between the two layers. From Fig.7, we know that the phases are unsynchronized when $c = 0.022$ for the homogeneous media. From Fig.9, we can observe that the phases are synchronized when the coupling strength $c = 0.008$ and $c = 0.025$ for homogeneous and inhomogeneous media, respectively.

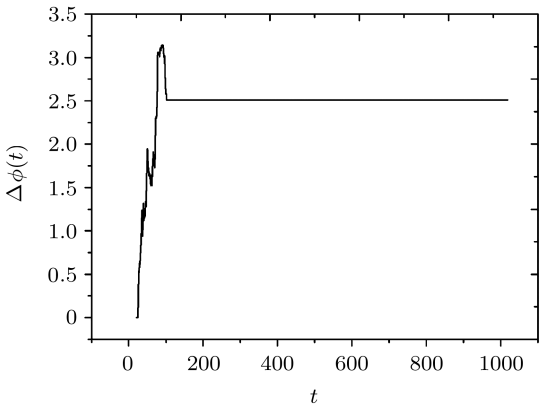


Fig.8. The average of phase difference between two layers against the evolution of time, parameter value: $c = 0.022$, inhomogeneous media.

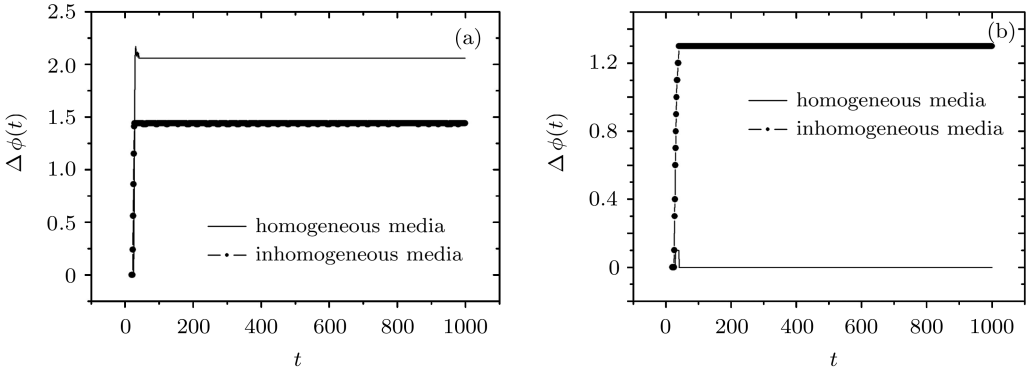


Fig.9. The average of phase difference between two layers against the evolution of time: (a) $c = 0.008$, (b) $c = 0.025$, the solid line corresponds to inhomogeneous media; the line with dots corresponds to homogeneous media.

5. Conclusion

In this paper, we studied the synchronization of spiral waves in a two-larger coupled inhomogeneous excitable system. At the initial time, we let the first layer be a spiral wave and the second layer in the rest state. Numerical simulation results show that phase synchronization and complete synchronization are exhibited between the two layers. Phase synchronization is observed when the coupling strength is weak and the complete synchronization is only observed when the coupling strength is strong. For some range of the coupling strength, neither the complete synchronization nor the phase synchronization can be observed. We also found that the inhomogeneity in excitable systems is helpful to the synchronization by comparing the different influence on synchronization between homogeneous and inhomogeneous media. The result is useful to understanding the behaviour of some real systems and the behaviour of scroll waves in three-dimensional inhomogeneous media.

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