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Exponential stability of cellular neural networks with multiple time delays and impulsive effects*

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In this work, the stability issues of the equilibrium points of the cellular neural networks with multiple time delays and impulsive effects are investigated. Based on the stability theory of Lyapunov–Krasovskii, the method of linear matrix inequality (LMI) and parametrized first-order model transformation, several novel conditions guaranteeing the delay-dependent and the delay-independent exponential stabilities are obtained. A numerical example is given to illustrate the effectiveness of our results.

Keywords: cellular neural networks (CNNs), multi-delays, exponential stability, linear matrix inequality (LMI)

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1. Introduction

Cellular neural networks (CNNs) were introduced by Chua and Yang[1] and Roska and Chua[2] in 1988 and 1992 respectively. They have found that the CNNs have important potential applications in signal processing, pattern recognition, optimization and associative memories, especially in static image treatment. However, some application requires the signals transmitted among the cells to have a delay in time. The delayed cellular neural networks (DCNNs) were first introduced in Ref.[2] and applied in some motion-related fields, such as speed detection of moving objects, processing of moving images and pattern classification. In order to achieve these applications, one must introduce the time delays in the signal transmission among the cells. It is well known that time delays may cause instability, divergence or oscillation in many systems.[3] In recent years, the stability of DCNNs has become an important topic of theoretical studies. So many sufficient conditions ensuring global asymptotic stability and global exponential stability for CNNs with constant or time-varying delays have been derived. Some of them can be found in Refs.[4–22].

For the cellular neural network mode with impulses, up till now, a few relevant studies can be found.[23,24] Recurrent neural networks are often subjected to impulsive perturbations which can affect dynamical behaviours of the systems, just as time delay. Therefore, it is necessary to consider both impulsive effect and delay effect on the periodic solution and stability of neural networks. To the best of our knowledge, no or few results for the problem of CNNs with multiple time delays and impulse effects have been reported in the literature. In this paper, we attempt to derive several delay-dependent and delay independent exponential stability criteria for CNNs with multiple time delays and impulsive effects, and discuss the conditions of the stability.

The remainder of the present paper is organized as follows. In Section 2, the problems to be investigated in this paper are formulated, and some preliminaries are presented. In Section 3, the exponential stability criteria for the equilibrium points of CNNs with multiple time delays and impulsive effects are derived. A numerical example and the simulations including a comparison analysis of conservatism are presented in Section 4. Finally, some conclusions are drawn in Section 5.

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2. Problem formulation and preliminaries

Let $\|\cdot\|$ denote the Euclidian norm in $\boldsymbol{R}^n$ or $\boldsymbol{R}^{n\times n}$ and $\|\boldsymbol{\phi}\|_\tau = \sup\limits_{-\tau\le\theta\le 0}\|\boldsymbol{\phi}(\theta)\|$. $\boldsymbol{A}^{\mathrm{T}}$ represent the transpose of matrix $\boldsymbol{A}\in\boldsymbol{R}^{n\times n}$, where $\boldsymbol{A}>0$ (<0), i.e. $\boldsymbol{A}$ is a positive definite (negative definite) matrix, and $\boldsymbol{A}\le\boldsymbol{B}$ means that $\boldsymbol{A}-\boldsymbol{B}\le 0$ is negatively semi-definite.

We consider the following cellular neural network model with impulsive effects:

$$\begin{cases}\dot{x}_i(t) = -d_i x_i(t) + \sum\limits_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum\limits_{k=1}^{m}\sum\limits_{j=1}^{n} b_{ij}^{(k)} f_j(x_j(t-\tau_k)),\ t\ne t_k,\\ \Delta x_i(t_k) = I_{ik}x(t_k),\quad t=t_k,\ k=1,2,\ldots,\end{cases} \tag{1}$$

or, in the matrix-vector notation

$$\begin{cases}\dot{\boldsymbol{x}}(t) = -\boldsymbol{D}\boldsymbol{x}(t) + \boldsymbol{A}f(\boldsymbol{x}(t)) + \sum\limits_{k=1}^{m}\boldsymbol{B}_k f(\boldsymbol{x}(t-\tau_k)), t\ne t_k,\\ \Delta\boldsymbol{x}(t_k) = \boldsymbol{I}_k\boldsymbol{x}(t_k),\quad t=t_k,\ k=1,2,\ldots,\end{cases} \tag{2}$$

where n is the number of neurons in the network, $x_i(t)$ is the state of the i-th neuron at time t, $f_j(x_j(t))$are the activation functions of the neurons at time t. d_i and τ_k are nonnegative numbers representing the neuron charging time constants and axonal signal transmission delays, respectively. a_{ij} and $b_{ij}^{(k)}$ denote the weights of the neuron interconnections. $\boldsymbol{D}=\operatorname{diag}(d_i)>0$, $\boldsymbol{A}=(a_{ij})_{n\times n}$, $\boldsymbol{B}_k=\left(b_{ij}^{(k)}\right)_{n\times n}$, and $\boldsymbol{f}(u)=(f_1(u_1),\ldots,f_n(u_n))^{\mathrm{T}}$. $\Delta x_i(t_k)=x_i\left(t_k^+\right)-x_i\left(t_k^-\right)=I_{ik}(x_i(t_k))$ are the impulses at moments t_k, and $t_1<t_2<\ldots$ is a strictly increasing sequence such that $\lim\limits_{k\to\infty}t_k=+\infty$.

The initial conditions associated with Eq.(1) are in the form

$$x_i(t)=\phi_i(t), t\in\left[-\max_{1\le k\le m}\{\tau_k\},0\right], \tag{3}$$

where $\phi_i(t)$ $(i=1,2,\ldots,n)$ are continuous functions.

Many stability results associated with system (2) have been derived recently under the assumption (**H**) that the activation functions satisfy the Liptschitzian continuous condition or the following condition:[4−24] $f_i(i=1,2,\ldots,n)$ are bounded on $\boldsymbol{R}$, and for any $x,y\in\boldsymbol{R}$, $i=1,2,\ldots,n$,

$$0\le\frac{f_i(x)-f_i(y)}{x-y}\le 1.$$

It is easy to see from the assumption (**H**) that the activation functions are continuous on $\boldsymbol{R}$.

Recall the assumption (**H**) and define

$$s_i(t)=\begin{cases}\dfrac{f_i(x_i(t))}{x_i(t)}, & x_i(t)\ne 0,\ i=1,2,\ldots,n,\\ 0, & x_i(t)=0.\end{cases} \tag{4}$$

From the assumption (**H**) and Eq.(4), we have

$$0\le s_i\le 1. \tag{5}$$

So system (2) can be rewritten as follows:

$$\begin{cases}\dot{\boldsymbol{x}}(t) = -\boldsymbol{D}\boldsymbol{x}(t) + \boldsymbol{A}\boldsymbol{S}(t)\boldsymbol{x}(t)\\ \qquad\qquad + \sum\limits_{k=1}^{m}\boldsymbol{B}_k\boldsymbol{S}(t-\tau_k)\boldsymbol{x}(t-\tau_k), t\ne t_k,\\ \Delta\boldsymbol{x}(t_k) = I_k\boldsymbol{x}(t_k),\quad t=t_k,\ k=1,2,\ldots,\end{cases} \tag{6}$$

where $\boldsymbol{S}(t)=\operatorname{diag}(s_i(t))_{n\times n}$.

System (6) can be seen as a linear non-autonomous delay system with impulsive effects.

From Ref.[3], the following equation holds:

$$\begin{aligned}\boldsymbol{x}(t-\tau) =& \boldsymbol{x}(t) - \int_{-\tau}^{0}\dot{\boldsymbol{x}}(t+\xi)\,\mathrm{d}\xi\\ =& \boldsymbol{x}(t) - \int_{-\tau}^{0}\left[A\boldsymbol{x}(t+\xi) + A_d\boldsymbol{x}(t+\xi-\tau)\right]\mathrm{d}\xi.\end{aligned} \tag{7}$$

We use it to transform the following system:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{A}_d\boldsymbol{x}(t-\tau). \tag{8}$$

So, a distributed delay system can be described as follows:

$$\dot{\boldsymbol{x}}(t) = (\boldsymbol{A}+\boldsymbol{C})\boldsymbol{x}(t) + (\boldsymbol{A}_d - \boldsymbol{C})\boldsymbol{x}(t-\tau) - \boldsymbol{C}\int_{-\tau}^{0}[\boldsymbol{A}\boldsymbol{x}(t+\theta) + \boldsymbol{A}_d\boldsymbol{x}(t+\theta-\tau)]\,\mathrm{d}\theta, \tag{9}$$

where $\boldsymbol{C}$ is the parameter matrix. Terminologically, such a process is called a parametrized first-order model transformation.

Obviously, new system (9) may exhibit some additional dynamics because the time delay in system (9) is extended to 2τ. We can obtain the stability conditions of system (8) form the stability analysis of system (9) as shown in Refs.[2, 3].

Applying the above model transformation to systems (6), we have

$$\begin{cases} \dot{\boldsymbol{x}}(t) = \left[-\boldsymbol{D} + \boldsymbol{A}\boldsymbol{S}(t) + \sum\limits_{k=1}^{m}\boldsymbol{C}_k\right]\boldsymbol{x}(t) + \sum\limits_{k=1}^{m}[\boldsymbol{B}_k\boldsymbol{S}(t-\tau_k) - \boldsymbol{C}_k]\boldsymbol{x}(t-\tau_k) \\ \qquad\qquad - \sum\limits_{k=1}^{m}\boldsymbol{C}_k\int_{t-\tau_k}^{t}\left[(-\boldsymbol{D} + \boldsymbol{A}\boldsymbol{S}(\xi))\boldsymbol{x}(\xi) + \sum\limits_{j=1}^{m}\boldsymbol{B}_j\boldsymbol{S}(\xi-\tau_j)\boldsymbol{x}(\xi-\tau_j)\right]\mathrm{d}\xi, \\ \Delta\boldsymbol{x}(t_k) = I_k\boldsymbol{x}(t_k), k = 1, 2, \ldots. \end{cases} \tag{10}$$

As the above analysis, we can obtain the stability conditions of system (2) from the stability analysis of system (10).

Firstly, some definitions and lemmas to be used in this paper are listed as follows.

Definition 1[3] The equilibrium point of system (2) is said to be stable and independent of delays, if it is stable for any $\tau_k \geq 0$ $(k = 1, 2, \ldots, m)$.

Definition 2[3] The equilibrium point of system (2) is said to be delay-dependent stable, if there exists a maximal connected set $\boldsymbol{\Theta} \subset \boldsymbol{R}_+^m$ containing the origin $\boldsymbol{0} \in \boldsymbol{R}^m$ of delay-parameter space such that it is stable for any $(\tau_1, \tau_2, \ldots, \tau_m) \in \boldsymbol{\Theta}$. The maximal connected set $\boldsymbol{\Theta}$ is called the stable region with respect to delay parameter.

Lemma 1[25,26] Given any real matrices $\boldsymbol{\Sigma}_1$, $\boldsymbol{\Sigma}_2$, and $\boldsymbol{\Sigma}_3$ of appropriate dimensions and a scalar $\varepsilon > 0$ such that $0 < \boldsymbol{\Sigma}_3 = \boldsymbol{\Sigma}_3^{\mathrm{T}}$, then the following inequality holds:

$$\boldsymbol{\Sigma}_1^{\mathrm{T}}\boldsymbol{\Sigma}_2 + \boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{\Sigma}_1 \leq \varepsilon\boldsymbol{\Sigma}_1^{\mathrm{T}}\boldsymbol{\Sigma}_3\boldsymbol{\Sigma}_1 + \varepsilon^{-1}\boldsymbol{\Sigma}_2^{\mathrm{T}}\boldsymbol{\Sigma}_3^{-1}\boldsymbol{\Sigma}_2.$$

Lemma 2[25] The following linear matrix inequality (LMI) holds:

$$\begin{bmatrix} \boldsymbol{Q}(\boldsymbol{x}) & \boldsymbol{S}(\boldsymbol{x}) \\ \boldsymbol{S}^{\mathrm{T}}(\boldsymbol{x}) & \boldsymbol{R}(\boldsymbol{x}) \end{bmatrix} > 0$$

where $\boldsymbol{Q}(\boldsymbol{x}) = \boldsymbol{Q}^{\mathrm{T}}(\boldsymbol{x})$, $\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{R}^{\mathrm{T}}(\boldsymbol{x})$, and $\boldsymbol{S}(\boldsymbol{x})$ depend on $\boldsymbol{x}$ and are equivalent to

$$\boldsymbol{R}(\boldsymbol{x}) > 0, \quad \text{and} \quad \boldsymbol{Q}(\boldsymbol{x}) - \boldsymbol{S}(\boldsymbol{x})\boldsymbol{R}^{-1}(\boldsymbol{x})\boldsymbol{S}^{\mathrm{T}}(\boldsymbol{x}) > 0.$$

3. The exponential stability criteria for CNNs with multiple time delays and impulsive effects

Theorem 3.1 Suppose that the assumption (**H**) is satisfied. Then, system (1) is globally exponentially stable for any delay $\tau \in [0, \bar{\tau}]$, if there exist a positive constant $\varepsilon > 0$, symmetrical and positive definite matrices $\boldsymbol{P}$, $\boldsymbol{Q}_{i0}$, $\boldsymbol{T}_i$, positive diagonal matrices $\boldsymbol{Q}_0$, $\boldsymbol{Q}_i$, $\boldsymbol{R}_i$, $\boldsymbol{Q}_{ij}$ and constant matrix $\boldsymbol{W}_i \in \boldsymbol{R}^{n\times n}$ $(i, j = 1, 2, \ldots, m)$, with $(\mathbf{H}_1)$ $I_k\boldsymbol{P}I_k - I_k\boldsymbol{P} - \boldsymbol{P}I_k < 0$, $k = 1, 2, \ldots$, satisfied and one of the following LMIs holding:

(i) $\boldsymbol{\Omega}_2 = \boldsymbol{\chi}_2 + \boldsymbol{\Gamma}_2 + \sum\limits_{i=1}^{m}\mathrm{e}^{2\varepsilon\bar{\tau}_i}\boldsymbol{W}_i\boldsymbol{Q}_{i0}^{-1}\boldsymbol{W}_i^{\mathrm{T}} < \boldsymbol{0}$,

(ii) $\boldsymbol{\Omega}_2^* = \boldsymbol{\chi}_2^* + \boldsymbol{\Gamma}_2 + \sum\limits_{i=1}^{m}\mathrm{e}^{2\varepsilon\bar{\tau}_i}(\boldsymbol{P}\boldsymbol{B}_i - \boldsymbol{W}_i)\boldsymbol{Q}_{i0}^{-1}(\boldsymbol{P}\boldsymbol{B}_i - \boldsymbol{W}_i)^{\mathrm{T}} < 0$,

where

$$\begin{aligned}\boldsymbol{\chi}_2 =& -\boldsymbol{P}\boldsymbol{D}-\boldsymbol{D}\boldsymbol{P}+2\varepsilon\boldsymbol{P}+\boldsymbol{Q}_0\\&+\sum_{i=1}^{m}\left(\boldsymbol{W}_i+\boldsymbol{W}_i^{\mathrm{T}}+\boldsymbol{Q}_i+\boldsymbol{Q}_{i0}+\bar{\tau}_i\boldsymbol{T}_i+\bar{\tau}_i\boldsymbol{R}_i\right)\\&+\sum_{i,j=1}^{m}\bar{\tau}_i\boldsymbol{Q}_{ij},\\ \boldsymbol{\chi}_2^{*} =&\boldsymbol{\chi}_2+\boldsymbol{P}\boldsymbol{A}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P},\end{aligned}$$

$$\begin{aligned}\boldsymbol{\Gamma}_2 =&\boldsymbol{P}\boldsymbol{A}\boldsymbol{Q}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\sum_{i=1}^{m}\mathrm{e}^{2\varepsilon\,\bar{\tau}_i}\boldsymbol{P}\boldsymbol{B}_i\boldsymbol{Q}_i^{-1}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}\\&+\frac{1}{2\varepsilon}\sum_{i,j=1}^{m}\left(\mathrm{e}^{2\varepsilon\bar{\tau}_i}-1\right)\boldsymbol{W}_i\boldsymbol{B}_j\boldsymbol{Q}_{ij}^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{W}_i^{\mathrm{T}}\\&+\frac{1}{2\varepsilon}\sum_{i=1}^{m}\left(\mathrm{e}^{2\varepsilon\bar{\tau}_i}-1\right)\\&\times\left(\boldsymbol{W}_i\boldsymbol{D}\boldsymbol{T}_i^{-1}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}_i^{\mathrm{T}}+\boldsymbol{W}_i\boldsymbol{A}\boldsymbol{R}_i^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}_i^{\mathrm{T}}\right).\end{aligned}$$

The corresponding parametrized model transformation matrices are given by $\boldsymbol{C}_i = P^{-1}\boldsymbol{W}_i$ $(i = 1,\ 2,\ \ldots,\ m)$.

Proof Considering the following Lyapunov–Krasovskii functional candidate:

$$V\left(\boldsymbol{x}\left(t\right)\right)=V_1\left(\boldsymbol{x}\left(t\right)\right)+V_2\left(\boldsymbol{x}\left(t\right)\right)+V_3\left(\boldsymbol{x}\left(t\right)\right),$$

where

$$\begin{aligned}V_1\left(\boldsymbol{x}\left(t\right)\right)=&\mathrm{e}^{2\varepsilon}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\boldsymbol{P}\boldsymbol{x}\left(t\right)+\sum_{i=1}^{m}\int_{t-\tau_i}^{t}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}\left(\xi\right)\left(\boldsymbol{Q}_i+\boldsymbol{Q}_{i0}\right)\boldsymbol{x}\left(\xi\right)\mathrm{d}\xi,\\V_2\left(\boldsymbol{x}\left(t\right)\right)=&\sum_{i=1}^{m}\int_{-\tau_i}^{0}\left(\int_{t+\theta}^{t}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}\left(\xi\right)\left(\boldsymbol{T}_i+\boldsymbol{R}_i\right)\boldsymbol{x}\left(\xi\right)\mathrm{d}\xi\right)\mathrm{d}\theta\\V_3\left(\boldsymbol{x}\left(t\right)\right)=&\sum_{i,j=1}^{m}\int_{-\tau_i-\tau_j}^{-\tau_j}\left(\int_{t+\theta}^{t}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}\left(\xi\right)\boldsymbol{Q}_{ij}\boldsymbol{x}\left(\xi\right)\mathrm{d}\xi\right)\mathrm{d}\theta.\end{aligned}$$

When $t \neq t_k$, the time derivative of Lyapunov–Krasovskii functional $V\left(\boldsymbol{x}\left(t\right)\right)$ along the trajectories of system (10) is derived as follows:

$$\dot{V}\left(\boldsymbol{x}\left(t\right)\right)=\dot{V}_1\left(\boldsymbol{x}\left(t\right)\right)+\dot{V}_2\left(\boldsymbol{x}\left(t\right)\right)+\dot{V}_3\left(\boldsymbol{x}\left(t\right)\right),\tag{11}$$

where

$$\begin{aligned}\dot{V}_2\left(\boldsymbol{x}\left(t\right)\right)=&\sum_{i=1}^{m}\left[\int_{-\tau_i}^{0}\mathrm{e}^{2\varepsilon t}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\left(\boldsymbol{T}_i+\boldsymbol{R}_i\right)\boldsymbol{x}\left(t\right)\mathrm{d}\xi-\int_{t-\tau_i}^{t}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}\left(\xi\right)\left(\boldsymbol{T}_i+\boldsymbol{R}_i\right)\boldsymbol{x}\left(\xi\right)\mathrm{d}\xi\right]\\=&\sum_{i=1}^{m}\left[\tau_i\mathrm{e}^{2\varepsilon t}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\left(\boldsymbol{T}_i+\boldsymbol{R}_i\right)\boldsymbol{x}\left(t\right)-\int_{t-\tau_i}^{t}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}\left(\xi\right)\left(\boldsymbol{T}_i+\boldsymbol{R}_i\right)\boldsymbol{x}\left(\xi\right)\mathrm{d}\xi\right],\end{aligned}\tag{12}$$

$$\dot{V}_3\left(\boldsymbol{x}\left(t\right)\right)=\sum_{i,j=1}^{m}\left[\tau_i\mathrm{e}^{2\varepsilon t}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\boldsymbol{Q}_{ij}\boldsymbol{x}\left(t\right)-\int_{t-\tau_i}^{t}\mathrm{e}^{2\varepsilon(\xi-\tau_j)}\boldsymbol{x}^{\mathrm{T}}\left(\xi-\tau_j\right)\boldsymbol{Q}_{ij}\boldsymbol{x}\left(\xi-\tau_j\right)\mathrm{d}\xi\right],\tag{13}$$

$$\begin{aligned}\dot{V}_1\left(\boldsymbol{x}\left(t\right)\right)=&2\varepsilon\,\mathrm{e}^{2\varepsilon t}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\boldsymbol{P}\boldsymbol{x}\left(t\right)+2\mathrm{e}^{2\varepsilon t}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\boldsymbol{P}\dot{\boldsymbol{x}}\left(t\right)\\&+\mathrm{e}^{2\varepsilon t}\sum_{i=1}^{m}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\left(\boldsymbol{Q}_i+\boldsymbol{Q}_{i0}\right)\boldsymbol{x}\left(t\right)-\sum_{i=1}^{m}\mathrm{e}^{2\varepsilon(t-\tau_i)}\boldsymbol{x}^{\mathrm{T}}\left(t-\tau_i\right)\left(\boldsymbol{Q}_i+\boldsymbol{Q}_{i0}\right)\boldsymbol{x}\left(t-\tau_i\right)\\=&2\varepsilon\,\mathrm{e}^{2\varepsilon t}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\boldsymbol{P}\boldsymbol{x}\left(t\right)+2\mathrm{e}^{2\varepsilon t}\boldsymbol{x}^{\mathrm{T}}\left(t\right)P\Bigg\{\left(-\boldsymbol{D}+\boldsymbol{A}\boldsymbol{S}\left(t\right)+\sum_{i=1}^{m}\boldsymbol{C}_i\right)\boldsymbol{x}\left(t\right)\\&+\left(\sum_{i=1}^{m}\left[\boldsymbol{B}_i\boldsymbol{S}\left(t-\tau_i\right)-\boldsymbol{C}_i\right]\right)\boldsymbol{x}\left(t-\tau_i\right)\\&-\sum_{i=1}^{m}\boldsymbol{C}_i\int_{t-\tau_i}^{t}\left[\left(-\boldsymbol{D}+\boldsymbol{A}\boldsymbol{S}\left(\xi\right)\right)\boldsymbol{x}\left(\xi\right)+\sum_{j=1}^{m}\boldsymbol{B}_j\boldsymbol{S}\left(\xi-\tau_j\right)\boldsymbol{x}\left(\xi-\tau_j\right)\right]\mathrm{d}\xi\Bigg\}\\&+\mathrm{e}^{2\varepsilon t}\sum_{i=1}^{m}\boldsymbol{x}^{\mathrm{T}}\left(t\right)\left(\boldsymbol{Q}_i+\boldsymbol{Q}_{i0}\right)\boldsymbol{x}\left(t\right)-\sum_{i=1}^{m}\mathrm{e}^{2\varepsilon(t-\tau_i)}\boldsymbol{x}^{\mathrm{T}}\left(t-\tau_i\right)\left(\boldsymbol{Q}_i+\boldsymbol{Q}_{i0}\right)\boldsymbol{x}\left(t-\tau_i\right).\end{aligned}\tag{14}$$

From Lemma 1, the following inequalities hold:

$$\begin{aligned}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PAS}(t)\boldsymbol{x}(t)\leq&\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PAQ}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Px}(t)+\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{S}(t)\boldsymbol{Q}_0\boldsymbol{S}(t)\boldsymbol{x}(t)\\ \leq&\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PAQ}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Px}(t)+\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{Q}_0\boldsymbol{x}(t).\end{aligned}\tag{15}$$

Moreover,

$$\begin{aligned}&\sum_{i=1}^{m}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}(\boldsymbol{B}_i\boldsymbol{S}(t-\tau_i)-\boldsymbol{C}_i)\boldsymbol{x}(t-\tau_i)\\ &=\sum_{i=1}^{m}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PB}_i\boldsymbol{S}(t-\tau_i)\boldsymbol{x}(t-\tau_i)-\sum_{i=1}^{m}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_i\boldsymbol{x}(t-\tau_i)\\ \leq&\sum_{i=1}^{m}\left[\mathrm{e}^{2\varepsilon\tau_i}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PB}_i\boldsymbol{Q}_i^{-1}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{Px}(t)+\mathrm{e}^{-2\varepsilon\tau_i}\boldsymbol{x}^{\mathrm{T}}(t-\tau_i)\boldsymbol{Q}_i\boldsymbol{x}(t-\tau_i)\right]\\ &+\sum_{i=1}^{m}\mathrm{e}^{2\varepsilon\tau_i}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_i\boldsymbol{Q}_{i0}^{-1}\boldsymbol{W}_i^{\mathrm{T}}\boldsymbol{x}(t)+\mathrm{e}^{-2\varepsilon\tau_i}\sum_{i=1}^{m}\boldsymbol{x}^{\mathrm{T}}(t-\tau_i)\boldsymbol{Q}_{i0}\boldsymbol{x}(t-\tau_i),\end{aligned}\tag{16}$$

$$\begin{aligned}&\sum_{i=1}^{m}\int_{t-\tau_i}^{t}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PC}_i\boldsymbol{D}\boldsymbol{x}(\xi)\,\mathrm{d}\xi\\ \leq&\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_i\boldsymbol{DT}_i^{-1}\boldsymbol{DW}_i^{\mathrm{T}}\boldsymbol{x}(t)\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\mathrm{e}^{2\varepsilon(t-\xi)}\mathrm{d}\xi+\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\mathrm{e}^{2\varepsilon(\xi-t)}\boldsymbol{x}^{\mathrm{T}}(\xi)\boldsymbol{T}_i\boldsymbol{x}(\xi)\,\mathrm{d}\xi\\ =&\sum_{i=1}^{m}\frac{\mathrm{e}^{2\varepsilon\bar{\tau}_i}-1}{2\varepsilon}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_i\boldsymbol{DT}_i^{-1}\boldsymbol{DW}_i^{\mathrm{T}}\boldsymbol{x}(t)+\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\mathrm{e}^{2\varepsilon(\xi-t)}\boldsymbol{x}^{\mathrm{T}}(\xi)\boldsymbol{T}_i\boldsymbol{x}(\xi)\,\mathrm{d}\xi,\end{aligned}\tag{17}$$

$$\begin{aligned}&-\sum_{i=1}^{m}\int_{t-\tau_i}^{t}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PC}_i\boldsymbol{AS}(\xi)\boldsymbol{x}(\xi)\,\mathrm{d}\xi\\ \leq&\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_i\boldsymbol{AR}_i^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}\boldsymbol{x}(t)\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\mathrm{e}^{2\varepsilon(t-\xi)}\mathrm{d}\xi+\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\mathrm{e}^{-2\varepsilon(t-\xi)}\boldsymbol{x}^{\mathrm{T}}(\xi)\boldsymbol{R}_i\boldsymbol{x}(\xi)\,\mathrm{d}\xi\\ =&\sum_{i=1}^{m}\frac{\mathrm{e}^{2\varepsilon\bar{\tau}_i}-1}{2\varepsilon}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_i\boldsymbol{AR}_i^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}\boldsymbol{x}(t)+\sum_{i=1}^{m}\int_{t-\bar{\tau}_i}^{t}\mathrm{e}^{-2\varepsilon(t-\xi)}\boldsymbol{x}^{\mathrm{T}}(\xi)\boldsymbol{R}_i\boldsymbol{x}(\xi)\,\mathrm{d}\xi,\end{aligned}\tag{18}$$

$$\begin{aligned}&-\sum_{i=1}^{m}\sum_{j=1}^{m}\int_{t-\tau_i}^{t}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PC}_i\boldsymbol{B}_j\boldsymbol{S}(\xi-\tau_j)\boldsymbol{x}(\xi-\tau_j)\,\mathrm{d}\xi\\ \leq&\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_i\boldsymbol{B}_j\boldsymbol{Q}_{ij}^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{W}_i^{\mathrm{T}}\boldsymbol{x}(t)\int_{t-\bar{\tau}_i}^{t}\mathrm{e}^{2\varepsilon(t-\xi+\bar{\tau}_i)}\mathrm{d}\xi\right.\\ &\left.+\int_{t-\tau_i}^{t}\mathrm{e}^{-2\varepsilon(t-\xi+\bar{\tau}_i)}\,\boldsymbol{x}^{\mathrm{T}}(\xi-\tau_j)\boldsymbol{Q}_{ij}\,\boldsymbol{x}(\xi-\tau_j)\,\mathrm{d}\xi\right]\\ =&\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{W}_i\boldsymbol{B}_j\boldsymbol{Q}_{ij}^{-1}\boldsymbol{B}_j^{\mathrm{T}}\boldsymbol{W}_i^{\mathrm{T}}\boldsymbol{x}(t)\frac{\mathrm{e}^{2\varepsilon\bar{\tau}_i}\left(\mathrm{e}^{2\varepsilon\bar{\tau}_i}-1\right)}{2\varepsilon}\right.\\ &\left.+\int_{t-\tau_i}^{t}\mathrm{e}^{-2\varepsilon(t-\xi+\bar{\tau}_i)}\,\boldsymbol{x}^{\mathrm{T}}(\xi-\tau_j)\boldsymbol{Q}_{ij}\,\boldsymbol{x}(\xi-\tau_j)\,\mathrm{d}\xi\right].\end{aligned}\tag{19}$$

From inequalities (12–19), we have

$$\dot{V}(\boldsymbol{x}(t))\leq\mathrm{e}^{2\varepsilon t}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Omega}_2\boldsymbol{x}(t).\tag{20}$$

On the other hand, inequality (15) is replaced by the following inequality:

$$\begin{aligned}2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PAS}(t)\boldsymbol{x}(t)=&2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}[\boldsymbol{A}+\boldsymbol{A}(\boldsymbol{S}(t)-\boldsymbol{I})]\boldsymbol{x}(t)\\ \leq&2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PAx}(t)+\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{PAR}_1^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Px}(t)+\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{R}_1\boldsymbol{x}(t).\end{aligned}\tag{21}$$

Similarly, inequality (16) can be rewritten as

$$\begin{aligned}
&\sum_{i=1}^{m} 2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\left(\boldsymbol{B}_i+\boldsymbol{B}_i\left(\boldsymbol{S}\left(t-\tau_i\right)-\boldsymbol{E}_0\right)-\boldsymbol{C}_i\right)\boldsymbol{x}\left(t-\tau_i\right)\\
=&\sum_{i=1}^{m} 2\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{B}_i\left(\boldsymbol{S}\left(t-\tau_i\right)-\boldsymbol{E}_0\right)\boldsymbol{x}\left(t-\tau_i\right)-\sum_{i=1}^{m} 2\boldsymbol{x}^{\mathrm{T}}(t)\left(\boldsymbol{W}_i-\boldsymbol{P}\boldsymbol{B}_i\right)\boldsymbol{x}\left(t-\tau_i\right)\\
\leq&\sum_{i=1}^{m}\left[\mathrm{e}^{2\varepsilon\tau_i}\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{B}_i\boldsymbol{Q}_i^{-1}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P}\boldsymbol{x}(t)+\mathrm{e}^{-2\varepsilon\tau_i}\boldsymbol{x}^{\mathrm{T}}\left(t-\tau_i\right)\boldsymbol{Q}_i\boldsymbol{x}\left(t-\tau_i\right)\right]\\
&+\sum_{i=1}^{m}\mathrm{e}^{2\varepsilon\tau_i}\boldsymbol{x}^{\mathrm{T}}(t)\left(\boldsymbol{W}_i-\boldsymbol{P}\boldsymbol{B}_i\right)_i\boldsymbol{Q}_{i0}^{-1}\left(\boldsymbol{W}_i-\boldsymbol{P}\boldsymbol{B}_i\right)^{\mathrm{T}}\boldsymbol{x}(t)+\mathrm{e}^{-2\varepsilon\tau_i}\sum_{i=1}^{m}\boldsymbol{x}^{\mathrm{T}}\left(t-\tau_i\right)\boldsymbol{Q}_{i0}\boldsymbol{x}\left(t-\tau_i\right). \qquad (22)
\end{aligned}$$

From Eqs.(11)–(14) and inequalities (16)–(18), (21), and (22), we have

$$\dot{V}\left(\boldsymbol{x}(t)\right)\leq\boldsymbol{x}^{\mathrm{T}}(t)\boldsymbol{\Omega}_2^{*}\boldsymbol{x}(t). \qquad (23)$$

From inequality (20) or (23) it follows that $\dot{V}\left(\boldsymbol{x}(t)\right)\leq 0$. And $\dot{V}\left(\boldsymbol{x}(t)\right)=0$ if and only if $\boldsymbol{x}(t)=0$.
When $t=t_k$, we have

$$\begin{aligned}
V\left(\boldsymbol{x}\left(t_k^{+}\right)\right)=&\mathrm{e}^{2\varepsilon t_k^{+}}\boldsymbol{x}^{\mathrm{T}}\left(t_k\right)\left(\boldsymbol{E}_0-\boldsymbol{I}_k\right)^{\mathrm{T}}\boldsymbol{P}\left(\boldsymbol{E}_0-\boldsymbol{I}_k\right)\boldsymbol{x}\left(t_k\right)\\
&+\sum_{i=1}^{m}\int_{t_k^{+}-\tau_i}^{t_k^{+}}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}(\xi)\left(\boldsymbol{Q}_i+\boldsymbol{Q}_{i0}\right)\boldsymbol{x}(\xi)\,\mathrm{d}\xi\\
&+\sum_{i=1}^{m}\int_{-\tau_i}^{0}\left(\int_{t_k^{+}+\theta}^{t_k^{+}}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}(\xi)\left(\boldsymbol{T}_i+\boldsymbol{R}_i\right)\boldsymbol{x}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\theta\\
&+\sum_{i,j=1}^{m}\int_{-\tau_i-\tau_j}^{-\tau_j}\left(\int_{t_k^{+}+\theta}^{t_k^{+}}\mathrm{e}^{2\varepsilon\xi}\ \boldsymbol{x}^{\mathrm{T}}(\xi)\boldsymbol{Q}_{ij}\boldsymbol{x}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\theta\\
=&\mathrm{e}^{2\varepsilon t_k}\boldsymbol{x}^{\mathrm{T}}\left(t_k\right)\left(\boldsymbol{I}_k^2-\boldsymbol{I}_k\boldsymbol{P}-\boldsymbol{P}\boldsymbol{I}_k\right)\boldsymbol{x}\left(t_k\right)+\mathrm{e}^{2\varepsilon t_k}\boldsymbol{x}^{\mathrm{T}}\left(t_k\right)\boldsymbol{P}\boldsymbol{x}\left(t_k\right)\\
&+\sum_{i=1}^{m}\int_{t_k-\tau_i}^{t_k}\mathrm{e}^{2\varepsilon\xi}\ \boldsymbol{x}^{\mathrm{T}}(\xi)\left(\boldsymbol{Q}_i+\boldsymbol{Q}_{i0}\right)\boldsymbol{x}(\xi)\,\mathrm{d}\xi\\
&+\sum_{i=1}^{m}\int_{-\tau_i}^{0}\left(\int_{t_k+\theta}^{t_k}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}(\xi)\left(\boldsymbol{T}_i+\boldsymbol{R}_i\right)\boldsymbol{x}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\theta\\
&+\sum_{i,j=1}^{m}\int_{-\tau_i-\tau_j}^{-\tau_j}\left(\int_{t_k+\theta}^{t_k}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}(\xi)\boldsymbol{Q}_{ij}\boldsymbol{x}(\xi)\,\mathrm{d}\xi\right)\mathrm{d}\theta\\
=&V\left(\boldsymbol{x}\left(t_k\right)\right)+\mathrm{e}^{2\varepsilon t_k}\boldsymbol{x}^{\mathrm{T}}\left(t_k\right)\left(\boldsymbol{I}_k^2-\boldsymbol{I}_k\boldsymbol{P}-\boldsymbol{P}\boldsymbol{I}_k\right)\boldsymbol{x}\left(t_k\right)\leq V\left(\boldsymbol{x}\left(t_k\right)\right). \qquad (24)
\end{aligned}$$

Hence, we have $\mathrm{e}^{2\varepsilon t}\lambda_m(P)\left\|\boldsymbol{x}(t)\right\|^2\leq V\left(\boldsymbol{x}(t)\right)\leq V\left(\boldsymbol{x}(0)\right)$.
On the other hand

$$\begin{aligned}
V\left(\boldsymbol{x}(0)\right)=&\boldsymbol{x}^{\mathrm{T}}(0)\boldsymbol{P}\boldsymbol{x}(0)+\int_{-\tau}^{0}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}(\xi)\left(\boldsymbol{R}_2+\boldsymbol{R}_3\right)\boldsymbol{x}(\xi)\,\mathrm{d}\xi\\
&+\int_{-\bar{\tau}}^{0}\left[\int_{\theta}^{0}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}(\xi)\left(\boldsymbol{R}_4+\boldsymbol{R}_5\right)\boldsymbol{x}(\xi)\,\mathrm{d}\xi\right]\mathrm{d}\theta+\int_{-2\bar{\tau}}^{-\bar{\tau}}\left[\int_{\theta}^{0}\mathrm{e}^{2\varepsilon\xi}\boldsymbol{x}^{\mathrm{T}}(\xi)\boldsymbol{R}_6\boldsymbol{x}(\xi)\,\mathrm{d}\xi\right]\mathrm{d}\theta\\
\leq&\lambda_M(\boldsymbol{P})\left\|\boldsymbol{\varphi}\right\|_{\bar{\tau}}^2+\lambda_M\left(\boldsymbol{R}_2+\boldsymbol{R}_3\right)\left\|\boldsymbol{\varphi}\right\|_{\bar{\tau}}^2\int_{-\bar{\tau}}^{0}\mathrm{e}^{2\varepsilon\xi}\mathrm{d}\xi\\
&+\lambda_M\left(\boldsymbol{R}_4+\boldsymbol{R}_5\right)\left\|\boldsymbol{\varphi}\right\|_{\bar{\tau}}^2\int_{-\bar{\tau}}^{0}\int_{\theta}^{0}\mathrm{e}^{2\varepsilon\xi}\mathrm{d}\xi\mathrm{d}\theta+\lambda_M\left(\boldsymbol{R}_6\right)\left\|\boldsymbol{\varphi}\right\|_{2\bar{\tau}}^2\int_{-2\bar{\tau}}^{-\bar{\tau}}\int_{\theta}^{0}\mathrm{e}^{2\varepsilon\xi}\mathrm{d}\xi\mathrm{d}\theta\\
\leq&\left[\lambda_M(\boldsymbol{P})+\lambda_M\left(\boldsymbol{R}_2+\boldsymbol{R}_3\right)\frac{1-\mathrm{e}^{-2\varepsilon\bar{\tau}}}{2\varepsilon}+\lambda_M\left(\boldsymbol{R}_4+\boldsymbol{R}_5\right)\frac{2\varepsilon\bar{\tau}-1+\mathrm{e}^{-2\varepsilon\bar{\tau}}}{4\varepsilon^2}\right.\\
&\left.+\lambda_M\left(\boldsymbol{R}_6\right)\frac{2\varepsilon\bar{\tau}-\mathrm{e}^{-2\varepsilon\bar{\tau}}+\mathrm{e}^{-4\varepsilon\bar{\tau}}}{2\varepsilon}\right]\left\|\boldsymbol{\varphi}\right\|_{2\bar{\tau}}^2. \qquad (25)
\end{aligned}$$

Hence, we obtain

$$\|\boldsymbol{x}(t)\| \leq \frac{1}{\sqrt{\lambda_m(\boldsymbol{P})}}\left[\lambda_M(\boldsymbol{P}) + \lambda_M(\boldsymbol{R}_2+\boldsymbol{R}_3)\frac{1-\mathrm{e}^{-2\varepsilon\bar{\tau}}}{2\varepsilon} + \lambda_M(\boldsymbol{R}_4+\boldsymbol{R}_5)\frac{2\varepsilon\bar{\tau}-1+\mathrm{e}^{-2\varepsilon\bar{\tau}}}{4\varepsilon^2} + \lambda_M(\boldsymbol{R}_6)\frac{2\varepsilon\bar{\tau}-\mathrm{e}^{-2\varepsilon\bar{\tau}}+\mathrm{e}^{-4\varepsilon\bar{\tau}}}{2\varepsilon}\right]^{\frac{1}{2}} \|\boldsymbol{\varphi}\|_{2\bar{\tau}}\,\mathrm{e}^{-\varepsilon t}. \tag{26}$$

It implies that the CNNs with multiple time delays and impulsive effects are exponentially stable. The proof is completed.

If we do not apply the parametrized first-order model transformation to the system, i.e. $\boldsymbol{W}_i = 0$, then the following Corollary 3.1 holds.

Corollary 3.1 Suppose that the assumption (**H**) is satisfied. Then, system (2) is globally exponentially stable, if there exist a positive constant $\varepsilon > 0$, symmetrical and positive definite matrix $\boldsymbol{P}$, positive diagonal matrices $\boldsymbol{Q}_j$ $(j = 0,\ 1,\ 2,\ \ldots,\ m)$, with $(\mathbf{H}_1)$ $\boldsymbol{I}_k\boldsymbol{P}\boldsymbol{I}_k - \boldsymbol{I}_k\boldsymbol{P} - \boldsymbol{P}\boldsymbol{I}_k < 0$, $k = 1, 2, \ldots$, satisfied and one of the following LMIs holding:

(i) $\boldsymbol{\Omega}_{21} = -\boldsymbol{P}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{P} + 2\varepsilon\boldsymbol{P} + \sum\limits_{j=0}^{m}\boldsymbol{Q}_j + \boldsymbol{P}\boldsymbol{A}\boldsymbol{Q}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \sum\limits_{i=1}^{m}\mathrm{e}^{2\varepsilon\bar{\tau}_i}\boldsymbol{P}\boldsymbol{B}_i\boldsymbol{Q}_i^{-1}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P} < 0,$

Or,

$$\begin{bmatrix} -\boldsymbol{P}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{P} + 2\varepsilon\boldsymbol{P} + \sum\limits_{j=0}^{m}\boldsymbol{Q}_j & -\boldsymbol{P}\boldsymbol{A} & -\mathrm{e}^{\varepsilon\bar{\tau}_1}\boldsymbol{P}\boldsymbol{B}_1 & \ldots & -\mathrm{e}^{\varepsilon\bar{\tau}_m}\boldsymbol{P}\boldsymbol{B}_m \\ -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} & -\boldsymbol{Q}_0 & 0 & \ldots & 0 \\ -\mathrm{e}^{\varepsilon\bar{\tau}_1}\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P} & 0 & -\boldsymbol{Q}_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -\mathrm{e}^{\varepsilon\bar{\tau}_m}\boldsymbol{B}_m^{\mathrm{T}}\boldsymbol{P} & 0 & 0 & \ldots & -\boldsymbol{Q}_m \end{bmatrix} < 0,$$

(ii) $\boldsymbol{\Omega}_{21}^{*} = -\boldsymbol{P}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{P} + 2\varepsilon\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \sum\limits_{j=0}^{m}\boldsymbol{Q}_j + \boldsymbol{P}\boldsymbol{A}\boldsymbol{Q}_0^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \sum\limits_{i=1}^{m}\mathrm{e}^{2\varepsilon\bar{\tau}_i}\boldsymbol{P}\boldsymbol{B}_i\boldsymbol{Q}_i^{-1}\boldsymbol{B}_i^{\mathrm{T}}\boldsymbol{P} < 0,$

Or,

$$\begin{bmatrix} -\boldsymbol{P}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{P} + 2\varepsilon\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} + \sum\limits_{j=0}^{m}\boldsymbol{Q}_j & -\boldsymbol{P}\boldsymbol{A} & -\mathrm{e}^{\varepsilon\bar{\tau}_1}\boldsymbol{P}\boldsymbol{B}_1 & \ldots & -\mathrm{e}^{\varepsilon\bar{\tau}_m}\boldsymbol{P}\boldsymbol{B}_m \\ -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} & -\boldsymbol{Q}_0 & 0 & \ldots & 0 \\ -\mathrm{e}^{\varepsilon\bar{\tau}_1}\boldsymbol{B}_1^{\mathrm{T}}\boldsymbol{P} & 0 & -\boldsymbol{Q}_1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ -\mathrm{e}^{\varepsilon\bar{\tau}_m}\boldsymbol{B}_m^{\mathrm{T}}\boldsymbol{P} & 0 & 0 & \ldots & -\boldsymbol{Q}_m \end{bmatrix} < 0.$$

Remark 3.1 If we have the known delays, from Corollary 3.1, we can estimate the degree of exponential convergence of the considered system by solving the following optimization problem:

$$(P_1)\quad \begin{cases} \max \varepsilon, \\ \text{s.t.}\quad (\mathbf{H}_1) \quad \text{and} \quad \boldsymbol{\Omega}_{21} < 0, \ \text{or} \ \boldsymbol{\Omega}_{21}^{*} < 0. \end{cases}$$

Considering the system with a single delay, we obtain the following Corollary 3.2.

Corollary 3.2 Suppose that the assumption (**H**) is satisfied. Then, for any delay $\tau \in [0,\ \bar{\tau}]$, system (2) with a single delay is globally exponentially stable, if there exist a positive constant $\varepsilon > 0$, symmetrical and positive definite matrix $\boldsymbol{P}$, $\boldsymbol{R}_3$, $\boldsymbol{R}_4$, positive diagonal matrices $\boldsymbol{R}_1$, $\boldsymbol{R}_2$, $\boldsymbol{R}_5$, $\boldsymbol{R}_6$, and constant matrix $\boldsymbol{W} \in \boldsymbol{R}^{n\times n}$, with $(\mathbf{H}_1)$ $\boldsymbol{I}_k\boldsymbol{P}\boldsymbol{I}_k - \boldsymbol{I}_k\boldsymbol{P} - \boldsymbol{P}\boldsymbol{I}_k < 0$, $k = 1, 2, \ldots$, satisfied and one of the following LMIs holding:

(i) $\boldsymbol{\Omega}_{23} = \boldsymbol{\chi}_2 + \boldsymbol{\Gamma}_2 + \mathrm{e}^{2\varepsilon\tau}\boldsymbol{W}\boldsymbol{R}_3^{-1}\boldsymbol{W}^{\mathrm{T}} < 0,$

(ii) $\boldsymbol{\Omega}_{23}^{*} = \boldsymbol{\chi}_2^{*} + \boldsymbol{\Gamma}_2 + \mathrm{e}^{2\varepsilon\tau}(\boldsymbol{P}\boldsymbol{B} - \boldsymbol{W})\boldsymbol{R}_3^{-1}(\boldsymbol{P}\boldsymbol{B} - \boldsymbol{W})^{\mathrm{T}} < 0,$

where

$$\begin{aligned} \boldsymbol{\chi}_2^{*} &= \boldsymbol{\chi}_2 + \boldsymbol{P}\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}, \\ \boldsymbol{\chi}_2 &= -\boldsymbol{P}\boldsymbol{D} - \boldsymbol{D}\boldsymbol{P} + 2\varepsilon\boldsymbol{P} + \boldsymbol{W} + \boldsymbol{W}^{\mathrm{T}} \\ &\quad + \boldsymbol{R}_1 + \boldsymbol{R}_2 + \boldsymbol{R}_3 + \bar{\tau}(\boldsymbol{R}_4 + \boldsymbol{R}_5 + \boldsymbol{R}_6), \end{aligned}$$

$$\begin{aligned}\boldsymbol{\Gamma}_2 =&\boldsymbol{PAR}_1^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\mathrm{e}^{2\varepsilon\tau}\boldsymbol{PBR}_2^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}\\&+\frac{\mathrm{e}^{2\varepsilon\tau}-1}{2\varepsilon}\left(\boldsymbol{WDR}_4^{-1}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}+\boldsymbol{WAR}_5^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}\right.\\&\left.+\mathrm{e}^{2\varepsilon\tau}\boldsymbol{WBR}_6^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{W}^{\mathrm{T}}\right).\end{aligned}$$

The corresponding parametrized model transformation matrix is given by $\boldsymbol{C}=\boldsymbol{P}^{-1}\boldsymbol{W}$.

Based on the Corollary 3.2, if $\boldsymbol{W}=0$, the following Corollary 3.3 holds.

Corollary 3.3 Suppose that the assumption (H) is satisfied. Then, for any delay $\tau\in[0,\ \bar{\tau}]$, system (2) with a single delay is globally exponentially stable, if there exist a positive constant $\varepsilon>0$, symmetrical and positive definite matrix $\boldsymbol{P}$, positive diagonal matrices $\boldsymbol{R}_1,\ \boldsymbol{R}_2$, with $(\mathbf{H}_1)$ $\boldsymbol{I}_k\boldsymbol{P}\boldsymbol{I}_k-\boldsymbol{I}_k\boldsymbol{P}-\boldsymbol{P}\boldsymbol{I}_k<0$, $k=1,2,\ldots$, satisfied and one of the following LMIs holding:

(i) $\boldsymbol{\Omega}_{24}=-\boldsymbol{PD}-\boldsymbol{DP}+2\varepsilon\boldsymbol{P}+\boldsymbol{R}_1+\boldsymbol{R}_2+\boldsymbol{PAR}_1^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\mathrm{e}^{2\varepsilon\tau}\boldsymbol{PBR}_2^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}<0$,
Or,

$$\begin{bmatrix}-\boldsymbol{PD}-\boldsymbol{DP}+2\varepsilon\boldsymbol{P}+\boldsymbol{R}_1+\boldsymbol{R}_2 & -\boldsymbol{PA} & -\mathrm{e}^{\varepsilon\tau}\boldsymbol{PB}\\ -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} & -\boldsymbol{R}_1 & 0\\ -\mathrm{e}^{\varepsilon\tau}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} & 0 & -\boldsymbol{R}_2\end{bmatrix}<0,$$

(ii) $\boldsymbol{\Omega}_{24}^{*}=-\boldsymbol{PD}-\boldsymbol{DP}+2\varepsilon\boldsymbol{P}+\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{R}_1+\boldsymbol{R}_2+\boldsymbol{PAR}_1^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\mathrm{e}^{2\varepsilon\tau}\boldsymbol{PBR}_2^{-1}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P}<0$,
Or,

$$\begin{bmatrix}-\boldsymbol{PD}-\boldsymbol{DP}+2\varepsilon\boldsymbol{P}+\boldsymbol{PA}+\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P}+\boldsymbol{R}_1+\boldsymbol{R}_2 & -\boldsymbol{PA} & -\mathrm{e}^{\varepsilon\tau}\boldsymbol{PB}\\ -\boldsymbol{A}^{\mathrm{T}}\boldsymbol{P} & -\boldsymbol{R}_1 & 0\\ -\mathrm{e}^{\varepsilon\tau}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{P} & 0 & -\boldsymbol{R}_2\end{bmatrix}<0.$$

Remark 3.2 If we have the known delays, from Corollary 3.3, we can estimate the degree of exponential convergence of the considered system by solving the following optimization problem:

$$(P_2)\quad\begin{cases}\max\varepsilon,\\ \text{s.t.}\quad(\mathbf{H}_1)\quad\text{and}\quad\boldsymbol{\Omega}_{23}<0,\quad\text{or}\quad\boldsymbol{\Omega}_{23}^{*}<0.\end{cases}$$

4. Numerical examples

Example 4.1 Consider the following CNNs with multiple time delays and impulsive effects:

$$\begin{cases}\dot{x}(t)=-\boldsymbol{D}\boldsymbol{x}(t)+Af(\boldsymbol{x}(t))+\boldsymbol{B}_1f(\boldsymbol{x}(t-\tau_1))\\ \qquad\quad+\boldsymbol{B}_2f(\boldsymbol{x}(t-\tau_2))\ ,\\ \Delta\boldsymbol{x}(t_k)=I_k\boldsymbol{x}(t_k^-)\ ,\end{cases}\tag{27}$$

where $f_1(x)=f_2(x)=\dfrac{1}{2}(|x+1|-|x-1|)$,

$$\boldsymbol{D}=\begin{bmatrix}2&0\\0&2\end{bmatrix},\quad\boldsymbol{A}=\begin{bmatrix}-0.10&0\\0&-0.050\end{bmatrix},$$

$$\boldsymbol{B}_1=\begin{bmatrix}0.05&-0.01\\-0.10&0.035\end{bmatrix}\quad\boldsymbol{B}_2=\begin{bmatrix}0.10&-0.03\\0&-0.035\end{bmatrix},$$

and

$$\boldsymbol{I}_k\equiv\boldsymbol{I}=\begin{bmatrix}0.2&0\\0&0.2\end{bmatrix}.$$

When $\tau_1=1$, and $\tau_2=2$, from Theorem 3.1, solving the optimization problem $(\mathbf{P}_2)$ yields that the system is exponentially stable when $\varepsilon\le0.9872$. When $\varepsilon=0.9872$, the corresponding feasible solution is shown as follows:

$$\boldsymbol{P}=\begin{bmatrix}10638&-235\\-235&8716\end{bmatrix},\quad\boldsymbol{Q}_0=\begin{bmatrix}1069.5&0\\0&487\end{bmatrix},$$

$$\boldsymbol{Q}_{10}=\begin{bmatrix}41.55&172.53\\172.53&720.87\end{bmatrix},\quad\boldsymbol{Q}_{20}=\begin{bmatrix}41.53&172.45\\172.45&721.06\end{bmatrix},$$

$$\boldsymbol{R}_1=\begin{bmatrix}0.296&0\\0&5.045\end{bmatrix},\quad\boldsymbol{R}_2=\begin{bmatrix}0.151&0\\0&2.530\end{bmatrix},$$

$$\boldsymbol{Q}_1=\begin{bmatrix}2057.9&0\\0&2107.1\end{bmatrix},\quad\boldsymbol{Q}_2=\begin{bmatrix}7702.1&0\\0&7196.9\end{bmatrix},$$

$$\boldsymbol{T}_1=\begin{bmatrix}41.537&172.574\\172.574&720.570\end{bmatrix},\quad\boldsymbol{T}_2=\begin{bmatrix}27.463&114.186\\114.186&477.300\end{bmatrix},$$

$$\boldsymbol{Q}_{11}=\begin{bmatrix}0.101&0\\0&1.689\end{bmatrix},$$

$$\boldsymbol{Q}_{12}=\boldsymbol{Q}_{22}=\begin{bmatrix}0.1024&0\\0&1.6892\end{bmatrix},$$

and

$$\boldsymbol{Q}_{21} = \begin{bmatrix} 0.2023 & 0 \\ 0 & 1.6892 \end{bmatrix}.$$

The time response curves are plotted in Fig.1.

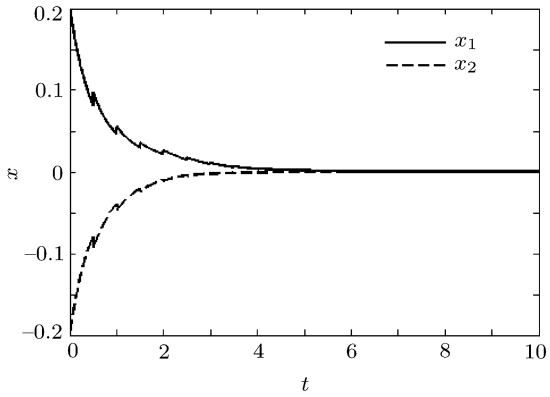


Fig.1. The time response curves of system (27) for $\tau_1 = 1$ and $\tau_2 = 2$.

5. Conclusions

In this paper, we have investigated the exponential stability of the cellular neural networks with multiple time delays and impulsive effects. The method combining Lyapunov–Krasovskii functional, parametrized first-order model transformation and the linearization of considered model has been adopted to study these issues. As a result, several novel delay-dependent and delay-independent exponential stability criteria for cellular neural networks with multiple time delays and impulsive effects have been exploited separately. The stability region with respect to the delays can be estimated. By comparing with the recent results, the illustrated example shows that the proposed results are effective.

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