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Synchronization of stochastically hybrid coupled neural networks with coupling discrete and distributed time-varying delays*

Tang Yang(唐 漾)[a)†], Zhong Hui-Huang(钟恢凰)[b)], and Fang Jian-An(方建安)[a)]

[a)] *Department of Automation, Donghua University, Shanghai* 201620, *China*

[b)] *Department of Communication, Donghua University, Shanghai* 201620, *China*

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A general model of linearly stochastically coupled identical connected neural networks with hybrid coupling is proposed, which is composed of constant coupling, coupling discrete time-varying delay and coupling distributed time-varying delay. All the coupling terms are subjected to stochastic disturbances described in terms of Brownian motion, which reflects a more realistic dynamical behaviour of coupled systems in practice. Based on a simple adaptive feedback controller and stochastic stability theory, several sufficient criteria are presented to ensure the synchronization of linearly stochastically coupled complex networks with coupling mixed time-varying delays. Finally, numerical simulations illustrated by scale-free complex networks verify the effectiveness of the proposed controllers.

Keywords: stochastically hybrid coupling, discrete and distributed time-varying delays, complex dynamical networks, chaotic neural networks

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1. Introduction

In the last decade the research interest has increasingly been aroused in the dynamic analysis of various complex networks including neural networks, regular graph, random graph, small-world network, scale-free network, evolving networks, etc.[1−27] Nowadays complex network lies in our daily life with examples including ecosystems, the World Wide Web (WWW), the World Trade Web, linguistic webs and food webs, etc.[1,2] The chaotic synchronization has been studied in Refs.[3–18, 22–42]. More recently, there has been increasing interest in the study of synchronization dynamics in complex dynamical networks composed of coupled chaotic dynamical systems.[3−10,14,24] As a particular class of complex networks, the synchronization of chaotic neural networks or an array of chaotic neural networks has been a hot topic now.[11−18,22−27]

Moreover, in most of the published literature on synchronization of complex networks, linear or nonlinear deterministic coupling has been investigated intensively. In reality the network coupling often occurs in random fashion and therefore stochastic coupling is ubiquitous in the evolution of complex networks. In fact, stochastic effects may be viewed as random fluctuations from the release of neurotransmitters, and other probabilistic causes.[14,19−21,26] Therefore, the transmitted signals may not be fully detected and received by the other neurons of the system. In order to investigate and simulate more realistic complex dynamical networks, the noise effect should be taken into consideration in modelling the complex networks.[14,19−21,23,25,26] In Ref.[14], a model of complex networks with time-varying delayed stochastically coupling has been introduced, but it has not considered the distributed time-varying stochastically coupling term.

On the other hand, the distributed delay in complex networks or/and neural networks has been an active subject nowadays.[19,25−29] Time delays often occur in complex networks due to the signals propagation through the links[19−21] and the frequently delayed couplings in biological neural networks, gene regulatory networks, communication networks and so on. According to the way it arises, the time-delay can be generally divided into two classes: discrete and distributed.[19,25,26,28] It is worth pointing out that continuously distributed delays have gained particu-

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†Corresponding author. E-mail: tany_yang@mail.dhu.edu.cn

lar attention since a network usually has a spatial nature due to the presence of a number of parallel pathways of a variety of axon sizes and lengths.[28] Therefore, the distributed delay should be taken into consideration for modelling a realistic complex network. Recently, the synchronization between only two chaotic neural networks with discrete and distributed delays has been studied in.[25−27] However, to the best of authors' knowledge, synchronization problems for stochastically hybrid coupled complex neural networks with coupling discrete and distributed time-varying delays have not been addressed in the literature, and so the purpose of this paper is to shorten such a gap.

Motivated by the above discussion, we deal with the synchronization problem for an array of stochastically hybrid coupled complex networks with both the discrete and distributed time-varying delays. By using the adaptive feedback techniques, several theoretical results guaranteeing the synchronization of the stochastically hybrid coupled complex networks with discrete and distributed time-varying delays are obtained. Moreover, the derived theoretical results are verified by scale-free complex networks.

Notation Throughout this paper, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote, respectively, the n-dimensional Euclidean space and the set of all real matrices. The superscript 'T' denotes matrix transposition and the notation $\boldsymbol{X} \geq \boldsymbol{Y}$ (respectively, $\boldsymbol{X} > \boldsymbol{Y}$) where $\boldsymbol{X}$ and $\boldsymbol{Y}$ are symmetric matrices, means that $\boldsymbol{X} - \boldsymbol{Y}$ is positive semi-definite (respectively, positive definite). $\lambda_{\max}(\cdot)$ or $\lambda_{\min}(\cdot)$ denotes the largest or smallest eigenvalue of a matrix, respectively. Let $h > 0$ and $\mathcal{C}([-h, 0; \mathbb{R}^n])$ denote the family of continuous functions ϕ from $[-h, 0]$ to $\mathbb{R}^n$ with the norm $\|\phi\|_2 = \sup_{-h\leq\theta\leq 0} \|\phi(\theta)\|$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^n$; $\mathbb{E}\{\cdot\}$ represents the mathematical expectation operator. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. the filtration contains all P-null sets and is right continuous). $L^{\mathcal{P}}_{\mathcal{F}_0}([-h, 0]; \mathbb{R}^n)$ denotes the family of all $\mathcal{F}_0$-measurable $\mathcal{C}([-h, 0]; \mathbb{R}^n)$- valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h\leq\theta\leq 0} \mathbb{E}\|\xi(\theta)\|^p < \infty$. $\boldsymbol{I}$ represents the identity matrix.

2. Model formulation and preliminaries

Consider the following stochastically hybrid complex dynamical neural networks with coupling discrete and distributed time-varying delays coupled by N identical nodes:

$$\begin{aligned}
\mathrm{d}\boldsymbol{x}_i(t) = &\left[-\boldsymbol{C}\boldsymbol{x}_i(t) + \boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{x}_i(t)) + \boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{x}_i(t-\tau_1(t))) + \boldsymbol{D}\int_{t-\tau_2(t)}^{t} \tilde{\boldsymbol{h}}(\boldsymbol{x}_i(s))\mathrm{d}s + \boldsymbol{U}_i\right]\mathrm{d}t \\
&+\alpha_i \sum_{j=1}^{N} G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\mathrm{d}w_{1i}(t) + \beta_i \sum_{j=1}^{N} G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau_1(t))\mathrm{d}w_{2i}(t) \\
&+\gamma_i \sum_{j=1}^{N} G_{3ij}\boldsymbol{\Gamma}_3 \int_{t-\tau_2(t)}^{t} \boldsymbol{x}_j(s)\mathrm{d}s\mathrm{d}w_{3i}(t), \quad i = 1, 2, \ldots, N, \qquad (1)
\end{aligned}$$

where $\boldsymbol{x}_i(t) = [x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t)]^{\mathrm{T}} \in \mathbb{R}^n$ is the state vector of the ith node; $\boldsymbol{U}_i = [u_1, u_2, \ldots, u_N]^{\mathrm{T}}$, $i = 1, 2 \ldots, N$ is the control input; $\boldsymbol{C} = \mathrm{diag}(c_1, c_2, \ldots, c_n) > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and the external inputs; $\boldsymbol{A} = (a_{ij})_{n\times n}$, $\boldsymbol{B} = (b_{ij})_{n\times n}$ and $\boldsymbol{D} = (d_{ij})_{n\times n}$ represent the connection weight matrix, the discrete time-varying connection weight matrix and distributed time-varying connection weight matrix respectively; $\tau_1(t)$ denotes the time-varying delay, and $\tau_2(t)$ represents the distributed time-varying delay; $\boldsymbol{\omega}_i = [\omega_{1i}(t), \omega_{2i}(t), \omega_{3i}(t)]^{\mathrm{T}}$ is 3-dimensional Brownian motion; α_i, β_i and γ_i are the noise intensities; The matrices $\boldsymbol{\Gamma}_1$, $\boldsymbol{\Gamma}_2$ and $\boldsymbol{\Gamma}_3$ describe inner couplings of the complex networks; $\tilde{f}$, $\tilde{g}$ and $\tilde{h}$ are the activation functions of neurons satisfying $\tilde{\boldsymbol{f}}(\boldsymbol{x}_i(\boldsymbol{t})) = (\tilde{f}_1(x_{i1}(t)), \ldots, \tilde{f}_n(x_{in}(t)))^{\mathrm{T}} \in \mathbb{R}^n$, $\tilde{\boldsymbol{g}}(\boldsymbol{x}_i(t-\tau_1(t))) = (\tilde{g}_1(x_{i1}(t-\tau_1(t))), \ldots, \tilde{g}_n(x_{in}(t-\tau_1(t))))^{\mathrm{T}} \in \mathbb{R}^n$,

and $\tilde{\boldsymbol{h}}(\boldsymbol{x}_i(t)) = (\tilde{h}_1(x_{i1}(t)), \ldots, \tilde{h}_n(x_{in}(t)))^{\mathrm{T}} \in \mathbb{R}^n$; The matrices $G_1 = (G_{1ij})_{N\times N}, G_2 = (G_{2ij})_{N\times N}$ and $G_3 = (G_{3ij})_{N\times N}$ are the coupling configuration matrices representing the coupling strength and the topological structure of the networks, and they satisfy

$$G_{kii} = -\sum_{j=1,j\neq i}^{N} G_{kij},$$
$$k = 1, 2, 3, i, j = 1, 2 \ldots, N. \tag{2}$$

The initial conditions associated with system (1) are given as follows:

$$\boldsymbol{x}_i(s) = \xi_i(s), \quad -\tau^* \leq s \leq 0, \quad i = 1, 2 \ldots, N, \tag{3}$$

where for any $\xi_i \in L^2_{\mathscr{F}_0}([-\tau^*, 0], \mathbb{R}^n), \tau^* = \sup_{t\in \mathrm{R}} \in \{\tau_1(t), \tau_2(t)\}$, with $L^2_{\mathscr{F}_0}([-\tau^*, 0], \mathbb{R}^n)$ being the family of all $\mathscr{F}_0$-measurable $\mathcal{C}([-\tau^*, 0]; \mathbb{R}^n)$-valued random variables satisfying that $\sup_{-\tau^*\leq\varphi\leq 0} \mathbb{E}\left|\xi_i(\varphi)\right|^2 < \infty$, and $\mathcal{C}([-\tau^*, 0]; \mathbb{R}^n)$ denotes the family of all continuous $\mathbb{R}^n$-valued functions $\xi_i(\varphi)$ on $[-\tau^*, 0]$ with the norm $\|\xi_i\| = \sup_{-\tau*\leq\varphi\leq 0} \left|\xi_i(\varphi)\right|$.

$\boldsymbol{s}(t) \in \mathbb{R}^n$ is a solution of an isolated node

$$\mathrm{d}\boldsymbol{s}(t) = \Big[-\boldsymbol{C}\boldsymbol{s}(t) + \boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{s}(t)) + \boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{s}(t-\tau_1(t))) + \boldsymbol{D}\int_{t-\tau_2(t)}^{t} \tilde{\boldsymbol{h}}(\boldsymbol{s}(\eta))\mathrm{d}\eta \Big]\mathrm{d}t \tag{4}$$

with $\boldsymbol{x}_0 \in \mathbb{R}^n$.

Here, let $\boldsymbol{e}_i(t) = \boldsymbol{x}_i(t) - \boldsymbol{s}(t)$ be the error states. It follows from systems (1) and (3) and condition (2) that

$$\begin{aligned}\mathrm{d}\boldsymbol{e}_i(t) = {} & \Big[-\boldsymbol{C}\boldsymbol{e}_i(t) + \boldsymbol{A}\boldsymbol{f}(\boldsymbol{e}_i(t)) + \boldsymbol{B}\boldsymbol{g}(\boldsymbol{e}_i(t-\tau_1(t))) \\ & + \boldsymbol{D}\int_{t-\tau_2(t)}^{t} \boldsymbol{h}(\boldsymbol{e}_i(s))\mathrm{d}s + \boldsymbol{U}_i \Big]\mathrm{d}t \\ & + \alpha_i \sum_{j=1}^{N} G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)\mathrm{d}w_{1i}(t) \\ & + \beta_i \sum_{j=1}^{N} G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\tau_1(t))\mathrm{d}w_{2i}(t) \\ & + \gamma_i \sum_{j=1}^{N} G_{3ij}\boldsymbol{\Gamma}_3 \int_{t-\tau_2(t)}^{t} \boldsymbol{e}_j(s)\mathrm{d}s\mathrm{d}w_{3i}(t), \\ & i = 1, 2, \ldots, N, \end{aligned} \tag{5}$$

where $\boldsymbol{f}(\boldsymbol{e}_i(t)) = \tilde{\boldsymbol{f}}(\boldsymbol{e}_i(t) + \boldsymbol{s}(t)) - \tilde{\boldsymbol{f}}(\boldsymbol{s}(t))$, $\boldsymbol{g}(\boldsymbol{e}_i(t)) = \tilde{\boldsymbol{g}}(\boldsymbol{e}_i(t)+\boldsymbol{s}(t)) - \tilde{\boldsymbol{g}}(\boldsymbol{s}(t))$ and $\boldsymbol{h}(\boldsymbol{e}_i(t)) = \tilde{\boldsymbol{h}}(\boldsymbol{e}_i(t)+\boldsymbol{s}(t)) - \tilde{\boldsymbol{h}}(\boldsymbol{s}(t))$.

Remark 1 The complex neural network model (1) is quite general, which contains the coupling mixed time-varying delays and stochastically coupling. It should be pointed out that the synchronization problem for complex networks with both coupling discrete and distributed time-varying delays has not received much attention in the literature, and it does not mention the case where the stochastically coupling distributed time-varying delays are also involved.

Throughout this paper, the following assumptions are needed:

(A_1). The activation functions $\tilde{f}_i(x)$ and $\tilde{g}_i(x)$ satisfy the Lipschitz condition, i.e. for all $i = 1, 2, \ldots, n$, there exist constants $\lambda_i > 0$ and $\phi_i > 0$ such that

$$\left|\tilde{f}_i(x) - \tilde{f}_i(y)\right| \leq \lambda_i \left|x - y\right|,$$
$$\left|\tilde{g}_i(x) - \tilde{g}_i(y)\right| \leq \phi_i \left|x - y\right|, \quad \forall x, y \in \mathbb{R}^n.$$

(A_2). The activation function $\tilde{\boldsymbol{h}}(\boldsymbol{x})$ satisfies the Lipschitz condition in a vector form

$$\left\|\tilde{\boldsymbol{h}}(\boldsymbol{x}) - \tilde{\boldsymbol{h}}(\boldsymbol{y})\right\| \leq \left\|\boldsymbol{H}(\boldsymbol{x} - \boldsymbol{y})\right\|, \quad \forall x, y \in \mathbb{R}^n,$$

where $\boldsymbol{H} \in \mathbb{R}^{n\times n}$ is a constant matrix.

(A_3). $\tilde{f}_i(0) = \tilde{g}_i(0) = \tilde{h}_i(0) = 0$.

(A_4). $\tau_1(t)$ and $\tau_2(t)$ are bounded and continuously differentiable functions, $0 < \tau_1(t) \leq \tau_1$, $0 < \tau_2(t) \leq \tau_2$ and $\dot{\tau}_1(t) \leq \mu < 1$, $\dot{\tau}_2(t) \leq \varrho < 1$.

Under the assumptions (A_3), it is easy to verify that $f_i(0) = g_i(0) = h_i(0) = 0$. Hence, error system (4) admits a trivial solution $\boldsymbol{e}_i(0) \equiv 0, i = 1, 2, \ldots, N$.

The following definition is needed in this paper:

Definition 1 Let $\boldsymbol{x}_i(t; t_0, X_0)(1 \leq i \leq N)$ be a solution of the array of linearly stochastically coupled complex networks with coupling discrete and distributed delays (1), where $\boldsymbol{X}_0 = (\boldsymbol{x}_1^0, \boldsymbol{x}_2^0, \ldots, \boldsymbol{x}_N^0)$. If there is a nonempty subset $\Omega \subseteq \mathbb{R}^n$, with $\boldsymbol{x}_i^0 \in \Omega(1 \leq i \leq N)$, such that $\boldsymbol{x}_i(t; t_0, \boldsymbol{X}_0) \in \mathbb{R}^n$ for all $t \geq t_0, 1 \leq i \leq N$, and

$$\lim_{t\to\infty}\mathbb{E}\left\|\boldsymbol{x}_i(t; t_0, X_0) - s(t; t_0, X_0)\right\|^2 = 0, \quad i = 1, 2, \ldots, N, \tag{6}$$

or

$$\lim_{t\to\infty}\mathbb{E}\left\|\boldsymbol{e}_i(t)\right\|^2 = 0, \quad i = 1, 2, \ldots, N, \tag{7}$$

then stochastically hybrid complex networks (1) with coupling mixed time-varying delays is said to achieve synchronization in mean square and $\Omega \times \cdots \times \Omega$ is called the region of synchrony for networks (1).

3. Main results

The following lemmas will be essential to achieving the main results.

Lemma 1 (See Ref.[43]). Let $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2$, and $\boldsymbol{\Omega}_3$ be real matrices of appropriate dimensions with $\boldsymbol{\Omega}_3 > 0$. Given any vector $\boldsymbol{x}, \boldsymbol{y}$ of appropriate dimensions, then the following inequality holds:

$$2\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\Omega}_1^{\mathrm{T}}\boldsymbol{\Omega}_2\boldsymbol{y} \leq \boldsymbol{x}^{\mathrm{T}}\boldsymbol{\Omega}_1^{\mathrm{T}}\boldsymbol{\Omega}_3\boldsymbol{\Omega}_1\boldsymbol{x} + \boldsymbol{y}^{\mathrm{T}}\boldsymbol{\Omega}_2^{\mathrm{T}}\boldsymbol{\Omega}_3^{-1}\boldsymbol{\Omega}_2\boldsymbol{y}.$$

Lemma 2 (See Ref.[20]). For any positive definite matrix $\boldsymbol{N} > 0$, scalar $\nu > \nu(t) > 0$, vector function $w : [0, \nu] \to \mathbb{R}^n$ such that the integrations concerned are well defined, the following inequality holds:

$$\left(\int_0^{v(t)} \boldsymbol{w}(s)\mathrm{d}s\right)^{\mathrm{T}} \boldsymbol{N} \left(\int_0^{v(t)} \boldsymbol{w}(s)\mathrm{d}s\right) \leq v(t)\left(\left(\int_0^{v(t)} \boldsymbol{w}^{\mathrm{T}}(s)\boldsymbol{N}\boldsymbol{w}(s)\right)\right)\mathrm{d}s.$$

Theorem 1 Under assumptions $(A_1)-(A_4)$, let the controller be $\boldsymbol{U}_i(t) = \boldsymbol{\varepsilon}_i(t)\boldsymbol{e}_i(t)$, where the feedback strengths $\boldsymbol{\varepsilon}_i(t) = (\varepsilon_{i1}(t), \varepsilon_{i2}(t), \ldots, \varepsilon_{in}(t))^{\mathrm{T}}$ with update laws are taken as

$$\begin{aligned} &\dot{\varepsilon}_{ij} = -\delta_{ij}e_{ij}^2(t), \\ &i = 1, 2, \ldots, N, \quad j = 1, 2, \ldots, n, \end{aligned} \tag{8}$$

in which δ_{ij} are arbitrary positive constants, i.e. $\delta_{ij} > 0$. Then, system (5) is said to be synchronized in mean square, that is, the trivial solution of error system (5) is asymptotically stable in mean square.

Proof 1 Construct the following nonnegative function:

$$\begin{aligned} V(t, \boldsymbol{e}(t)) = &\frac{1}{2}\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) \\ &+ \sum_{i=1}^{N}\int_{t-\tau_1(t)}^{t} \boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{P}\boldsymbol{e}_i(s)\mathrm{d}s \\ &+ \sum_{i=1}^{N}\int_{t-\tau_2(t)}^{t}\int_{s}^{t} \boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{Q}\boldsymbol{e}_i(\theta)\mathrm{d}\theta\mathrm{d}s \\ &+ \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{1}{\delta_{ij}}(\varepsilon_{ij} + l)^2, \end{aligned} \tag{9}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are positive definite matrices and l is a constant, $\boldsymbol{P}$, $\boldsymbol{Q}$ and l are to be determined in the following, separately.

By using the well-known Itô-differential formula,[44] the stochastic derivative of V along the trajectory of error system (5) can be obtained as follows:

$$\begin{aligned} &\mathrm{d}V\left(t, \boldsymbol{e}\left(t\right)\right) \\ &= \mathscr{L}V(t, \boldsymbol{e}(t))\mathrm{d}t \\ &\quad + \sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}\left(t\right)\left[\alpha_i \sum_{j=1}^{N} G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j\left(t\right)\mathrm{d}w_{1i}\left(t\right)\right. \\ &\quad + \beta_i \sum_{j=1}^{N} G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t - \tau_1\left(t\right))\mathrm{d}w_{2i}(t) \\ &\quad \left. + \gamma_i \sum_{j=1}^{N} G_{3ij}\boldsymbol{\Gamma}_3 \int_{t-\tau_2(t)}^{t} \boldsymbol{e}_j\left(s\right)\mathrm{d}s\mathrm{d}w_{3i}\left(t\right)\right], \end{aligned} \tag{10}$$

where the weak infinitesimal operator $\mathscr{L}$ is given by

$$\begin{aligned} \mathscr{L}V(t, \boldsymbol{e}(t)) = &\sum_{i=1}^{N} \boldsymbol{e}_i^{\mathrm{T}}\left(t\right)\left[-\boldsymbol{C}\boldsymbol{e}_i\left(t\right) + \boldsymbol{A}\boldsymbol{f}\left(\boldsymbol{e}_i\left(t\right)\right) + \boldsymbol{B}\boldsymbol{g}\left(\boldsymbol{e}_i\left(t - \tau_1\left(t\right)\right)\right) + \boldsymbol{D}\int_{t-\tau_2(t)}^{t} \boldsymbol{h}\left(\boldsymbol{e}_i\left(s\right)\right)\mathrm{d}s + \boldsymbol{\varepsilon}_i\boldsymbol{e}_i\left(t\right)\right] \\ &+ \sum_{i=1}^{N}\left[\boldsymbol{e}_i^{\mathrm{T}}\left(t\right)\boldsymbol{P}\boldsymbol{e}_i\left(t\right) - \left(1 - \dot{\tau}_1\left(t\right)\right)\boldsymbol{e}_i^{\mathrm{T}}\left(t - \tau_1\left(t\right)\right)\boldsymbol{P}\boldsymbol{e}_i\left(t - \tau_1\left(t\right)\right)\right] \\ &+ \sum_{i=1}^{N}\left[\tau_2\left(t\right)\boldsymbol{e}_i^{\mathrm{T}}\left(t\right)\boldsymbol{Q}\boldsymbol{e}_i\left(t\right) - \left(1 - \dot{\tau}_2\left(t\right)\right)\int_{t-\tau_2(t)}^{t} \boldsymbol{e}_i^{\mathrm{T}}\left(s\right)\boldsymbol{Q}\boldsymbol{e}_i\left(s\right)\mathrm{d}s\right] \\ &+ \sum_{i=1}^{N}\sum_{j=1}^{n}\left(\varepsilon_{ij} + l\right)e_{ij}^2\left(t\right) + \frac{\alpha_i^2}{2}\sum_{i=1}^{N}\left[\sum_{j=1}^{N} G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j\left(t\right)\right]^{\mathrm{T}}\left[\sum_{j=1}^{N} G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j\left(t\right)\right] \\ &+ \frac{\beta_i^2}{2}\sum_{i=1}^{N}\left[\sum_{j=1}^{N} G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j\left(t - \tau_1\left(t\right)\right)\right]^{\mathrm{T}}\left[\sum_{j=1}^{N} G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j\left(t - \tau_1\left(t\right)\right)\right] \end{aligned}$$

$$
\begin{aligned}
&+\frac{\gamma_i^2}{2}\sum_{i=1}^{N}\left[\sum_{j=1}^{N}G_{3ij}\boldsymbol{\Gamma}_3\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j(s)\,\mathrm{d}s\right]^{\mathrm{T}}\left[\sum_{j=1}^{N}G_{3ij}\boldsymbol{\Gamma}_3\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j(s)\,\mathrm{d}s\right]\\
=&\sum_{i=1}^{N}\Bigg\{-\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{C}\boldsymbol{e}_i(t)+\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{A}\boldsymbol{f}(\boldsymbol{e}_i(t))+\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{B}\boldsymbol{g}(\boldsymbol{e}_i(t-\tau_1(t)))\\
&+\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{D}\int_{t-\tau_2(t)}^{t}\boldsymbol{h}(\boldsymbol{e}_i(s))\,\mathrm{d}s+\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_i(t)-(1-\dot{\tau}_1(t))\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{P}\boldsymbol{e}_i(t-\tau_1(t))\\
&+\tau_2(t)\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}_i(t)-l\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)-(1-\dot{\tau}_2(t))\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{Q}\boldsymbol{e}_i(s)\,\mathrm{d}s\\
&+\frac{\alpha_i^2}{2}\left[\sum_{j=1}^{N}G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)\right]^{\mathrm{T}}\left[\sum_{j=1}^{N}G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)\right]\\
&+\frac{\beta_i^2}{2}\left[\sum_{j=1}^{N}G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\tau_1(t))\right]^{\mathrm{T}}\left[\sum_{j=1}^{N}G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\tau_1(t))\right]\\
&+\frac{\gamma_i^2}{2}\left[\sum_{j=1}^{N}G_{3ij}\boldsymbol{\Gamma}_3\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j(s)\,\mathrm{d}s\right]^{\mathrm{T}}\left[\sum_{j=1}^{N}G_{3ij}\boldsymbol{\Gamma}_3\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j(s)\,\mathrm{d}s\right]\Bigg\}. \qquad (11)
\end{aligned}
$$

According to assumption (A_4), we have

$$-(1-\dot{\tau}_1(t))\le-(1-\mu), \qquad (12)$$

and

$$-(1-\dot{\tau}_2(t))\le-(1-\rho). \qquad (13)$$

From assumption (A_1), we can obtain

$$
\begin{aligned}
\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{e}_i(t))\boldsymbol{f}(\boldsymbol{e}_i(t))&=\sum_{j=1}^{n}f_j^2(e_{ij}(t))\le\sum_{j=1}^{n}\gamma_j^2e_{ij}^2(t)\\
&\le\Lambda\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t), \qquad (14)
\end{aligned}
$$

and

$$
\begin{aligned}
\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{e}_i(t))\boldsymbol{g}(\boldsymbol{e}_i(t))&=\sum_{j=1}^{n}g_j^2(e_{ij}(t))\le\sum_{j=1}^{n}\phi_j^2e_{ij}^2(t)\\
&\le\Phi\boldsymbol{e}_i^{\mathrm{T}}\boldsymbol{e}_i(t), \qquad (15)
\end{aligned}
$$

where $\Lambda=\max\{\lambda_j^2:j=1,2,\ldots,n\}$ and $\Phi=\max\{\phi_j^2:j=1,2,\ldots,n\}$.

It follows from Lemma 1 and inequalities (14) and (15) that

$$
\begin{aligned}
&\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{A}\boldsymbol{f}(\boldsymbol{e}_i(t))\\
\le&\frac{1}{2}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{e}_i(t)+\frac{1}{2}\boldsymbol{f}^{\mathrm{T}}(\boldsymbol{e}_i(t))\boldsymbol{f}(\boldsymbol{e}_i(t))\\
\le&\frac{1}{2}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{e}_i(t)+\frac{1}{2}\Lambda\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t), \qquad (16)
\end{aligned}
$$

and

$$
\begin{aligned}
&\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{B}\boldsymbol{g}(\boldsymbol{e}_i(t-\tau_1(t)))\\
\le&\frac{1}{2}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{e}_i(t)\\
&+\frac{1}{2}\boldsymbol{g}^{\mathrm{T}}(\boldsymbol{e}_i(t-\tau_1(t)))\boldsymbol{g}(\boldsymbol{e}_i(t-\tau_1(t)))\\
\le&\frac{1}{2}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{e}_i(t)\\
&+\frac{1}{2}\Phi\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_i(t-\tau_1(t)). \qquad (17)
\end{aligned}
$$

Let $\boldsymbol{\Omega}_1,\boldsymbol{\Omega}_2$ and $\boldsymbol{\Omega}_3$ in Lemma 1 be the compatible dimensions identity matrix, then we will have

$$
\begin{aligned}
&\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{D}\int_{t-\tau_2(t)}^{t}\boldsymbol{h}(\boldsymbol{e}_i(s))\mathrm{d}s\\
\le&\frac{1}{2}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{e}_i(t)\\
&+\frac{1}{2}\int_{t-\tau_2(t)}^{t}\boldsymbol{h}^{\mathrm{T}}(\boldsymbol{e}_i(s))\mathrm{d}s\int_{t-\tau_2(t)}^{t}\boldsymbol{h}(\boldsymbol{e}_i(s))\mathrm{d}s. \qquad (18)
\end{aligned}
$$

From assumption (A_2), the following inequality:

$$\|\boldsymbol{h}(\boldsymbol{e}_i(t))\|\le\|\boldsymbol{H}\boldsymbol{e}_i(t)\| \qquad (19)$$

holds, then $\boldsymbol{h}^{\mathrm{T}}(\boldsymbol{e}_i(t))\boldsymbol{h}(\boldsymbol{e}_i(t))\le\boldsymbol{e}_i^{\mathrm{T}}(t)H^{\mathrm{T}}H\boldsymbol{e}_i(t)$.

According to Lemma 2 and inequality (19), we have

$$
\begin{aligned}
&\frac{1}{2}\int_{t-\tau_2(t)}^{t}\boldsymbol{h}^{\mathrm{T}}(\boldsymbol{e}_i(s))\mathrm{d}s\int_{t-\tau_2(t)}^{t}\boldsymbol{h}(\boldsymbol{e}_i(s))\mathrm{d}s\\
\le&\frac{\tau_2(t)}{2}\int_{t-\tau_2(t)}^{t}\boldsymbol{h}^{\mathrm{T}}(\boldsymbol{e}_i(s))\boldsymbol{h}(\boldsymbol{e}_i(s))\mathrm{d}s\\
\le&\frac{\tau_2}{2}\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_i^{T}(s)\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{e}_i(s)\mathrm{d}s. \qquad (20)
\end{aligned}
$$

It is easy to check that

$$\left[\sum_{j=1}^{N} G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)\right]^{\mathrm{T}}\left[\sum_{j=1}^{N} G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)\right] \le N\sum_{j=1}^{N} G_{1ij}^2\boldsymbol{e}_j^{\mathrm{T}}(t)\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1\boldsymbol{e}_j(t)$$

$$\le NI\lambda_{\max}(\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1)\sum_{j=1}^{N} G_{1ij}^2\boldsymbol{e}_j^{\mathrm{T}}(t)\boldsymbol{e}_j(t), \tag{21}$$

$$\left[\sum_{j=1}^{N} G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\tau_1(t))\right]^{\mathrm{T}}\left[\sum_{j=1}^{N} G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\tau_1(t))\right]$$

$$\le N\sum_{j=1}^{N} G_{2ij}^2\boldsymbol{e}_j^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{\Gamma}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2\boldsymbol{e}_j(t-\tau_1(t)) \le N\boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2)\sum_{j=1}^{N} G_{2ij}^2\boldsymbol{e}_j^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_j(t-\tau_1(t)), \tag{22}$$

and

$$\left[\sum_{j=1}^{N} G_{3ij}\boldsymbol{\Gamma}_3\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j(s)\mathrm{d}s\right]^{\mathrm{T}}\left[\sum_{j=1}^{N} G_{3ij}\boldsymbol{\Gamma}_3\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j(s)\mathrm{d}s\right]$$

$$\le N\sum_{j=1}^{N} G_{3ij}^2\boldsymbol{\Gamma}_3^{\mathrm{T}}\boldsymbol{\Gamma}_3\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j^{\mathrm{T}}(s)\mathrm{d}s\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j(s)\mathrm{d}s \le N\sum_{j=1}^{N} G_{3ij}^2\boldsymbol{\Gamma}_3^{\mathrm{T}}\boldsymbol{\Gamma}_3\tau_2(t)\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j^{\mathrm{T}}(s)\boldsymbol{e}_j(s)\mathrm{d}s$$

$$\le \tau_2 N\boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_3^{\mathrm{T}}\boldsymbol{\Gamma}_3)\sum_{j=1}^{N} G_{3ij}^2\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j^{\mathrm{T}}(s)\boldsymbol{e}_j(s)\mathrm{d}s. \tag{23}$$

Using inequalities (12)–(23) yields from expression (11)

$$\begin{aligned}
\mathscr{L}V(t,e(t)) \le \sum_{i=1}^{N}\Bigg\{ & -\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{C}\boldsymbol{e}_i(t)+\frac{1}{2}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{e}_i(t)+\frac{1}{2}\Lambda\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)+\frac{1}{2}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{e}_i(t)\\
& +\frac{1}{2}\Phi\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_i(t-\tau_1(t))+\frac{1}{2}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{e}_i(t)+\frac{\tau_2}{2}\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_i^{\mathrm{T}}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}\boldsymbol{e}_i(s)\mathrm{d}s\\
& +\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}_i(t)-(1-\mu)\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{P}\boldsymbol{e}_i(t-\tau_1(t))+\tau_2\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{Q}\boldsymbol{e}_i(t)-l\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)\\
& -(1-\varrho)\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{Q}\boldsymbol{e}_i(s)\mathrm{d}s+\frac{\alpha^2}{2}\sigma_1 N\boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1)\sum_{n=1}^{N}\boldsymbol{e}_j^{\mathrm{T}}(t)\boldsymbol{e}_j(t)\\
& +\frac{\beta^2}{2}\sigma_2 N\boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2)\sum_{j=1}^{N}\boldsymbol{e}_j^{\mathrm{T}}(t-\tau_1(t))\boldsymbol{e}_j(t-\tau_1(t))\\
& +\tau_2\frac{\gamma^2}{2}\sigma_3 NI\lambda_{\max}(\boldsymbol{\Gamma}_3^{\mathrm{T}}\boldsymbol{\Gamma}_3)\sum_{j=1}^{N}\int_{t-\tau_2(t)}^{t}\boldsymbol{e}_j^{\mathrm{T}}(s)\boldsymbol{e}_j(s)\mathrm{d}s\Bigg\}\\
= \sum_{i=1}^{N}\Bigg\{ & \boldsymbol{e}_i^{\mathrm{T}}(t)\Bigg[-\boldsymbol{C}+\frac{1}{2}\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}+\frac{1}{2}\Lambda+\frac{1}{2}\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}}+\frac{1}{2}\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}}+\boldsymbol{P}+\tau_2\boldsymbol{Q}\\
& +\frac{\alpha^2}{2}\sigma_1 N^2\boldsymbol{I}\gamma_{\max}(\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1)-l\boldsymbol{I}\Bigg]\boldsymbol{e}_i^{\mathrm{T}}+\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\Bigg[\frac{1}{2}\Phi\boldsymbol{I}-(1-\mu)\boldsymbol{P}\\
& +\frac{\beta^2}{2}\sigma_2 N^2\boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2)\Bigg]\boldsymbol{e}_i(t-\tau_1(t))\\
& +\int_{(t-\tau_2(t))}^{t}\boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{e}_i(s)\mathrm{d}s\left[\frac{\tau_2}{2}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}-(1-\varrho)\boldsymbol{Q}+\tau_2\frac{\gamma^2}{2}\sigma_3 N^2\boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_3^{\mathrm{T}}\boldsymbol{\Gamma}_3)\right]\Bigg\}\\
\le \sum_{i=1}^{N}\Bigg\{ & \boldsymbol{e}_i^{\mathrm{T}}(t)[\lambda_{\max}(-\boldsymbol{C})+\frac{1}{2}\lambda_{\max}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})+\frac{1}{2}\Lambda+\frac{1}{2}\lambda_{\max}(\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}})\frac{1}{2}\lambda_{\max}(\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}})+\lambda_{\max}(\boldsymbol{P})\\
& +\tau_2\lambda_{\max}(\boldsymbol{Q})+\frac{\alpha^2}{2}\sigma_1 N^2\boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1)-l\boldsymbol{I}]\boldsymbol{e}_i(t)+\boldsymbol{e}_i^{\mathrm{T}}(t-\tau_1(t))\Bigg[\frac{1}{2}\Phi\boldsymbol{I}-(1-\mu)\boldsymbol{P}
\end{aligned}$$

$$+\frac{\beta^2}{2}\sigma_2 N^2 \boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2)\Big]\boldsymbol{e}_i(t-\tau_1(t))$$
$$+\int_{(t-\tau_2(t))}^{t}\boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{e}_i(s)\mathrm{d}s\left[\frac{\tau_2}{2}\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}-(1-\varrho)\boldsymbol{Q}+\tau_2\frac{\gamma^2}{2}\sigma_3 N^2\boldsymbol{I}\lambda_{\max}(\boldsymbol{\Gamma}_3^{\mathrm{T}}\boldsymbol{\Gamma}_3)\right]\Big\}, \tag{24}$$

where

$$\alpha=\max_{1\le i\le N}\{\alpha_i\},$$
$$\beta=\max_{1\le i\le N}\{\beta_i\},$$
$$\gamma=\max_{1\le i\le N}\{\gamma_i\},$$
$$\sigma_1=\max_{1\le i,j\le N}\{G_{1ij}^2\},$$
$$\sigma_2=\max_{1\le i,j\le N}\{G_{2ij}^2\},$$
$$\sigma_3=\max_{1\le i,j\le N}\{G_{3ij}^2\}.$$

Let

$$\boldsymbol{P}=\frac{1}{2(1-\mu)}[\Phi\boldsymbol{I}+\beta^2\sigma_2 N^2\lambda_{\max}(\boldsymbol{\Gamma}_2^{\mathrm{T}}\boldsymbol{\Gamma}_2)I], \tag{25}$$
$$\boldsymbol{Q}=\frac{\tau_2}{2(1-\rho)}[\boldsymbol{H}^{\mathrm{T}}\boldsymbol{H}+\gamma^2\sigma_3 N^2\lambda_{\max}(\boldsymbol{\Gamma}_3^{\mathrm{T}}\boldsymbol{\Gamma}_3)I], \tag{26}$$

and

$$l=\lambda_{\max}(-\boldsymbol{C})+\frac{1}{2}\lambda_{\max}(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}})+\frac{1}{2}\Lambda$$
$$+\frac{1}{2}\lambda_{\max}(\boldsymbol{B}\boldsymbol{B}^{\mathrm{T}})+\frac{1}{2}\lambda_{\max}(\boldsymbol{D}\boldsymbol{D}^{\mathrm{T}})+\lambda_{\max}(\boldsymbol{P})$$
$$+\tau_2\lambda_{\max}(\boldsymbol{Q})+\frac{\alpha^2}{2}\sigma_1 N^2\lambda_{\max}(\boldsymbol{\Gamma}_1^{\mathrm{T}}\boldsymbol{\Gamma}_1)+1, \tag{27}$$

then it will be obtained that

$$\mathscr{L}V(t,\boldsymbol{e}(t))\le-\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{e}(t). \tag{28}$$

According to the invariant principle of stochastic differential equation proposed in Ref.[45], it can be derived that $\mathbb{E}\left\|\boldsymbol{e}_i(t;\xi)\right\|^2\to 0$, and at the same time $\varepsilon_{ij}(t)\to\varepsilon_{ij}(i=1,2,\ldots,N,j=1,2,\ldots,n)$ (constants). This completes the proof.

If the discrete and distributed time-varying delays in stochastically hybrid coupled complex networks (1) become constant delays, we can obtain the following results.

Theorem 2 Under assumptions $(A_1)-(A_4)$, let the controller be $\boldsymbol{U}_i(t)=\boldsymbol{\varepsilon}_i(t)\boldsymbol{e}_i(t)$, where the feedback strengths $\boldsymbol{\varepsilon}_i(t)=(\varepsilon_{i1}(t),\varepsilon_{i2}(t),\ldots,\varepsilon_{in}(t))^{\mathrm{T}}$ with update laws are taken as

$$\dot{\varepsilon}_{ij}=-\delta_{ij}e_{ij}^2(t),$$
$$i=1,2,\ldots,N,\quad j=1,2,\ldots,n, \tag{29}$$

in which δ_{ij} are arbitrary positive constants, i.e. $\delta_{ij}>0$. Then, system (5) is said to be synchronized in mean square, that is, the trivial solution of error system (5) is asymptotically stable in mean square.

Proof 2 Construct the following nonnegative function:

$$V(t,\boldsymbol{e}(t))=\frac{1}{2}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t)$$
$$+\sum_{i=1}^{N}\int_{t-\tau_1}^{t}\boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{P}\boldsymbol{e}_i(s)\mathrm{d}s$$
$$+\sum_{i=1}^{N}\int_{-\tau_2}^{0}\int_{t-s}^{t}\boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{Q}\boldsymbol{e}_i(\theta)\mathrm{d}(\theta)\mathrm{d}s$$
$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{1}{\delta_{ij}}(\varepsilon_{ij}+l)^2 \tag{30}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are positive definite matrices and l is a constant, $\boldsymbol{P}$, $\boldsymbol{Q}$ and l are to be determined in the following, separately. The rest of the proof is similar to that of Theorem 1, and hence omitted.

If the stochastic coupling distributed delay term is removed from system (1), we can obtain

$$\mathrm{d}\boldsymbol{x}_i(t)=[-\boldsymbol{C}\boldsymbol{x}_i(t)+\boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{x}_i(t))+\boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{x}_i(t-\tau_1(t)))$$
$$+\boldsymbol{D}\int_{t-\tau_2(t)}^{t}\tilde{\boldsymbol{h}}(\boldsymbol{e}_i(s))\mathrm{d}s+\boldsymbol{U}_i]\mathrm{d}t$$
$$+\alpha_i\sum_{j=1}^{N}G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\mathrm{d}\omega_{1i}(t)$$
$$+\beta_i\sum_{j=1}^{N}G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau_1(t))\mathrm{d}\omega_{2i}(t). \tag{31}$$

Then, we have the following results.

Corollary 1 Under assumptions $(A_1)-(A_4)$, let the controller be $\boldsymbol{U}_i(t)=\varepsilon_i(t)\boldsymbol{e}_i(t)$, where the feedback strengths $\varepsilon_i(t)=(\varepsilon_{i1}(t),\varepsilon_{i2}(t),\ldots,\varepsilon_{in}(t))^{\mathrm{T}}$ with update laws are taken as

$$\dot{\varepsilon}_{ij}=-\delta_{ij}e_{ij}^2(t),$$
$$i=1,2,\ldots,N,\quad j=1,2,\ldots,n, \tag{32}$$

in which δ_{ij} are arbitrary positive constants, i.e. $\delta_{ij}>0$. Then, system (5) is said to be synchronized in mean square, that is, the trivial solution of error system (5) is asymptotically stable in mean square.

Proof 3 Construct the following nonnegative function:

$$\begin{aligned} V(t,\boldsymbol{e}(t)) = & \frac{1}{2}\sum_{i=1}^{N}\boldsymbol{e}_i^{\mathrm{T}}(t)\boldsymbol{e}_i(t) \\ & +\sum_{i=1}^{N}\int_{t-\tau_1(t)}^{t}\boldsymbol{e}_i^{\mathrm{T}}(s)\boldsymbol{P}\boldsymbol{e}_i(s)\mathrm{d}s \\ & +\sum_{i=1}^{N}\int_{t-\tau_2(t)}^{t}\int_{s}^{t}\boldsymbol{e}_i^{\mathrm{T}}(\theta)\boldsymbol{Q}\boldsymbol{e}_i(\theta)\mathrm{d}\theta\mathrm{d}s \\ & +\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{n}\frac{1}{\delta_{ij}}(\varepsilon_{ij}+l)^2, \end{aligned} \tag{33}$$

where $\boldsymbol{P}$ and $\boldsymbol{Q}$ are positive definite matrices and l is a constant, $\boldsymbol{P},\boldsymbol{Q}$ and l are to be determined in the following, separately. The rest of the proof is similar to that of Theorem 1, and hence omitted.

If both distributed delay and coupling distributed delay terms are removed from expression (1), we can have

$$\begin{aligned} \mathrm{d}\boldsymbol{x}_i(t) = & [-\boldsymbol{C}\boldsymbol{x}_i(t)+\boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{x}_i(t)) \\ & +\boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{x}_i(t-\tau_1(t)))+\boldsymbol{U}_i]\mathrm{d}t \\ & +\alpha_i\sum_{j=1}^{N}G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\mathrm{d}w_{1i}(t) \\ & +\beta_i\sum_{j=1}^{N}G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau_1(t))\mathrm{d}w_{2i}(t). \end{aligned} \tag{34}$$

Then, the following results can be obtained.

Corollary 2 Under assumptions $(A_1)-(A_4)$, let the controller be $\boldsymbol{U}_i(t)=\boldsymbol{\varepsilon}_i(t)\boldsymbol{e}_i(t)$, where the feedback strengths $\boldsymbol{\varepsilon}_i(t)=(\varepsilon_{i1}(t),\varepsilon_{i2}(t),\ldots,\varepsilon_{in}(t))^{\mathrm{T}}$ with update laws are taken as

$$\begin{aligned} & \dot{\varepsilon}_{ij}=-\delta_{ij}\boldsymbol{e}_{ij}^2(t), \\ & i=1,2,\ldots,N,\quad j=1,2,\ldots,n, \end{aligned} \tag{35}$$

in which δ_{ij} are arbitrary positive constants, i.e. $\delta_{ij}>0$. Then, system (5) is said to be synchronized in mean square, that is, the trivial solution of error system (5) is asymptotically stable in mean square.

If the stochastic coupling terms are removed from system (1) and only the model with general linear coupling and coupling mixed time-varying delays is considered, then the network reduces to

$$\begin{aligned} \mathrm{d}\boldsymbol{x}_i(t) = & \Big[-\boldsymbol{C}\boldsymbol{x}_i(t)+A\tilde{\boldsymbol{f}}(\boldsymbol{x}_i(t)) \\ & +\boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{x}_i(t-\tau_1(t))) \\ & +\boldsymbol{D}\int_{t-\tau_2(t)}^{t}\tilde{\boldsymbol{h}}(\boldsymbol{x}_i(s))\mathrm{d}s \\ & +\alpha_i\sum_{j=1}^{N}G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t) \\ & +\beta_i\sum_{j=1}^{N}G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau_1(t)) \\ & +\gamma_i\sum_{j=1}^{N}G_{3ij}\boldsymbol{\Gamma}_3\int_{t-\tau_2(t)}^{t}\boldsymbol{x}_j(s)\mathrm{d}s+\boldsymbol{U}_i\Big]\mathrm{d}t, \\ & i=1,2,\ldots,N. \end{aligned} \tag{36}$$

We can obtain the following results.

Corollary 3 Under assumptions $(A_1)-(A_4)$, let the controller be $\boldsymbol{U}_i(t)=\boldsymbol{\varepsilon}_i(t)\boldsymbol{e}_i(t)$, where the feedback strengths $\boldsymbol{\varepsilon}_i(t)=(\varepsilon_{i1}(t),\varepsilon_{i2}(t),\ldots,\varepsilon_{in}(t))^{\mathrm{T}}$ with update laws are taken as

$$\begin{aligned} & \varepsilon_{ij}=-\delta_{ij}\boldsymbol{e}_{ij}^2(t), \\ & i=1,2,\ldots,N,\ \mathrm{J}=1,2,\ldots,n, \end{aligned} \tag{37}$$

in which δ_{ij} are arbitrary positive constants, i.e. $\delta_{ij}>0$. Then, system (5) is said to be synchronized, that is, the trivial solution of error system (5) is asymptotically stable.

Remark 2 If the linear coupling term is removed from the coupled chaotic neural networks (1), in the case where $N=1$, the complete synchronization of chaotic neural networks with discrete and distributed delays can also be realized via the adaptive feedback scheme, and the results derived here are equivalent to the results in Refs.[25, 26]. Moreover, the coupled system can be a general complex network, that is, it is not restricted to the chaotic neural networks. One can easily extend our results derived in this paper to more general cases.

Remark 3 Inspired by the ideas in Ref.[14], we investigate in detail, a more general model of stochastically hybrid coupled networks with coupling discrete and distributed time-varying delays in this paper. Note that continuously distributed delays have gained particular attention since a network usually has a spatial nature due to the presence of a number of parallel pathways of a variety of axon sizes and lengths. If both distributed delay and coupling distributed delay are removed from system (1), corollary 2 obtained in this paper is equivalent to the main theorem in Ref.[14].

4. Numerical simulation

In this section, an illustrative example is given to support the obtained results.

Example Consider the following chaotic neural networks with discrete and distributed time-varying delays:[25,26]

$$\begin{aligned}\mathrm{d}\boldsymbol{x}(t) = \Big[&- \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{x}(t)) + \boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{x}(t-\tau_1(t))) \\ &+ \boldsymbol{D}\int_{t-\tau_2(t)}^{t} \tilde{\boldsymbol{h}}(\boldsymbol{x}(s))\mathrm{d}s\Big]\mathrm{d}t \end{aligned} \tag{38}$$

with $\boldsymbol{x}(t) = (x_1(t), x_2(t))^{\mathrm{T}}$, $\tilde{\boldsymbol{f}}(\boldsymbol{x}(t)) = \tilde{\boldsymbol{g}}(\boldsymbol{x}(t)) = \tilde{\boldsymbol{h}}(\boldsymbol{x}(t)) = \tanh(\boldsymbol{x}(t)) = (\tanh(x_1),\ \tanh(x_2))^{\mathrm{T}}$, $\tau_1(t) = \dfrac{\boldsymbol{e}^t}{\boldsymbol{e}^t+1}$, $\tau_2(t) = 1$,

$$\boldsymbol{C} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \boldsymbol{A} = \begin{pmatrix} 1.8 & -0.15 \\ -5.2 & 3.5 \end{pmatrix},$$

$$\boldsymbol{B} = \begin{pmatrix} -1.7 & -0.12 \\ -0.26 & -2.5 \end{pmatrix}, \text{ and } \boldsymbol{D} = \begin{pmatrix} 0.6 & 0.15 \\ -2 & -0.12 \end{pmatrix}.$$

Then, the neural network model has a chaotic attractor with initial values $x_1(t) = 0.9, x_2(t) = 0.9$, and $\forall t \in [-1, 0]$ (see Fig.1).

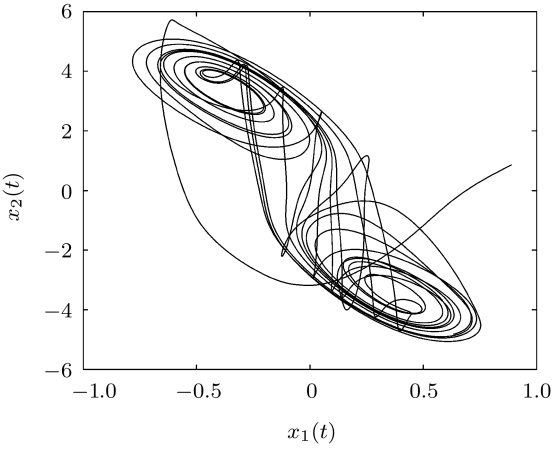


Fig.1. Chaotic trajectory of model described by expression (38).

The linearly stochastically hybrid coupled identical chaotic neural networks are considered as follows:

$$\begin{aligned}\mathrm{d}\boldsymbol{x}_i(t) = \Big[&-\boldsymbol{C}\boldsymbol{x}_i(t) + \boldsymbol{A}\tilde{\boldsymbol{f}}(\boldsymbol{x}_i(t)) + \boldsymbol{B}\tilde{\boldsymbol{g}}(\boldsymbol{x}_i(t-\tau_1(t))) \\ &+ \boldsymbol{D}\int_{t-\tau_2(t)}^{t} \tilde{\boldsymbol{h}}(\boldsymbol{x}_i(s))\mathrm{d}s + \boldsymbol{U}_i\Big]\mathrm{d}t \\ &+\alpha_i \sum_{j=1}^{N} G_{1ij}\boldsymbol{\Gamma}_1\boldsymbol{x}_j(t)\mathrm{d}w_{1i}(t) \\ &+\beta_i \sum_{j=1}^{N} G_{2ij}\boldsymbol{\Gamma}_2\boldsymbol{x}_j(t-\tau_1(t))\mathrm{d}w_{2i}(t) \\ &+\gamma_i \sum_{j=1}^{N} G_{3ij}\boldsymbol{\Gamma}_3 \int_{t-\tau_2(t)}^{t} \boldsymbol{x}_j(s)\mathrm{d}s\mathrm{d}w_{3i}(t) \\ &i = 1, 2, \ldots, N. \end{aligned} \tag{39}$$

where $\boldsymbol{x}_i(t) = [x_{i1}(t), x_{i2}(t)]^{\mathrm{T}}$ are the state variables of the ith chaotic neural networks. Here, we let the number of nodes is $N = 30$. Here we assume that the network structure of (1) obeys the scale-free distribution of the Barabási–Albert model.[2] For simplicity, the inner linking matrices are taken as

$$\boldsymbol{\Gamma}_1 = \boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The noise intensity is taken as $\alpha_i = \beta_i = \gamma_i = 0.03$, for $i = 1, 2, \ldots, 30$. In the simulations, the Euler–Maruyama numerical scheme is employed to simulate the stochastically hybrid coupled networks. The initial parameters are given as follows: $\sigma_{ij} = 30$, $x_{i1}(0) = 1 - 0.01 * i$, $x_{i2}(0) = 1 - 0.01 * i$; $\varepsilon_{i1}(0) = 3 + 0.1 * i$; $\varepsilon_{i2}(0) = 2 + 0.1 * i$.

The chaotic behaviour of system (38) is depicted in Fig.1. Figure 2 shows the state trajectories of linearly stochastically hybrid coupled networks (39) without control. One can easily see from Fig.3 that the state trajectories of stochastically coupled chaotic neural networks with coupling discrete and distributed delays. From Fig.4, the synchronization errors e_{i1} and $e_{i2}(i = 1, 2, \ldots, 30)$ between the stochastically hybrid coupled networks (39) with coupling discrete and distributed time-varying delays and the isolated chaotic neural networks (38). Figure 5 are the time evolution of the adaptive feedback ε_{i1} and ε_{i2} $(i = 1, 2, \ldots, 30)$. From these simulations, one can find that the synchronization in stochastically hybrid coupled networks with coupling discrete and distributed time-varying delays is achieved in mean square via the adaptive feedback scheme, and these simulations match the theoretical results perfectly.

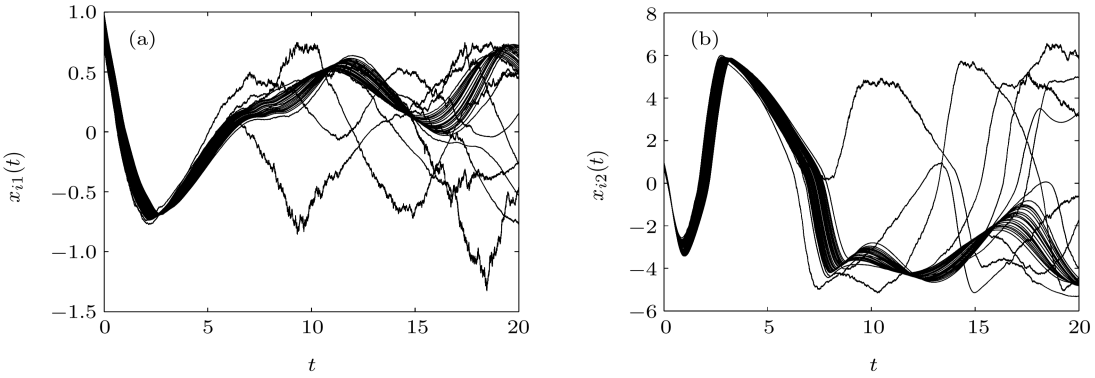


Fig.2. State trajectories of x_{i1} without control (a) and x_{i2} without control (b).

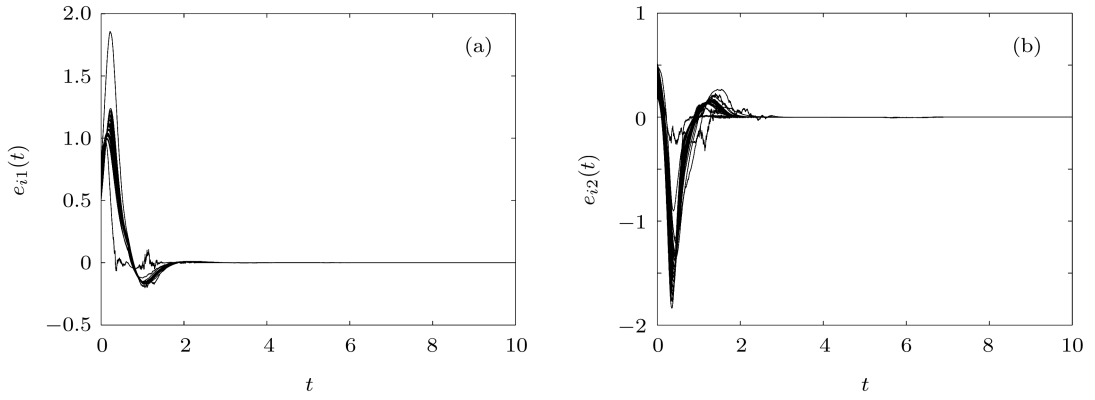


Fig.3. (a) Synchronization errors of e_{i1}, (b) synchronization errors of e_{i2}.

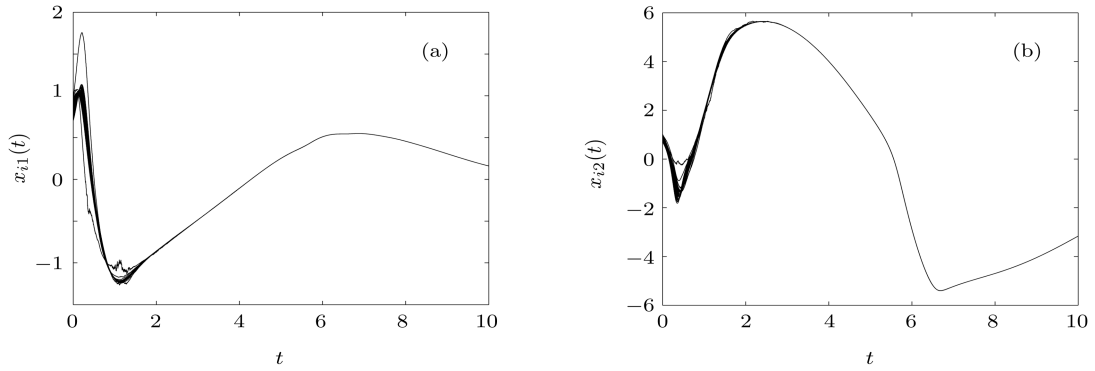


Fig.4. (a) State trajectory of x_{i1} under control, (b) state trajectory of x_{i2} under control.

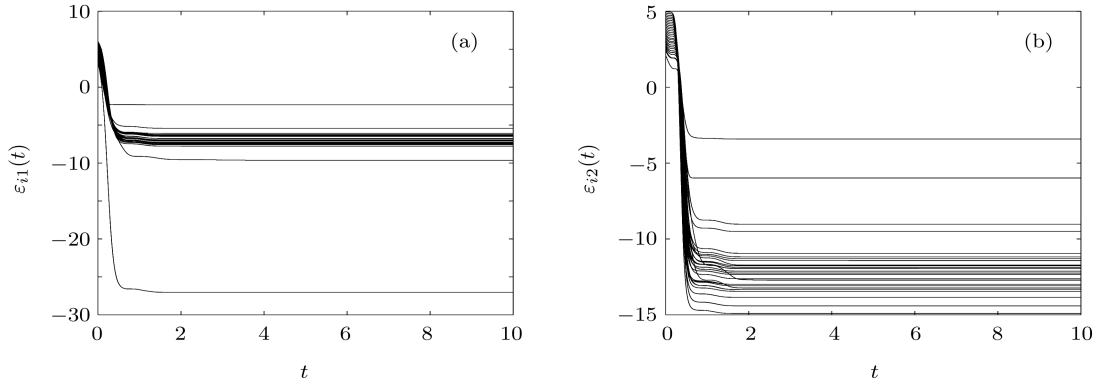


Fig.5. (a) Time evolution of ε_{i1}, (b) time evolution of ε_{i2}.

5. Conclusion

In this Letter, a more general model of linearly stochastically hybrid coupled networks is introduced, which reflects more realistic and complicated dynamical behaviours of complex networks. With the adaptive feedback approach, several sufficient conditions have been developed to guarantee the complete synchronization in an array of linearly stochastically hybrid coupled networks with both coupling discrete delay and distributed time-varying delay. The stochastic coupling terms are subjected to stochastic disturbances described in terms of Brownian motion. The variable feedback strength will be automatically adapted to a suitable strength. In the end, the numerical simulations are given to show the feasibility of the proposed approach.

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