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A unified approach to fuzzy modelling and robust synchronization of different hyperchaotic systems*

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In this paper, a Takagi–Sugeno (T–S) fuzzy model-based method is proposed to deal with the problem of synchronization of two identical or different hyperchaotic systems. The T–S fuzzy models with a small number of fuzzy IF–THEN rules are employed to represent many typical hyperchaotic systems exactly. The benefit of employing the T–S fuzzy models lies in mathematical simplicity of analysis. Based on the T–S fuzzy hyperchaotic models, two fuzzy controllers are designed via parallel distributed compensation (PDC) and exact linearization (EL) techniques to synchronize two identical hyperchaotic systems with uncertain parameters and two different hyperchaotic systems, respectively. The sufficient conditions for the robust synchronization of two identical hyperchaotic systems with uncertain parameters and the asymptotic synchronization of two different hyperchaotic systems are derived by applying the Lyapunov stability theory. This method is a universal one of synchronizing two identical or different hyperchaotic systems. Numerical examples are given to demonstrate the validity of the proposed fuzzy model and hyperchaotic synchronization scheme.

Keywords: hyperchaotic synchronization, fuzzy modelling, T–S fuzzy model, exact linearization (EL)

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1. Introduction

Since the pioneer work of Carroll and Pecora,[1] the synchronization of two chaotic dynamical systems becomes one of the most important applications of chaos and has received more and more attention due to its potential applications in creating secure communication systems.[2] Recently, many new hyperchaotic systems were found or created by researchers.[3−23] It should be noted that the presence of more than one positive Lyapunov exponent clearly improves the security by generating more complex dynamics. Therefore, the synchronization of two hyperchaotic systems is more meaningful and has been studied extensively in the last few years. Various control methods have been applied to synchronizing two hyperchaotic (or chaotic) systems, such as sampled-data feedback control,[19] adaptive control,[24−28] delayed feedback control,[29] impulsive control,[30,31] active backstepping control,[32] H_∞ control,[33] etc. Among these synchronization methods, some need to satisfy the Lipschitz conditions of nonlinear terms[34] and sometimes require very high gains.[35] Most of them deal with one or two kinds of specified hyperchaotic systems. There are few unified methods of synchronizing hyperchaotic systems.

Takagi–Sugeno (T–S) fuzzy model proposed in Ref.[36] is widely applied to many fields because of its simple structure with local dynamics. The analysis of a chaotic or hyperchaotic system via a fuzzy model allows a better understanding of the system itself, and even generating a new type of chaotic or hyperchaotic system. It is well-known that chaotic or hyperchaotic systems are extremely sensitive to the system parameters. Therefore, it is meaningless to use the T–S fuzzy models to represent chaotic or hyper-

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chaotic systems approximately. Fortunately, Tanaka *et al*[37] presented an exact T–S fuzzy model of many kinds of typical chaotic systems and Lian *et al*[38] presented a synthesis approach to the synchronization of chaotic systems by using the T–S fuzzy model-based design methods. This provides a theoretical foundation for applying the fuzzy control method to chaos control and the synchronization. But for hyperchaotic systems, to the best of authors' knowledge, there are few results about this work except Refs.[20, 39], in which only one or two particular hyperchaotic systems were represented by using the T–S fuzzy models. Furthermore, they did not present a general method to describe hyperchaotic systems by using the T–S fuzzy models exactly. Therefore, in this paper, we will give a systematic scheme to model many well-known hyperchaotic systems by using the T–S fuzzy models exactly.

In this paper we will solve the following problem: how to choose appropriate membership functions and how to set up appropriate fuzzy rules, such that an exact T–S fuzzy model of a hyperchaotic systems can be derived. This paper shows that a simple T–S fuzzy system with few fuzzy sets and fuzzy rules can model a complex hyperchaotic system exactly. The second contribution of the paper is that the theoretical foundation for applying the fuzzy control to the control and the synchronization of hyperchaotic systems is provided. Based on these T–S fuzzy hyperchaotic models, we design two fuzzy controllers to realize the synchronization of two identical and different hyperchaotic systems, respectively. By applying the Lyapunov stability theory and the exact linearization (EL) technique, we derive the sufficient conditions of the robust synchronization of two identical hyperchaotic systems with uncertain parameters and the asymptotic synchronization of two different hyperchaotic systems, respectively.

The rest of this paper is organized as follows. In Section 2, some preliminaries for the T–S fuzzy model are presented. Moreover, a systematic modelling method is presented to describe hyperchaotic systems by the T–S fuzzy model exactly. The exact T–S fuzzy models of many kinds of well-known hyperchaotic systems can be derived by using this method. Based on these T–S fuzzy hyperchaotic models, the robust synchronization of two identical hyperchaotic systems with uncertain parameters and the asymptotic synchronization of two different are achieved by designing fuzzy controllers in Section 3. The effectiveness of the proposed fuzzy modelling and synchronization approach is demonstrated through some numerical examples in Section 4. Some conclusions are finally drawn in Section 5.

2. Fuzzy modelling of hyperchaotic systems

2.1. T–S fuzzy model

Consider the continuous-time T–S fuzzy rule base described as follows:

R^{i}: IF $p_1(t)$ is M_{i1}, ..., and $p_q(t)$ is M_{iq}, THEN

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{b}, \tag{1}$$

where $i = 1, 2, \ldots, r$ (r is the number of fuzzy rules), $\boldsymbol{x}(t) \in \boldsymbol{R}^n$ and $\boldsymbol{u}(t) \in \boldsymbol{R}^m$ denote the state vector and the input vector, respectively, $\boldsymbol{A}_i \in \boldsymbol{R}^{n\times n}$ and $\boldsymbol{B}_i \in \boldsymbol{R}^{n\times m}$ are known system matrix and input matrix with appropriate dimensions, respectively, $p_1(t)$, $p_2(t)$, ..., $p_q(t)$ are premise variables, M_{ij} is a fuzzy set ($j = 1, 2, \ldots, q$), and $\boldsymbol{b} \in \boldsymbol{R}^n$ is a constant vector. By taking a standard fuzzy inference strategy, i.e. using a singleton fuzzifier, product fuzzy inference and centre average defuzzifier, the final continuous-time fuzzy T–S system is inferred as follows:

$$\dot{\boldsymbol{x}}(t) = \frac{\sum_{i=1}^{r} \omega_i(\boldsymbol{p}(t)) \left(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{b}\right)}{\sum_{i=1}^{r} \omega_i(\boldsymbol{p}(t))}, \tag{2}$$

where

$$\omega_i(\boldsymbol{p}(t)) = \prod_{j=1}^{q} M_{ij}(p_j(t)),$$

with $M_{ij}(p_j(t))$ being the degree of membership of $p_j(t)$ in M_{ij}, and

$$\begin{cases} \sum\limits_{i=1}^{r} \omega_i(\boldsymbol{p}(t)) > 0, \\ \omega_i(\boldsymbol{p}(t)) \geq 0. \end{cases} \quad i = 1, 2, \ldots, r.$$

By using $\mu_i(\boldsymbol{p}(t))(= \omega_i(\boldsymbol{p}(t))/\sum_{i=1}^{r} \omega_i(\boldsymbol{p}(t)))$ instead of $\omega_i(\boldsymbol{p}(t))$, Eq.(2) is rewritten as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \mu_i(\boldsymbol{p}(t)) \left(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{B}_i\boldsymbol{u}(t) + \boldsymbol{b}\right). \tag{3}$$

Note that

$$\begin{cases} \sum\limits_{i=1}^{r} \mu_i(\boldsymbol{p}(t)) = 1, \\ \mu_i(\boldsymbol{p}(t)) \geq 0, \end{cases} \quad i = 1, 2, \ldots, r,$$

where $\mu_i(\boldsymbol{p}(t))$ can be regarded as the firing strength of the ith IF–THEN rules.

2.2. A unified fuzzy modelling method of hyperchaotic systems

In this subsection, we investigate the problem of T–S fuzzy modelling of hyperchaotic systems. In the following hyperchaotic systems, $x_1(t)$, $x_2(t)$, $x_3(t)$ and $x_4(t)$ are state variables, a, c, d, l and k are system parameters. Assume that $x_1(t) \in [\tilde{c}_1 - \tilde{d}_1, \tilde{c}_1 + \tilde{d}_1]$, $x_2(t) \in [\tilde{c}_2 - \tilde{d}_2, \tilde{c}_2 + \tilde{d}_2]$ and $\tilde{d}_1 > 0$, $\tilde{d}_2 > 0$. According to the property of boundedness of hyperchaotic systems, it is obvious that the above assumption is reasonable. The exact values of $\tilde{c}_1$, $\tilde{d}_1$, $\tilde{c}_2$ and $\tilde{d}_2$ can be obtained by simulation. Since there is no input in the autonomous hyperchaotic systems, the term $\boldsymbol{B}_i\boldsymbol{u}(t)$ in the T–S fuzzy model is omitted in the process of fuzzy modelling of hyperchaotic systems. Note that most hyperchaotic systems can be expressed as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{f}(\boldsymbol{x}(t)) + \tilde{\boldsymbol{b}}, \tag{4}$$

where $\boldsymbol{x}(t)$ is the state vector of the hyperchaotic systems, $\boldsymbol{A}\boldsymbol{x}(t)$ and $\boldsymbol{f}(\boldsymbol{x}(t))$ denote the linear part and the nonlinear part of the hyperchaotic systems respectively, $\boldsymbol{A}$ is a known matrix, $\tilde{\boldsymbol{b}}$ is the constant vector of the hyperchaotic systems. In the following, the systematic T–S fuzzy modelling methods of hyperchaotic systems are summarized into three cases according to the different forms of the nonlinear part $f(x(t))$.

Case 1 Only one nonlinear term lies in $f(x)$. For this case, we can choose any state variable in the nonlinear term as the premise variable. For example, in the following modified Rössler hyperchaotic system:[21]

$$\begin{aligned}
\dot{x}_1(t) &= -x_2(t) - x_3(t),\\
\dot{x}_2(t) &= x_1(t) + ax_2(t) + x_4(t),\\
\dot{x}_3(t) &= k + lx_1(t) + x_1(t)x_3(t),\\
\dot{x}_4(t) &= -cx_3(t) + dx_4(t),
\end{aligned} \tag{5}$$

the unique nonlinear term is $x_1(t)x_3(t)$. Then we can choose either $x_1(t)$ or $x_3(t)$ as the premise variable. Without loss of generality, we choose $x_1(t)$ as the premise variable. Then the membership functions are defined as $F_{11}(x_1(t)) = \dfrac{1}{2}\left(1 - \dfrac{\tilde{c}_1 - x_1(t)}{\tilde{d}_1}\right)$, and $F_{21}(x_1(t)) = \dfrac{1}{2}\left(1 + \dfrac{\tilde{c}_1 - x_1(t)}{\tilde{d}_1}\right)$, and the fuzzy IF–THEN rules are defined as

R^1: IF $x_1(t)$ is F_{11}, THEN $\dot{x}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{b}$, and

R^2: IF $x_1(t)$ is F_{21}, THEN $\dot{x}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{b}$, where $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^{\mathrm{T}}$. For the above fuzzy rule base, by taking a standard fuzzy inference strategy, i.e. using a singleton fuzzifier, product fuzzy inference and centre average defuzzifier, we can obtain the following T–S fuzzy system:

$$\dot{\boldsymbol{x}}(t) = \frac{\sum_{i=1}^{2} F_{i1}(x_1(t))(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{b})}{\sum_{i=1}^{2} F_{i1}(x_1(t))}. \tag{6}$$

The remaining task is to choose appropriate values of $\boldsymbol{A}_1$, $\boldsymbol{A}_2$ and $\boldsymbol{b}$ such that Eq.(6) can be equivalent to Eq.(4). For system (5), each part of Eq.(4) is presented as follows:

$$\boldsymbol{A} = \begin{pmatrix} 0 & -1 & -1 & 0\\ 1 & a & 0 & 1\\ l & 0 & 0 & 0\\ 0 & 0 & -c & d \end{pmatrix}, \quad \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} 0\\ 0\\ x_1(t)x_3(t)\\ 0 \end{pmatrix},$$

and

$$\tilde{\boldsymbol{b}} = (0 \;\; 0 \;\; k \;\; 0)^{\mathrm{T}}.$$

Denote $\boldsymbol{A}_1 = \boldsymbol{A}_1' + \boldsymbol{A}_1''$, and $\boldsymbol{A}_2 = \boldsymbol{A}_2' + \boldsymbol{A}_2''$, and let $\boldsymbol{A}_1' = \boldsymbol{A}_2' = \boldsymbol{A}$, and $\boldsymbol{b} = \tilde{\boldsymbol{b}}$. Substituting them into Eq.(6), and equating Eq.(6) to Eq.(4), we can obtain the values of $\boldsymbol{A}_1''$ and $\boldsymbol{A}_2''$ as follows:

$$\boldsymbol{A}_1'' = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \tilde{c}_1 + \tilde{d}_1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and

$$\boldsymbol{A}_2'' = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & \tilde{c}_1 - \tilde{d}_1 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Consequently, we derive the exact T–S fuzzy model of hyperchaotic system (5).

Case 2 More than one nonlinear term lies in $f(x)$ with a common factor in these nonlinear terms. For this case, we can choose the common factor in the nonlinear terms as the premise variable. The remaining process of T–S fuzzy modelling of this class of hyperchaotic system is similar to that in Case 1, and therefore it is omitted.

Case 3 More than one nonlinear term lies in $f(x)$ without a common factor in these nonlinear terms. For this case, we have to choose two state variables occurring in $f(x)$ as the premise variables. For exam-

ple, in the following Chen hyperchaotic system:[3,25,28]

$$\begin{aligned}
\dot{x}_1(t) &= a(x_2(t) - x_1(t)) + x_4(t),\\
\dot{x}_2(t) &= dx_1(t) - x_1(t)x_3(t) + cx_2(t),\\
\dot{x}_3(t) &= x_1(t)x_2(t) - kx_3(t),\\
\dot{x}_4(t) &= x_2(t)x_3(t) + lx_4(t),
\end{aligned} \tag{7}$$

the nonlinear terms are $x_1(t)x_3(t)$, $x_1(t)x_2(t)$ and $x_2(t)x_3(t)$. Then we can choose any two state variables among from $x_1(t)$, $x_2(t)$ and $x_3(t)$. Without loss of generality, we choose x_1 and x_2 as the premise variables and the membership functions are defined as

$$\begin{aligned}
F_{11}(x_1(t)) &= \frac{1}{2}\left(1 - \frac{\tilde{c}_1 - x_1(t)}{\tilde{d}_1}\right),\\
F_{21}(x_1(t)) &= \frac{1}{2}\left(1 + \frac{\tilde{c}_1 - x_1(t)}{\tilde{d}_1}\right),\\
F_{32}(x_2(t)) &= \frac{1}{2}\left(1 - \frac{\tilde{c}_2 - x_2(t)}{\tilde{d}_2}\right),\\
F_{42}(x_2(t)) &= \frac{1}{2}\left(1 + \frac{\tilde{c}_2 - x_2(t)}{\tilde{d}_2}\right),\\
F_{12}(x_2(t)) &= F_{22}(x_2(t)) = 1,
\end{aligned}$$

and

$$F_{31}(x_1(t)) = F_{41}(x_1(t)) = 1.$$

Furthermore, the following fuzzy IF–THEN rules are presented:

R^1: IF $x_1(t)$ is F_{11} and $x_2(t)$ is F_{12}, THEN $\dot{x}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{b}$,

R^2: IF $x_1(t)$ is F_{21} and $x_2(t)$ is F_{22}, THEN $\dot{x}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{b}$,

R^3: IF $x_1(t)$ is F_{31} and $x_2(t)$ is F_{32}, THEN $\dot{x}(t) = \boldsymbol{A}_3\boldsymbol{x}(t) + \boldsymbol{b}$,

and

R^4: IF $x_1(t)$ is F_{41} and $x_2(t)$ is F_{42}, THEN $\dot{x}(t) = \boldsymbol{A}_4\boldsymbol{x}(t) + \boldsymbol{b}$,

where $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^{\mathrm{T}}$. For the above fuzzy rule base, by taking a standard fuzzy inference strategy, i.e. using a singleton fuzzifier, product fuzzy inference and centre average defuzzifier, we can obtain the following T–S fuzzy system:

$$\dot{\boldsymbol{x}}(t) = \frac{\sum_{i=1}^{4}\prod_{j=1}^{2}F_{ij}\left(x_j(t)\right)\left(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{b}\right)}{\sum_{i=1}^{4}\prod_{j=1}^{2}F_{ij}\left(x_j(t)\right)}. \tag{8}$$

The rest of task is to choose appropriate values of $\boldsymbol{A}_i$ and $\boldsymbol{b}_i$ $(i = 1, 2, 3, 4)$ such that Eq.(8) can be equivalent to Eq.(4). For system (7), each part of Eq.(4) is presented as follows:

$$\boldsymbol{A} = \begin{pmatrix} -a & a & 0 & 1\\ d & c & 0 & 0\\ 0 & 0 & -k & 0\\ 0 & 0 & 0 & l \end{pmatrix},\ \boldsymbol{f}(\boldsymbol{x}) = \begin{pmatrix} 0\\ -x_1(t)x_3(t)\\ x_1(t)x_2(t)\\ x_2(t)x_3(t) \end{pmatrix},$$

and

$$\tilde{\boldsymbol{b}} = (0\ \ 0\ \ 0\ \ 0)^{\mathrm{T}}.$$

Denote $\boldsymbol{A}_i = \boldsymbol{A}_i' + \boldsymbol{A}_i''$, and let $\boldsymbol{A}_i' = \boldsymbol{A}$ $(i = 1, 2, 3, 4)$, and $\boldsymbol{b} = \tilde{\boldsymbol{b}}$. Substituting them into Eq.(8), and equating Eq.(8) to Eq.(4), we can obtain the values of $\boldsymbol{A}_i''$ as follows:

$$\boldsymbol{A}_1'' = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 2(-\tilde{c}_1 - \tilde{d}_1) & 0\\ 0 & 2(\tilde{c}_1 + \tilde{d}_1) & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{A}_2'' = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 2(-\tilde{c}_1 + \tilde{d}_1) & 0\\ 0 & 2(\tilde{c}_1 - \tilde{d}_1) & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\boldsymbol{A}_3'' = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 2(\tilde{c}_2 + \tilde{d}_2) & 0 \end{pmatrix},$$

and

$$\boldsymbol{A}_4'' = \begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 2(\tilde{c}_2 - \tilde{d}_2) & 0 \end{pmatrix}.$$

Consequently, we derive the exact T–S fuzzy model of hyperchaotic system (7).

To summarize, by using the proposed fuzzy modelling method, most hyperchaotic systems can be represented by the following T–S fuzzy models exactly:

$\boldsymbol{R}^i$:

IF $p_1(t)$ is F_{i1}, ..., and $p_q(t)$ is F_{iq}, THEN $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{b}$,

where $i = 1, 2, \ldots, r$, the premise variable $p_j(t)$ is an proper state variable, $\boldsymbol{b}$ is a constant vector, r is the number of fuzzy IF–THEN rules. Then, applying the product-inference rule, singleton fuzzifier, and the centre of gravity defuzzifier to the above fuzzy rule base,

an overall fuzzy hyperchaotic system can be inferred as

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \mu_i(\boldsymbol{p}(t))(\boldsymbol{A}_i\boldsymbol{x}(t) + \boldsymbol{b}), \tag{9}$$

where

$$\mu_i(\boldsymbol{p}(t)) = \prod_{j=1}^{q} F_{ij}(p_j(t)) \Big/ \sum_{i=1}^{r} \left(\prod_{j=1}^{q} F_{ij}(p_j(t)) \right).$$

Remark 1 Most hyperchaotic systems can be represented by the T–S fuzzy models exactly using the proposed fuzzy modelling method in this paper. For example, the original Rössler hyperchaotic system[18,20] and other modified Rössler hyperchaotic systems[21] belong in Case 1. The Lorenz hyperchaotic system,[5] the generalized Lorenz hyperchaotic system,[39] the Liu hyperchaotic system,[13] the Lü hyperchaotic system (i),[14,19] the Lü hyperchaotic system (ii)[4] and a new hyperchaotic system[15] belong in Case 2. The modified Lü hyperchaotic system,[22] the generalized Henon–Heiles system[16] and another new hyperchaotic system[23] belong in Case 3. Therefore, the exact T–S fuzzy models of most hyperchaotic systems can be achieved by using the fuzzy modelling method in this paper. This plays an important role in applying the fuzzy control theory to the control and the synchronization of hyperchaotic systems.

Remark 2 Presented in Ref.[38] are some fuzzy modelling methods of chaotic systems, which are suitable for many typical chaotic systems. They can be applied to Case 1 and Case 2 in this paper. But they are ineffective for Case 3 in this paper.

Remark 3 In Ref.[20], the authors assumed that $x \in [c-d, c+d]$, which is reasonable. But they chose the membership functions as $M_1(x) = \frac{1}{2}\left(1 + \frac{x}{d}\right)$ and $M_2(x) = \frac{1}{2}\left(1 - \frac{x}{d}\right)$. Obviously this is not reasonable. If $c > 0$, when $x = c + d$, $M_1(x) = \frac{1}{2}\left(1 + \frac{c+d}{d}\right)$ will exceed 1. If $c < 0$, when $x = c - d$, $M_2(x) = \frac{1}{2}\left(1 - \frac{c-d}{d}\right)$ will exceed 1. In fact, as membership functions, $M_1(x)$ and $M_2(x)$ cannot exceed 1.

3. Robust synchronization of two hyperchaotic systems via T–S fuzzy models

In this section, we first investigate the robust synchronization of two identical hyperchaotic systems with uncertain parameters based on the T–S fuzzy models. The interval matrix theory is applied to dealing with the parametric uncertainty. Then, we discuss the asymptotic synchronization of two different hyperchaotic systems based on the T–S fuzzy models.

3.1. Robust synchronization of two identical hyperchaotic systems with uncertain parameters

Consider a fuzzy hyperchaotic system with uncertain parameters as a drive system

$$\dot{\boldsymbol{x}}(t) = \sum_{i=1}^{r} \mu_i(\boldsymbol{p}(t)) \left(\tilde{\boldsymbol{A}}_i\boldsymbol{x}(t) + \boldsymbol{b} \right), \tag{10}$$

and the following fuzzy hyperchaotic system as a response system

$$\dot{\hat{\boldsymbol{x}}}(t) = \sum_{i=1}^{r} \mu_i(\hat{\boldsymbol{p}}(t)) \left(\tilde{\boldsymbol{A}}'_i\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{b} \right), \tag{11}$$

where $\tilde{\boldsymbol{A}}_i$ and $\tilde{\boldsymbol{A}}'_i$ are interval matrices, which denote the uncertain parameters. In this paper, we assume that $\tilde{\boldsymbol{A}}_i$ and $\tilde{\boldsymbol{A}}'_i$ have the same upper and lower bounds for each i $(i = 1, 2, \ldots, r)$, i.e.

$$\tilde{\boldsymbol{A}}_i \in N[\boldsymbol{U}_i, \boldsymbol{V}_i] = \left\{ \tilde{\boldsymbol{A}}_i \in \boldsymbol{R}^{n\times n} | u_{ijk} \le a_{ijk} \le v_{ijk} \right\},$$

and

$$\tilde{\boldsymbol{A}}'_i \in N[\boldsymbol{U}_i, \boldsymbol{V}_i] = \left\{ \tilde{\boldsymbol{A}}'_i \in \boldsymbol{R}^{n\times n} | u_{ijk} \le a'_{ijk} \le v_{ijk} \right\},$$

where $j, k = 1, 2, \ldots, n$, u_{ijk} and v_{ijk} are the lower and upper bounds of a_{ijk} or a'_{ijk}, respectively.

Definition 1 Drive system (10) and response system (11) are robustly synchronized if drive system (10) and response system (11) are asymptotically synchronized for any uncertain parameters $\tilde{\boldsymbol{A}}_i, \tilde{\boldsymbol{A}}'_i \in N[\boldsymbol{U}_i, \boldsymbol{V}_i]$ $(i = 1, 2, \ldots, r)$.

Lemma 1[40] The interval matrix $\tilde{\boldsymbol{A}}_i \in N[\boldsymbol{U}_i, \boldsymbol{V}_i]$ is equivalent to the following form:

$$\tilde{\boldsymbol{A}}_i = \boldsymbol{A}_{0i} + \boldsymbol{E}_i\boldsymbol{\Sigma}\boldsymbol{F}_i, \ \boldsymbol{\Sigma} \in \boldsymbol{\Sigma}^*,$$

where $A_{0i} = (\boldsymbol{U}_i + \boldsymbol{V}_i)/2$, $\boldsymbol{G}_i = (g_{ijk})_{n\times n} = (\boldsymbol{V}_i - \boldsymbol{U}_i)/2$, $\boldsymbol{E}_i = (\sqrt{g_{i11}}e_1, \ldots, \sqrt{g_{i1n}}e_1, \ldots, \sqrt{g_{in1}}e_n, \ldots, \sqrt{g_{inn}}e_n)$, and $\boldsymbol{F}_i = (\sqrt{g_{i11}}e_1, \ldots, \sqrt{g_{i1n}}e_n, \ldots, \sqrt{g_{in1}}e_1, \ldots, \sqrt{g_{inn}}e_n)^{\mathrm{T}}$, with $\boldsymbol{e}_j$ $(j = 1, 2, \ldots, n)$ being the jth column vector of the identity matrix $\boldsymbol{I}_{n\times n}$, and $\boldsymbol{\Sigma}^* = \{\boldsymbol{\Sigma} \in \boldsymbol{R}^{n\times n} | \boldsymbol{\Sigma} = \mathrm{diag}\,(\varepsilon_{11}, \ldots, \varepsilon_{1n}, \ldots, \varepsilon_{n1}, \ldots, \varepsilon_{nn}), |\varepsilon_{jk}| \le 1\}$, in which $j, k = 1, \ldots, n$.

Assumption 1 In this paper, all the matrices $\boldsymbol{E}_i$ are identical and all the matrices $\boldsymbol{F}_i$ are identical.

From the process of fuzzy modelling of hyperchaotic systems, we can see that the above assumption holds for most hyperchaotic systems. Therefore, we can denote $\boldsymbol{E}_i = \boldsymbol{E}$ and $\boldsymbol{F}_i = \boldsymbol{F}$.

Lemma 2[40] For any given column vectors with appropriate dimensions ξ and η, there exists an arbitrary constant $\gamma > 0$ such that

$$2\xi^{\mathrm{T}}\boldsymbol{\Sigma}\eta \leq \gamma\xi^{\mathrm{T}}\xi + \gamma^{-1}\eta^{\mathrm{T}}\eta,\ \forall \boldsymbol{\Sigma} \in \boldsymbol{\Sigma}^*, \tag{12}$$

where $\boldsymbol{\Sigma}^*$ is defined as the same as that of Lemma 1.

Based on the parallel distributed compensation (PDC) technique, a fuzzy controller can be designed to realize the synchronization as follows:

Subcontroller $\boldsymbol{u}_{\mathrm{d}}(t)$:

R^i: IF $p_1(t)$ is F_{i1}, ..., and $p_q(t)$ is F_{iq}, THEN $\boldsymbol{u}_{\mathrm{d}}(t) = \bar{\boldsymbol{K}}_i\boldsymbol{x}(t),\ i = 1, 2, \ldots, r$.

Subcontroller $\hat{u}(t)$:

R^i: IF $\hat{p}_1(t)$ is F_{i1}, ..., and $\hat{p}_q(t)$ is F_{iq}, THEN $\hat{\boldsymbol{u}}(t) = -\bar{\boldsymbol{K}}_i\hat{\boldsymbol{x}}(t),\ i = 1, 2, \ldots, r$.

An overall fuzzy controller is constructed by the parallel connection, i.e.

$$\begin{aligned}\boldsymbol{u}(t) &= \boldsymbol{u}_{\mathrm{d}}(t) + \hat{\boldsymbol{u}}(t)\\ &= \sum_{i=1}^{r}\mu_i(\boldsymbol{p}(t))\bar{\boldsymbol{K}}_i\boldsymbol{x}(t)\\ &\quad -\sum_{i=1}^{r}\mu_i\left(\hat{\boldsymbol{p}}(t)\right)\bar{\boldsymbol{K}}_i\hat{\boldsymbol{x}}(t).\end{aligned} \tag{13}$$

Denote the error signal as $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$. Substituting expression (13) into Eq.(11), we can derive a closed-loop synchronization error system as follows:

$$\begin{aligned}\dot{\boldsymbol{e}}(t) &= \sum_{i=1}^{r}\mu_i(\boldsymbol{p}(t))\left(\tilde{\boldsymbol{A}}_i - \boldsymbol{B}\bar{\boldsymbol{K}}_i\right)\boldsymbol{x}(t)\\ &\quad -\sum_{i=1}^{r}\mu_i\left(\hat{\boldsymbol{p}}(t)\right)\left(\tilde{\boldsymbol{A}}'_i - \boldsymbol{B}\bar{\boldsymbol{K}}_i\right)\hat{\boldsymbol{x}}(t).\end{aligned} \tag{14}$$

According to Lemma 1 and Assumption 1, the error dynamics can be transformed into

$$\begin{aligned}\dot{\boldsymbol{e}}(t) &= \sum_{i=1}^{r}\mu_i(\boldsymbol{p}(t))\left(\boldsymbol{A}_{0i} - \boldsymbol{B}\bar{\boldsymbol{K}}_i + \boldsymbol{E}\boldsymbol{\Sigma}\boldsymbol{F}\right)\boldsymbol{x}(t)\\ &\quad -\sum_{i=1}^{r}\mu_i\left(\hat{\boldsymbol{p}}(t)\right)\left(\boldsymbol{A}_{0i} - \boldsymbol{B}\bar{\boldsymbol{K}}_i + \boldsymbol{E}\boldsymbol{\Sigma}\boldsymbol{F}\right)\hat{\boldsymbol{x}}(t).\end{aligned} \tag{15}$$

Assume that $\boldsymbol{B}$ is nonsingular. Then we always can find out a matrix $\bar{\boldsymbol{G}}$ by choosing appropriate control gain matrices $\bar{\boldsymbol{K}}_i$, such that

$$\boldsymbol{A}_{01} - \boldsymbol{B}\bar{\boldsymbol{K}}_1 = \boldsymbol{A}_{0i} - \boldsymbol{B}\bar{\boldsymbol{K}}_i = \bar{\boldsymbol{G}}, \tag{16}$$

where $i = 2, 3, \ldots, r$. Then the error dynamics (15) can be exactly linearized into

$$\dot{\boldsymbol{e}}(t) = \left(\bar{\boldsymbol{G}} + \boldsymbol{E}\boldsymbol{\Sigma}\boldsymbol{F}\right)\boldsymbol{e}(t). \tag{17}$$

Theorem 1 The error dynamics is robustly stable if the following linear matrix inequality (LMI) is satisfied:

$$\begin{pmatrix}\boldsymbol{X} + \boldsymbol{X}^{\mathrm{T}} + \gamma\boldsymbol{E}\boldsymbol{E}^{\mathrm{T}} & \boldsymbol{Q}\boldsymbol{F}^{\mathrm{T}}\\ \boldsymbol{F}\boldsymbol{Q} & -\gamma\boldsymbol{I}\end{pmatrix} < 0, \tag{18}$$

where $\boldsymbol{Q} = \boldsymbol{P}^{-1}$, $\boldsymbol{P}$ is a positive definite matrix, $\boldsymbol{X} = \bar{\boldsymbol{G}}\boldsymbol{Q}$, $\boldsymbol{I}$ denotes the identity matrix and γ is a given positive constant i.e. $\gamma > 0$.

Proof Now, we choose the Lyapunov function as

$$V(t) = \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{e}(t).$$

The time derivative of $V(t)$ along the trajectory of Eq.(17) is given as

$$\begin{aligned}\dot{V}(t) &= 2\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\dot{\boldsymbol{e}}(t)\\ &= \boldsymbol{e}^{\mathrm{T}}(t)\left(\boldsymbol{P}\bar{\boldsymbol{G}} + \bar{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P}\right)\boldsymbol{e}(t) + 2\boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{E}\boldsymbol{\Sigma}\boldsymbol{F}\boldsymbol{e}(t).\end{aligned}$$

Denote $\xi^{\mathrm{T}} = \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{P}\boldsymbol{E}$, and $\eta = \boldsymbol{F}\boldsymbol{e}(t)$. According to Lemma 2, we can derive

$$\dot{V}(t) \leq \boldsymbol{e}^{\mathrm{T}}(t)\boldsymbol{\Psi}\boldsymbol{e}(t), \tag{19}$$

where $\boldsymbol{\Psi} = \boldsymbol{P}\bar{\boldsymbol{G}} + \bar{\boldsymbol{G}}^{\mathrm{T}}\boldsymbol{P} + \gamma\boldsymbol{P}\boldsymbol{E}\boldsymbol{E}^{\mathrm{T}}\boldsymbol{P} + \gamma^{-1}\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}$, ν is any given positive constant, i.e. $\gamma > 0$. From inequality (19), we can see that if $\boldsymbol{\Psi} < 0$ holds, we have $\dot{V}(t) < 0$. Therefore, the error dynamics (17) is robustly stable. According to the Schur complement lemma, the inequality $\boldsymbol{\Psi} < 0$ can be transformed into the LMI (18). This completes the proof.

Therefore, we can choose $\bar{\boldsymbol{K}}_i = \boldsymbol{B}^{-1}\left(\boldsymbol{A}_{0i} - \bar{\boldsymbol{G}}\right)$ to realize the robust synchronization of two identical hyperchaotic systems with uncertain parameters, i.e. drive system (10) and response system (11).

Remark 4 If $\boldsymbol{B}$ is nonsingular, the problem of the synchronization of two identical hyperchaotic systems with uncertain parameters can be presented as optimization problem (20), where $\boldsymbol{Q} = \boldsymbol{P}^{-1}$, $\boldsymbol{X}_i = \boldsymbol{K}_i\boldsymbol{Q}$, $i = 1, 2, \ldots, r$, and $k = 2, 3, \ldots, r$. If we can find out such a small positive value of ε, i.e. $\varepsilon > 0$, the two identical hyperchaotic systems with uncertain parameters will be synchronized.

$$\min_{\boldsymbol{Q},\boldsymbol{X}_1,\ldots,\boldsymbol{X}_r} \varepsilon$$
$$\text{s.t. } \varepsilon > 0,\ \boldsymbol{Q} > 0,$$
$$\begin{pmatrix} \boldsymbol{Q}\boldsymbol{A}_{0i}^{\mathrm{T}} + \boldsymbol{A}_{0i}\boldsymbol{Q} - \boldsymbol{X}_i^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}} - \boldsymbol{B}\boldsymbol{X}_i + \gamma\boldsymbol{E}\boldsymbol{E}^{\mathrm{T}} & \boldsymbol{Q}\boldsymbol{F}^{\mathrm{T}} \\ \boldsymbol{F}\boldsymbol{Q} & -\gamma\boldsymbol{I} \end{pmatrix} < 0,$$

$$\begin{pmatrix} -\varepsilon\boldsymbol{I} & (\boldsymbol{A}_{01}\boldsymbol{Q} - \boldsymbol{B}\boldsymbol{X}_1)^{\mathrm{T}} - (\boldsymbol{A}_{0k}\boldsymbol{Q} - \boldsymbol{B}\boldsymbol{X}_k)^{\mathrm{T}} \\ (\boldsymbol{A}_{01}\boldsymbol{Q} - \boldsymbol{B}\boldsymbol{X}_1) - (\boldsymbol{A}_{0k}\boldsymbol{Q} - \boldsymbol{B}\boldsymbol{X}_k) & -\boldsymbol{I} \end{pmatrix} < 0. \tag{20}$$

3.2. Asymptotic synchronization of two different hyperchaotic systems

Consider fuzzy hyperchaotic system (9) as a drive system, and the following fuzzy hyperchaotic system as a response system:

$$\dot{\hat{\boldsymbol{x}}}(t) = \sum_{k=1}^{l} h_k\left(\hat{\boldsymbol{p}}(t)\right)\left(\hat{\boldsymbol{A}}_k\hat{\boldsymbol{x}}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \hat{\boldsymbol{b}}\right). \tag{21}$$

Assumption 2 The constant term $\boldsymbol{b}$ in Eq.(9) is equal to $\hat{\boldsymbol{b}}$ in Eq.(21).

Since the constant terms $\boldsymbol{b}$ and $\hat{\boldsymbol{b}}$ are equal to zero in most hyperchaotic systems, the above assumption holds in most cases. Based on the PDC technique, a fuzzy controller can be designed to realize the synchronization as follows:

Subcontroller $\boldsymbol{u}_{\mathrm{d}}(t)$:

R^i: IF $p_1(t)$ is F_{i1}, ..., and $p_q(t)$ is F_{iq}, THEN $\boldsymbol{u}_{\mathrm{d}}(t) = \boldsymbol{K}_i\boldsymbol{x}(t), \quad i = 1, 2, \ldots, r.$

Subcontroller $\hat{\boldsymbol{u}}(t)$:

R^k: IF $\hat{p}_1(t)$ is $\hat{F}_{k1}$, ..., and $\hat{p}_g(t)$ is $\hat{F}_{kg}$, THEN $\hat{\boldsymbol{u}}(t) = -\boldsymbol{H}_k\hat{\boldsymbol{x}}(t), \quad k = 1, 2, \ldots, l.$

An overall fuzzy controller is constructed by the parallel connection, i.e.

$$\begin{aligned} \boldsymbol{u}(t) &= \boldsymbol{u}_{\mathrm{d}}(t) + \hat{\boldsymbol{u}}(t) \\ &= \sum_{i=1}^{r} \mu_i(\boldsymbol{p}(t))\boldsymbol{K}_i\boldsymbol{x}(t) \\ &\quad - \sum_{k=1}^{l} h_k\left(\hat{\boldsymbol{p}}(t)\right)\boldsymbol{H}_k\hat{\boldsymbol{x}}(t). \end{aligned} \tag{22}$$

Denote the error signal as $\boldsymbol{e}(t) = \boldsymbol{x}(t) - \hat{\boldsymbol{x}}(t)$. Substituting expression (22) into Eq.(21), we can derive a closed-loop synchronization error system as follows:

$$\begin{aligned} \dot{\boldsymbol{e}}(t) &= \sum_{i=1}^{r} \mu_i(\boldsymbol{p}(t))(\boldsymbol{A}_i - \boldsymbol{B}\boldsymbol{K}_i)\boldsymbol{x}(t) \\ &\quad - \sum_{k=1}^{l} h_k\left(\hat{\boldsymbol{p}}(t)\right)\left(\hat{\boldsymbol{A}}_k - \boldsymbol{B}\boldsymbol{H}_k\right)\hat{\boldsymbol{x}}(t). \end{aligned} \tag{23}$$

Assume that $\boldsymbol{B}$ is nonsingular. Then we always can find out a Hurwitz stable matrix $\boldsymbol{G}$ by choosing appropriate control gain matrices $\boldsymbol{K}_i$ and $\boldsymbol{H}_k$, such that

$$\boldsymbol{A}_1 - \boldsymbol{B}\boldsymbol{K}_1 = \boldsymbol{A}_i - \boldsymbol{B}\boldsymbol{K}_i = \hat{\boldsymbol{A}}_k - \boldsymbol{B}\boldsymbol{H}_k = \boldsymbol{G}, \tag{24}$$

where $i = 2, 3, \ldots, r$ and $k = 1, 2, \ldots, l$. Then, error dynamics (23) can be exactly linearized into:

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{G}\boldsymbol{e}(t). \tag{25}$$

Since $\boldsymbol{G}$ is the Hurwitz stable matrix, Eq.(25) is asymptotically stable. Therefore error dynamics (23) is also asymptotically stable. Here, $\boldsymbol{K}_i = \boldsymbol{B}^{-1}(\boldsymbol{A}_i - \boldsymbol{G})$ and $\boldsymbol{H}_k = \boldsymbol{B}^{-1}(\hat{\boldsymbol{A}}_k - \boldsymbol{G})$.

Remark 5 If $\boldsymbol{B}$ is not nonsingular, we can realize the synchronization of two different hyperchaotic systems by solving a similar optimization problem to that in Remark 4.

Remark 6 Since the secure communication system composed of two different hyperchaotic systems is more difficult to break down, the derived asymptotic synchronization results in this paper are significant not only in theory but also in practical application. Moreover, a linear closed-loop synchronous error system can be obtained by the proposed fuzzy synchronous controller in this paper, and therefore many satisfactory performance indices can be achieved, such as rapid response, short transient time, etc.

Remark 7 Investigated in Refs.[41–43] is the synchronization of chaotic systems via the T–S fuzzy model. However, there exist some inaccuracies in them. In Ref.[41], the authors derived the following error system:

$$\begin{cases} \dot{\boldsymbol{e}}(t) = \sum\limits_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{A}_i\boldsymbol{e}(t), & t \neq t_k, \\ \Delta\boldsymbol{e}(t) = \boldsymbol{K}\boldsymbol{e}(t), & t = t_k. \end{cases} \quad k = 1, 2, \ldots, \tag{26}$$

It is obvious that the authors considered that $\boldsymbol{z}(t) = \hat{\boldsymbol{z}}(t)$. However, before reaching the synchronization between the drive system and the response system, $\hat{\boldsymbol{z}}(t)$ is not equal to $\boldsymbol{z}(t)$. Therefore, Eq.(26) should be

$$\begin{cases} \dot{\boldsymbol{e}}(t) = \sum\limits_{i=1}^{r} h_i(\boldsymbol{z}(t))\boldsymbol{A}_i\boldsymbol{e}(t) + \omega, \ t \neq t_k, \\ \Delta\boldsymbol{e}(t) = \boldsymbol{K}\boldsymbol{e}(t), \ t = t_k, \end{cases} \quad k = 1, 2, \ldots, \tag{27}$$

where $\omega = \sum\limits_{i=1}^{r} (h_i(\boldsymbol{z}(t)) - h_i(\hat{\boldsymbol{z}}(t)))\boldsymbol{A}_i\hat{\boldsymbol{x}}(t)$. There are similar errors to those in Refs.[42, 43].

4. Illustrative examples

To visualize the effectiveness of the theoretical analysis and design, some examples are included for illustration. Firstly, we investigate the T–S fuzzy modelling of the Lü hyperchaotic system (i) and the Lorenz hyperchaotic system. The Lü hyperchaotic system (i) is as follows:

$$\begin{aligned} \dot{x}_1(t) &= a(x_2(t) - x_1(t)) + x_4(t), \\ \dot{x}_2(t) &= -x_1x_3(t) + cx_2(t), \\ \dot{x}_3(t) &= x_1x_2(t) - kx_3(t), \\ \dot{x}_4(t) &= x_1x_3(t) + lx_4(t). \end{aligned} \tag{28}$$

The Lorenz hyperchaotic system is bellow:

$$\begin{aligned} \dot{x}_1(t) &= -a(x_1(t) - x_2(t)) + x_4(t), \\ \dot{x}_2(t) &= lx_1(t) - x_2(t) - x_1(t)x_3(t), \\ \dot{x}_3(t) &= x_1(t)x_2(t) - kx_3(t), \\ \dot{x}_4(t) &= -x_1(t)x_3(t) + dx_4(t). \end{aligned} \tag{29}$$

According to Ref.[19], when the parameters are chosen as $a = 36$, $c = 20$, $k = 3$ and $l = 1$, the Lü hyperchaotic system (i) has two positive Lyapunov exponents, which are shown in Fig.1.

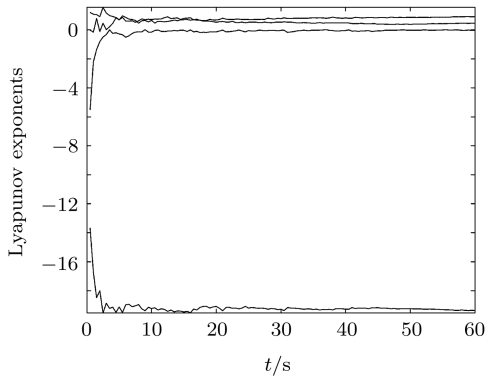


Fig.1. The Lyapunov exponents of Lü hyperchaotic system (i).

Thus, the Lü hyperchaotic system is hyperchaotic in this case, which is shown in Figs.2 and 3. The initial state vector is chosen as $\boldsymbol{x}(0) = (10, 10, 10, 10)^{\mathrm{T}}$. From the simulation we can have $x_1 \in [\tilde{c}_1 - \tilde{d}_1, \tilde{c}_1 + \tilde{d}_1]$, where $\tilde{c}_1 = 0.1409$, and $\tilde{d}_1 = 20.8651$. Then we can derive the T–S fuzzy models of the Lü hyperchaotic system (i) as follows:

R^1: IF $x_1(t)$ is F_{11}, THEN $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{b}$,

R^2: IF $x_1(t)$ is F_{21}, THEN $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{b}$,

where $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^{\mathrm{T}}$,

$$\boldsymbol{A}_1 = \begin{pmatrix} -36 & 36 & 0 & 1 \\ 0 & 20 & -21.0060 & 0 \\ 0 & 21.0060 & -3 & 0 \\ 0 & 0 & 21.0060 & 1 \end{pmatrix},$$

$$\boldsymbol{A}_2 = \begin{pmatrix} -36 & 36 & 0 & 1 \\ 0 & 20 & 20.7242 & 0 \\ 0 & -20.7242 & -3 & 0 \\ 0 & 0 & -20.7242 & 1 \end{pmatrix},$$

and

$$\boldsymbol{b} = (0 \ \ 0 \ \ 0 \ \ 0)^{\mathrm{T}}.$$

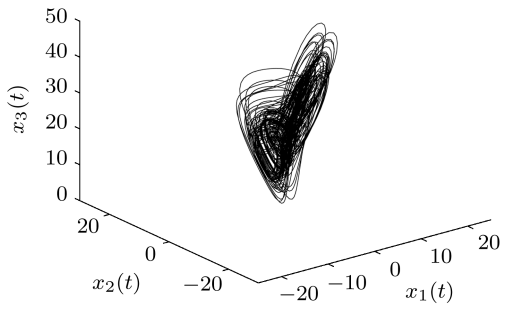


Fig.2. The projection of Lü hyperchaotic system (i) on $x_1 - x_2 - x_3$.

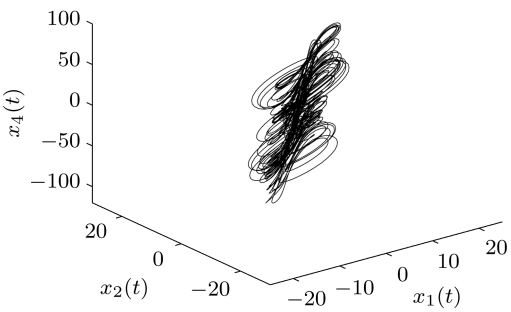


Fig.3. The projection of Lü hyperchaotic system (i) on $x_1 - x_2 - x_4$.

The membership functions are chosen as

$$F_{11}(x_1(t)) = \frac{1}{2}\left(1 - \frac{0.1409 - x_1(t)}{20.8651}\right),$$

and

$$F_{21}(x_1(t)) = \frac{1}{2}\left(1 + \frac{0.1409 - x_1(t)}{20.8651}\right).$$

According to Ref.[3], when the parameters are chosen as $a = 10$, $l = 28$, $k = 8/3$ and $d = 1.3$, the Lorenz hyperchaotic system has two positive Lyapunov exponents, which are shown in Fig.4.

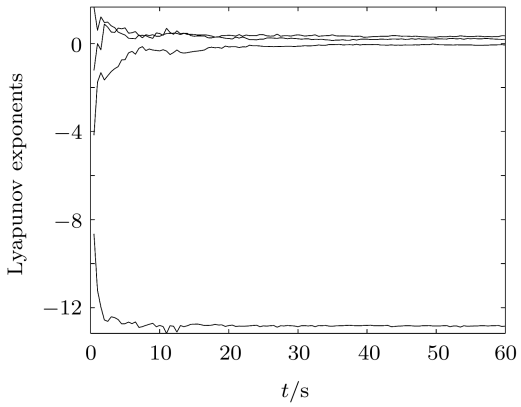


Fig.4. The Lyapunov exponents of Lorenz hyperchaotic system.

Thus, the Lorenz hyperchaotic system is hyperchaotic in this case, which is shown in Figs.5 and 6. The initial state vector is chosen as $\boldsymbol{x}(0) = (30, 30, 30, 30)^{\mathrm{T}}$. From the simulation we can obtain $x_1 \in [\tilde{c}_1 - \tilde{d}_1, \tilde{c}_1 + \tilde{d}_1]$, where $\tilde{c}_1 = 3.5625$ and $\tilde{d}_1 = 26.5831$. Then we can derive the T–S fuzzy models of the Lorenz hyperchaotic system as follows:
R^1: IF $x_1(t)$ is F_{11}, THEN $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_1\boldsymbol{x}(t) + \boldsymbol{b}$,
R^2: IF $x_1(t)$ is F_{21}, THEN $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}_2\boldsymbol{x}(t) + \boldsymbol{b}$,
where $\boldsymbol{x}(t) = (x_1(t), x_2(t), x_3(t), x_4(t))^{\mathrm{T}}$,

$$\boldsymbol{A}_1 = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 28 & -1 & -30.1455 & 0 \\ 0 & 30.1455 & -8/3 & 0 \\ 0 & 0 & -30.1455 & 1.3 \end{pmatrix},$$

$$\boldsymbol{A}_2 = \begin{pmatrix} -10 & 10 & 0 & 1 \\ 28 & -1 & 23.0206 & 0 \\ 0 & -23.0206 & -8/3 & 0 \\ 0 & 0 & 23.0206 & 1.3 \end{pmatrix},$$

and

$$\boldsymbol{b} = (0 \;\; 0 \;\; 0 \;\; 0)^{\mathrm{T}}.$$

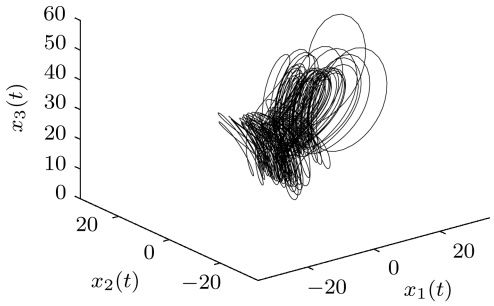


Fig.5. The projection of Lorenz hyperchaotic system on $x_1 - x_2 - x_3$.

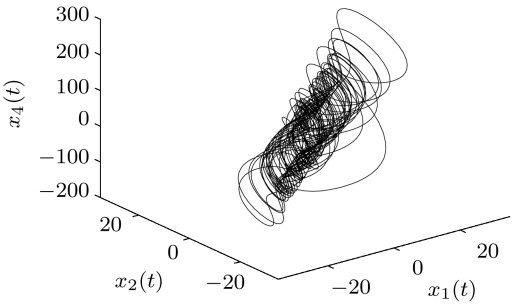


Fig.6. The projection of Lorenz hyperchaotic system on $x_1 - x_2 - x_4$.

The membership functions are chosen as

$$F_{11}(x_1(t)) = \frac{1}{2}\left(1 - \frac{3.5625 - x_1(t)}{26.5831}\right),$$

and

$$F_{21}(x_1(t)) = \frac{1}{2}\left(1 + \frac{3.5625 - x_1(t)}{26.5831}\right).$$

In the following, based on the above T–S fuzzy hyperchaotic models, we first investigate the synchronization of the Lü hyperchaotic system (i) with uncertain parameters. Assume that the uncertain parameters of the drive system is $\tilde{a} = 36 + 0.2\sin(t)$, $\tilde{c} = 20 + 0.1\cos(t)$, $\tilde{k} = 3 + 0.03\sin(t)$ and $\tilde{l} = 1 + 0.01\cos(t)$. And the uncertain parameters of the response system are $\tilde{a}' = 36 - 0.2\cos(t)$, $\tilde{c}' = 20 + 0.1\sin(t)$, $\tilde{k}' = 3 - 0.03\cos(t)$ and $\tilde{l}' = 1 + 0.01\sin(t)$. The control matrix $\boldsymbol{B}$ is chosen as $\boldsymbol{B} = \boldsymbol{I}$, where $\boldsymbol{I}$ denotes the identity matrix. The initial conditions for the drive system and the response system are $\boldsymbol{x}(0) = (5, 10, 5, 20)^{\mathrm{T}}$ and $\hat{\boldsymbol{x}}(0) = (20, -5, -10, 5)^{\mathrm{T}}$. According to Theorem 1, choosing $\gamma = 5$, the following results are derived by LMI toolbox in Matlab 7.0:

$$\boldsymbol{P} = \mathrm{diag}(0.26127, 0.29859, 0.19782, 0.19035),$$

and

$$\bar{\boldsymbol{G}} = \mathrm{diag}(-14.082, -9.8101, -13.923, -14.206),$$

and then the following feedback gains are derived as

$$\bar{\boldsymbol{K}}_1 = \begin{pmatrix} -21.918 & 36 & 0 & 1 \\ 0 & 29.8101 & -21.0060 & 0 \\ 0 & 21.0060 & 10.923 & 0 \\ 0 & 0 & 21.0060 & 15.206 \end{pmatrix},$$

and

$$\bar{\boldsymbol{K}}_2 = \begin{pmatrix} -21.918 & 36 & 0 & 1 \\ 0 & 29.8101 & 20.7242 & 0 \\ 0 & -20.7242 & 10.923 & 0 \\ 0 & 0 & -20.7242 & 15.206 \end{pmatrix}.$$

The synchronization error curves are presented in Fig.7. From Fig.7, we can see that the robust synchronization of two identical hyperchaotic systems with uncertain parameters is derived.

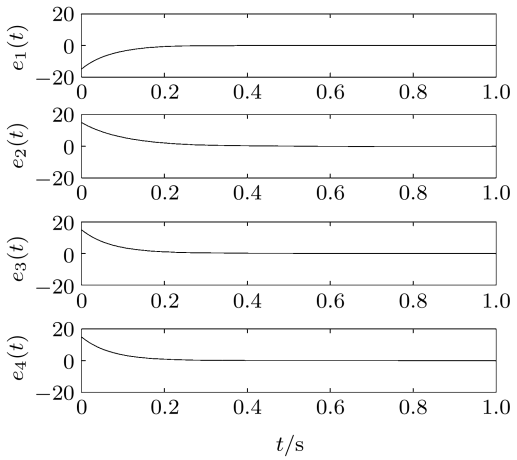


Fig.7. Synchronization error curves of two identical hyperchaotic systems with uncertain parameters.

Next, we discuss the problem of synchronization between the Lü hyperchaotic system and the Lorenz hyperchaotic system. We consider the Lü hyperchaotic system (i) as a drive system, and the Lorenz hyperchaotic system as a response system. The control matrix $\boldsymbol{B}$ is chosen as $\boldsymbol{B} = \boldsymbol{I}$, where $\boldsymbol{I}$ denotes the identity matrix. The initial conditions for the drive system and the response system are $\boldsymbol{x}(0) = (10, 10, 10, 10)^{\mathrm{T}}$ and $\hat{\boldsymbol{x}}(0) = (30, 30, 30, 30)^{\mathrm{T}}$ respectively. According to relationship (24), choosing

$$\boldsymbol{G} = \mathrm{diag}(-6, -1, -2, -5),$$

we can obtain the feedback gains $\boldsymbol{K}_i$, and $\boldsymbol{H}_k$ $(i, k = 1, 2)$ as follows:

$$\boldsymbol{K}_1 = \begin{pmatrix} -30 & 36 & 0 & 1 \\ 0 & 21 & -21.0060 & 0 \\ 0 & 21.0060 & -1 & 0 \\ 0 & 0 & 21.0060 & 6 \end{pmatrix},$$

$$\boldsymbol{K}_2 = \begin{pmatrix} -30 & 36 & 0 & 1 \\ 0 & 21 & 20.7242 & 0 \\ 0 & -20.7242 & -1 & 0 \\ 0 & 0 & -20.7242 & 6 \end{pmatrix},$$

$$\boldsymbol{H}_1 = \begin{pmatrix} -4 & 10 & 0 & 1 \\ 28 & 0 & -30.1455 & 0 \\ 0 & 30.1455 & -2/3 & 0 \\ 0 & 0 & -30.1455 & 6.3 \end{pmatrix},$$

and

$$\boldsymbol{H}_2 = \begin{pmatrix} -4 & 10 & 0 & 1 \\ 28 & 0 & 23.0206 & 0 \\ 0 & -23.0206 & -2/3 & 0 \\ 0 & 0 & 23.0206 & 6.3 \end{pmatrix}.$$

The synchronization error curves are presented in Fig.8. From Fig.8, we can see that asymptotic synchronization between two different hyperchaotic systems is derived.

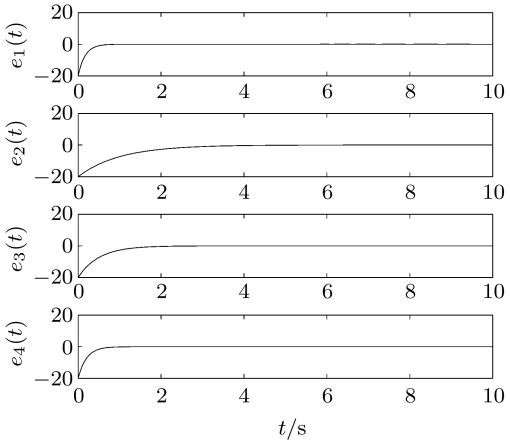


Fig.8. Synchronization error curves of two different hyperchaotic systems.

5. Conclusions

In this work, a T–S fuzzy modelling method of hyperchaotic systems is proposed, which provides a foundation to apply the fuzzy control theory to the

synchronization of hyperchaotic systems. The exact T–S fuzzy models of many well-known hyperchaotic systems can be derived. Based on these fuzzy hyperchaotic models, the robust synchronization of two identical hyperchaotic systems with uncertain parameters and the asymptotic synchronization of two different hyperchaotic systems are achieved via the Lyapunov stability theory and the PDC technique. Numerical simulation results are given to demonstrate the effectiveness of the proposed scheme. This implies that fuzzy model-based synchronization approach is very flexible and useful in practical applications.

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