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Formation and instability of spiral wave induced by Gaussian coloured noise*

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In this paper, we studied the effect of Gaussian coloured noise on the formation and instability of spiral waves described by one class of modified FitzHugh–Nagumo equation. It was found that Gaussian coloured noise plays a constructive role in the formation, transition and instability of spiral wave. Too weak or too strong noise may act against the formation of spiral waves. At a certain noise level, spiral wave is maintained in a medium, in which spiral wave cannot be observed in the absence of the noise. It is difficult to make a stable spiral wave into unstable state by Gaussian coloured noise, unless the noise level is very high. The parameter regions of Gaussian coloured noise for spiral forming and spiral instability were given and discussed with numerical simulations.

Keywords: spiral wave, FitzHugh–Nagumo model, Gaussian coloured noise
PACC: 0545

1. Introduction

For reaction–diffusion systems, the spatiotemporal dynamics of spiral waves is always one of the noticeable themes.[1,2] In the reaction–diffusion systems far from thermodynamic equilibrium spiral waves can be induced and develop under suitable conditions.[3−6] The investigation of spatiotemporal self-organization has led to the fundamental understanding of complex dynamics in many chemical and physicochemical systems.[7−16] In last decades, the formation and instability of spiral waves have been extensively studied in many excitable systems.[17−20] Especially, it has been used to experimentally and theoretically study the Belousov–Zhabotinsky reaction, Bernard convection, and CO oxidation on platinum.[20−25] Recent studies also indicate the existence of similar structures in numerous biochemical and biological media.

Most previous works on the spiral waves were based on the uniform medium or medium with parameters that are supposed to be invariable with time and are the same on each site in the studied systems.[26,27] But the real system is always influenced by random forcing. The random forcing is usually pictured by the so-called noise. The effect of noise, especially the Gaussian white noise, has been extensively studied in nonlinear systems, such as synchronization, stochastic resonance, etc. It is also shown that a weak noise can play a key role in spiral formation. However, a much higher noise is necessary to destroy the spiral wave.

Gaussian white noise is an ideal approximation to real noise. However, in many cases Gaussian white noise is insufficient and unreasonable. In fact, white noise cannot be achieved because it is produced by infinite power. All real noises are coloured. Most real fluctuations in actual systems are described by Gaussian coloured noise which are driven by stochastic processes.[28] Therefore, it is necessary to study spiral wave more carefully by means of Gaussian coloured noises.[29] In this paper, we investigate the formation and instability of spiral wave induced by the Gaussian coloured noise and a modified FitzHugh–Nagumo (FHN) model will be used.

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2. Model and problem

FitzHugh–Nagumo model is usually used to describe the spatiotemporal dynamics of the Brillouin zone reaction, neural networks, and CO oxidation on catalytic surface,[20,25] etc. The FHN model[29,30] is given by

$$\frac{\partial u}{\partial t} = f(u, v) + D_u \nabla^2 u, \tag{1}$$

$$\frac{\partial v}{\partial t} = g(u, v) + D_v \nabla^2 v, \tag{2}$$

where the two reaction terms are $f(u, v) = \frac{1}{\varepsilon} u(1 - u)\left(u - \frac{\nu + b}{a}\right)$ and $g(u, v) = u - v$. The reaction terms $f(u, v)$ and $g(u, v)$ determine the local kinetics of the two relevant variables. The fast variable u and the slow variable v are known as the excitation variable and recovery variable, respectively. D_u is the spatial diffusion coefficient of u and D_v is that of v, and a, b, and ε are parameters. When $u = 0$ and $v = 0$, it has a stable excitable state. We can obtain two other steady states by solving equations $f(u, v) = 0$ and $g(u, v) = 0$.[20−23] When a and b are constants, with $0 < a < 1 + b$ and $0 < \varepsilon < 1$, the case of $b > 0$ or $b \leq 0$ yields excitable or oscillatory dynamic behaviours of the system, respectively.

In real systems modelled by FHN equations, such as the CO oxidation system, the noises mostly result from thermal fluctuations or fluctuations of CO partial pressure. The parameter b in FHN model is a random value perturbed by the noise.[29] The noise can be well described by Gaussian coloured noise. Parameter b is rewritten as [29,30]

$$b = b_0 + \xi(t), \tag{3}$$

where $\xi(t)$ is stochastic Gaussian coloured noise with mean $\langle \xi(t) \rangle = 0$, and correlation $\langle \xi(t)\xi(t') \rangle = C(t-t')$. In our following discussions, we take $\xi(t)$ as exponentially correlated Gaussian coloured noise, which is also called Ornstein–Uhlenbeck noise or is simply called Gaussian coloured noise. The term $C(t - t')$ is given by[28]

$$C(t - t') = \frac{D_0}{\tau_0} \exp\left(- \frac{|t - t'|}{\tau_0} \right), \tag{4}$$

where D_0 is noise time intensity, τ_0 is noise correlation time. It is obvious that $\xi(t)$ tends to Gaussian white noise when $\tau_0 \to 0$. In other words, Gaussian white noise can be looked upon as a special case of the Gaussian coloured noise with zero correlation time.[29] Correspondingly, correlation of Gaussian white noise is described by

$$C(t - t') = 2D_0 \delta(t - t'), \tag{5}$$

where $\delta(t - t')$ is Dirac δ function.

There are several methods to analyse the system driven by Gaussian coloured noise. Our method is based on adiabatic elimination of fast variables. We write the equivalent equation for coloured noise as[28]

$$\frac{\mathrm{d}\xi(t)}{\mathrm{d}t} = -\frac{1}{\tau_0}\xi(t) + \frac{1}{\tau_0}\Gamma(t), \tag{6}$$

where $\Gamma(t)$ is Gaussian white noise with $\langle \Gamma(t) \rangle = 0$ and $\langle \Gamma(t)\Gamma(t') \rangle = 2D_0\delta(t-t')$. From the above we can describe $\xi(t)$ as $\xi(t+\Delta t) = \xi(t) + (-\xi(t) + \Gamma(t))\frac{\Delta t}{\tau_0} + O(\Delta t^2)$, where Δt is the time step. In our numerical simulations, time step Δt is the smallest time unit. Coloured noise correlation time τ_0 is usually larger than the time step Δt. If we lower τ_0 to Δt, then $\xi(t + \Delta t) \to \xi(t) + (-\xi(t) + \Gamma(t)) + O(\Delta t^2) \to \Gamma(t)$, where Δt is rather small. So the noise can be regarded as white noise as long as $\tau_0 \to \Delta t$ in numerical computations.

In the following numerical simulations, we use two-dimensional square domain with no-flux boundary condition and the propagation wave front vanishes when it moves out of the lattice. Lattice size $N_1 = N_2 = 256$, time step $\Delta t = 0.02$, spatial step $\Delta x = 0.5$, $\Delta y = 0.5$. We change variable u and variable v into $u_{i,j}$ and $v_{i,j}$, where $i, j = 1, 2, \ldots, 256$. Then the initial conditions of vertical gradient distribution of spiral formation are $u_{i,j} = 0.9$ for $i = 126$, 127 and $j = 1, 2, \ldots 128$; $u_{i,j} = 0.7$ and $v_{i,j} = 0.7$ for $i = 128$, 129 and $j = 1, 2, \ldots 128$; $v_{i,j} = 0.9$ for $i = 130$, 131 and $j = 1, 2, \ldots 128$; and $u_{i,j} = 0.0$, $v_{i,j} = 0.0$ for other values of i and j.

3. Formation of a stable rotating spiral wave

In this section, the model parameters $a = 1.052$, $b_0 = 0.02$, $\varepsilon = 0.02$, $D_u = 1.0$, and $D_v = 0.$ are used.

Firstly, the model parameters are not perturbed by any noise. Although the model is close to the excitable fixed point, there is no propagation of spiral wave in the domain. The patterns are shown in Figs.1(a)–1(d). Figures 1(e, f) show the time series of the mean value U of $u_{i,j}$ and the standard deviation S of $u_{i,j}$. Note that U and S reach the maximal values at $t = 17.4$, and decrease monotonically with increasing time.

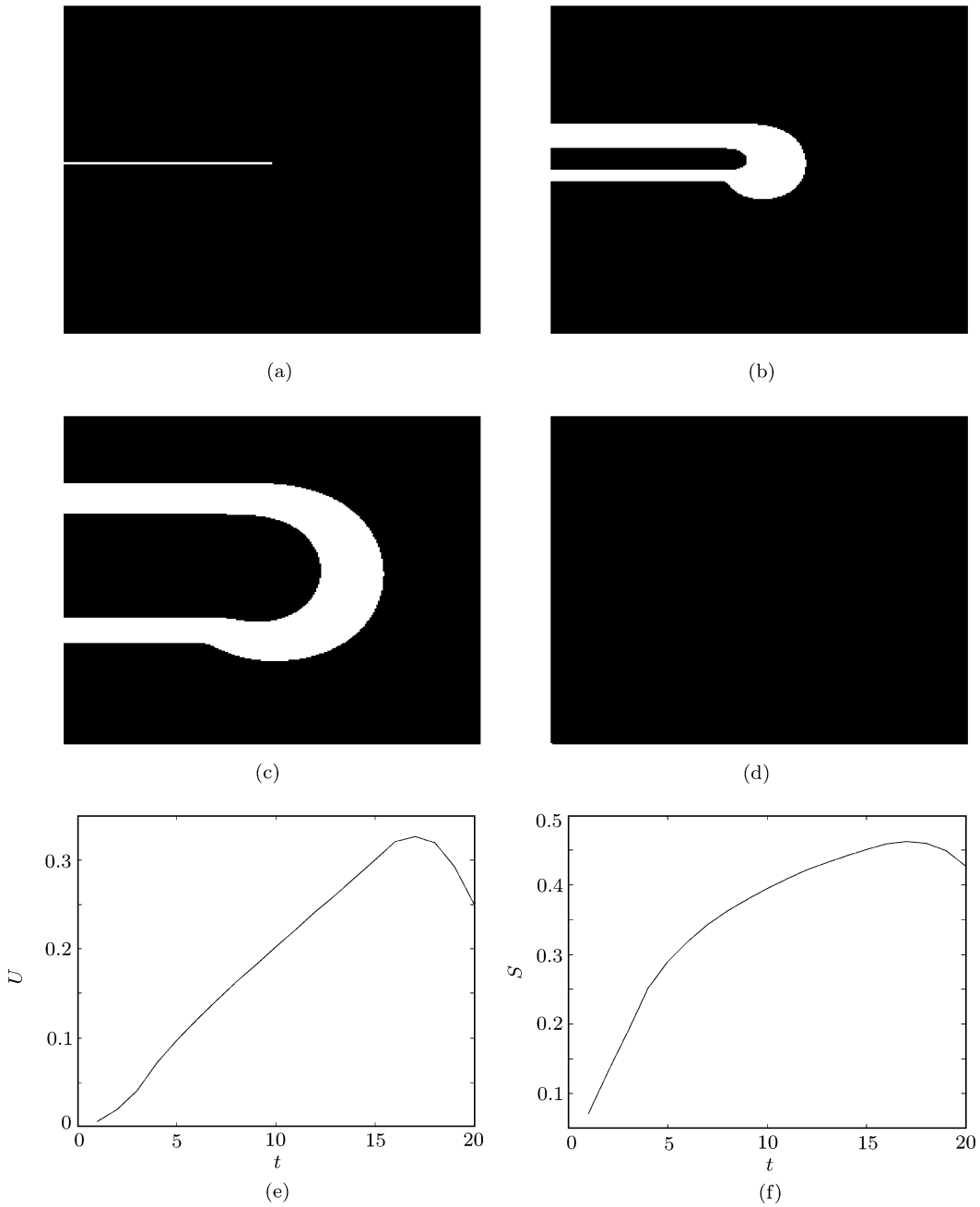


Fig.1. Spiral development sequence with $a = 1.052$, $b_0 = 0.02$, $\varepsilon = 0.02$, $D_u = 1.0$, $D_v = 0$ and no noise. (a) snapshot at time t=0, (b) snapshot at time $t = 5$, (c) snapshot at time $t = 10$, (d) snapshot at time $t = 20$. (e) change of U (the mean value of $u_{i,j}$). (f) change of S (the standard deviation of $u_{i,j}$).

Now we suppose that the model parameter b is driven by exponentially correlated Gaussian colour noise. It is found that the spiral waves can be obtained under suitable conditions. For example, if $D_0 = 6.25 \times 10^{-12}$ and noise correlation time $\tau_0 = 10\Delta t$, spiral wave is developed successfully at time $t = 60$ (see Fig.2(a)–2(f)).

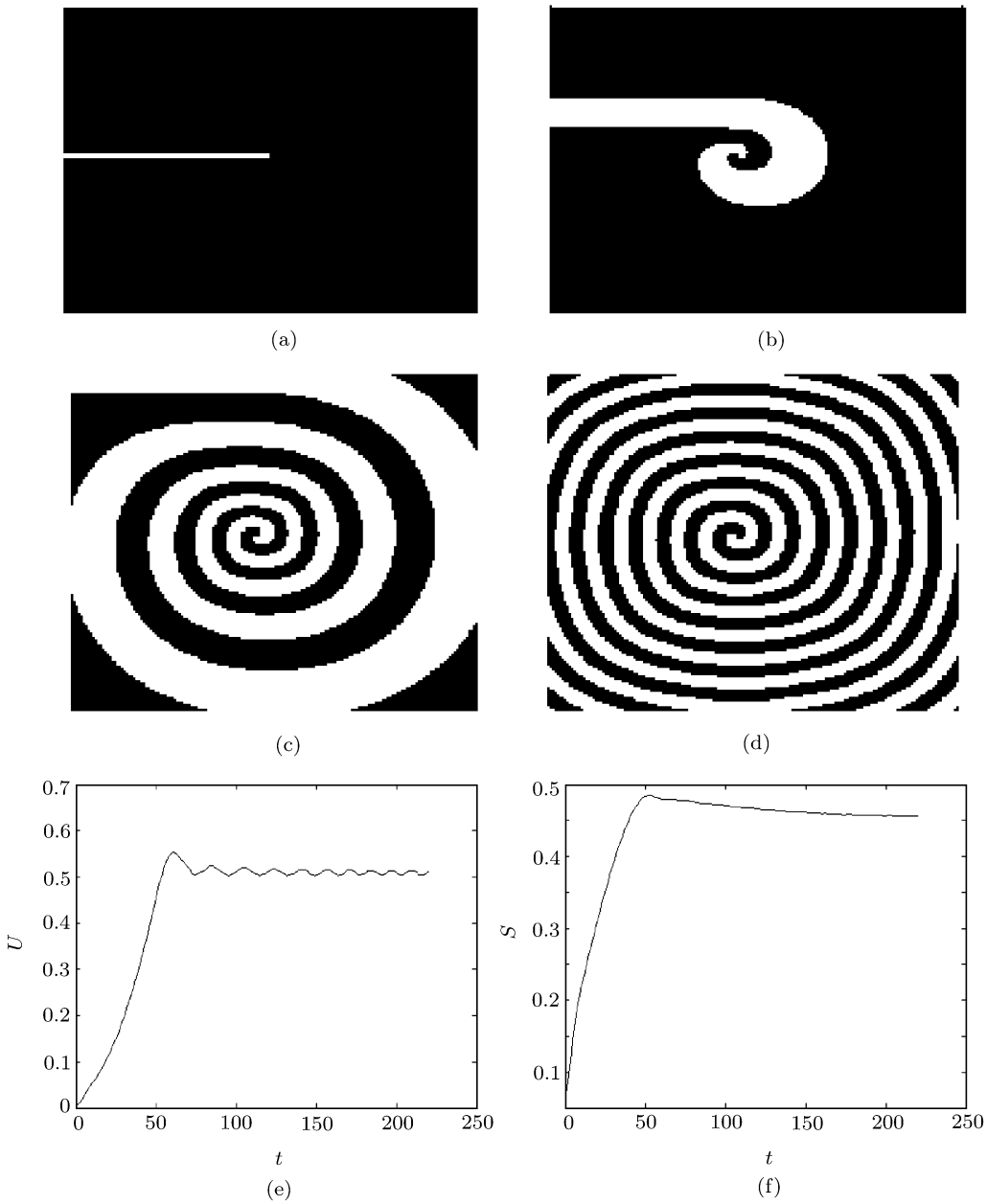


Fig.2. Spiral development sequence with $a = 1.052$, $b_0 = 0.02$, $\varepsilon = 0.02$, $D_u = 1.0$ and $D_v = 0$. Parameter $b = b_0 + \xi(t)$, where $\xi(t)$ is exponentially correlated Gaussian coloured noise. Noise time intensity $D_0 = 6.25 \times 10^{-12}$, noise correlation time $\tau_0 = 10\Delta t$, $\Delta t = 0.02$. (a) snapshot at time $t = 0$, (b) snapshot at time $t = 20$, (c) snapshot at time $t = 60$, (d) snapshot at time $t = 220$. (e) change of U (the mean value of $u_{i,j}$). (f) change of S (the standard deviation of $u_{i,j}$).

Furthermore, we have performed extensive simulations to study the cases of much wider regions of parameters. For example, spiral wave could not be observed for intensity $D_0 = 1.296 \times 10^{-11}$ and noise correlation time $\tau_0 = 20\Delta t$ (see Fig.3(a)–3(d)). Figures 3(e, f) show that the mean value of $u_{i,j}$ and the standard deviation of $u_{i,j}$ begin to fluctuate violently from $t = 8$.

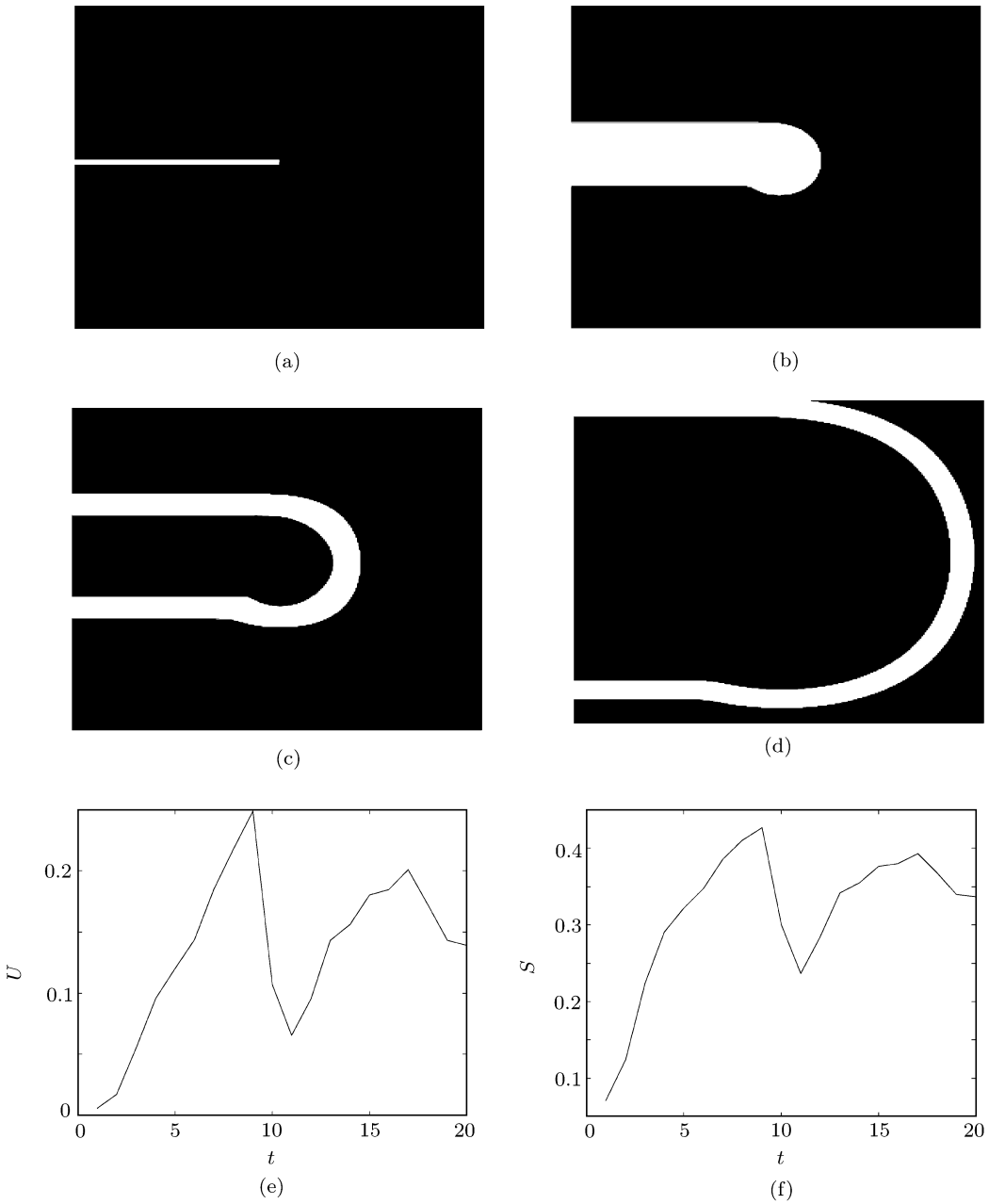


Fig.3. Spiral development sequence with $a = 1.052$, $b_0 = 0.02$, $\varepsilon = 0.02$, $D_u = 1.0$ and $D_v = 0$. Parameter $b = b_0 + \xi(t)$, where $\xi(t)$ is exponentially correlated Gaussian coloured noise. Noise time intensity $D_0 = 1.296 \times 10^{-11}$, noise correlation time $\tau_0 = 20\Delta t$, $\Delta t = 0.02$. (a) snapshot at time $t = 0$, (b) snapshot at time $t = 5$, (c) snapshot at time $t = 10$, (d) snapshot at time $t = 20$. (e) change of U (the mean value of $u_{i,j}$). (f) change of S (the standard deviation of $u_{i,j}$).

It is obvious that there exists a parameter domain of Gaussian coloured noise for successfully inducing spiral wave. This domain is surrounded by the two 50% probability spiral formation lines (see Fig.4). The two lines are the upper limit and the lower limit of spiral formation with 50% probability, respectively. The probability of spiral formation is larger than 50% for points in the domain. In Fig.4, parameter $T = \tau_0/\Delta t$, where $\Delta t = 0.02$. Parameter $A = \sqrt{2D_0/\Delta t}$, where unit length $\Delta x = 0.5$ and D_0 is the noise time intensity. It shows that too weak or too strong noise would act against the formation and propagation of spiral waves. With the increase of noise correlation time τ_0, spiral wave can be induced by very low noise time intensity D_0.

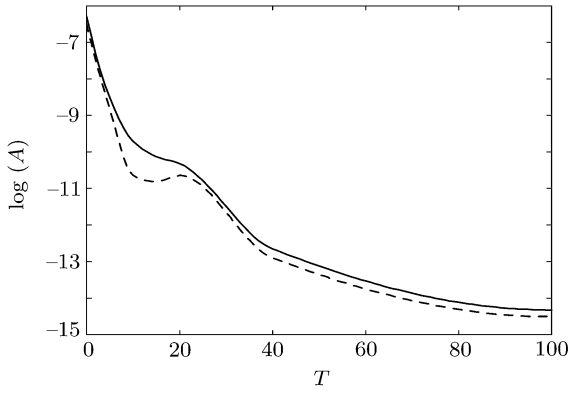
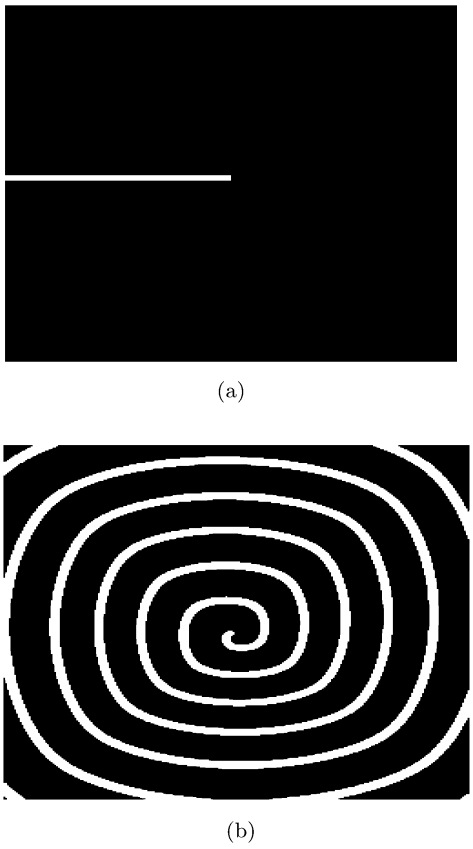


Fig.4. The upper limit (solid line) and the lower limit (dashed line) of spiral formation with 50% probability. $T = \tau_0/\Delta t$ with $\Delta t = 0.02$ and τ_0 is the noise correlation time. $A = \sqrt{2D_0/\Delta t}$ and $\log(A) = \frac{1}{2}\log(2D_0/\Delta t)$ where D_0 is the noise time intensity. Clearly, the parameter domain of spiral formation is the narrow region between the two lines.

4. Instability of spiral wave

In this section, the parameters used are $a = 0.84$, $b_0 = 0.12$, $\varepsilon = 0.015$, $D_u = 1.0$, and $D_v = 0$. A spiral wave is formed under these parameters without noises in Fig.5(a) at time $t = 80$.

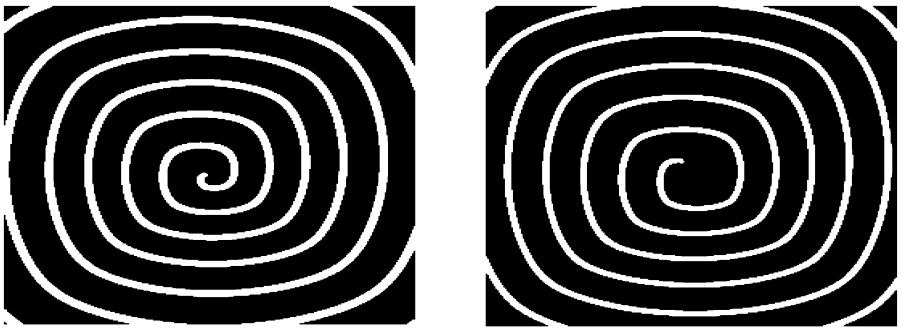


(a)

(b)

Fig.5. Spiral formation sequences with $a = 0.84$, $b_0 = 0.12$, $\varepsilon = 0.015$, $D_u = 1.0$, $D_v = 0$ and without any noise. (a) snapshot at time $t = 0$, (b) snapshot is at time $t = 80$.

Now we suppose that parameter b is perturbed by Gaussian coloured noise at time $t = 0$, as shown in Fig.6. That is to say, time $t = 0$ in Fig.6 or in Fig.7 just corresponds to time $t = 80$ in Fig.5 by setting $D_0 = 0.003481$ and $\tau_0 = 100\Delta t$. From Fig.6(e, f), we observe that spiral wave comes into segment spiral waves and chaos at time $t = 26$. At the same time, the mean value of and the standard deviation of $u_{i,j}$ begins to fluctuate violently.



(a) (b)

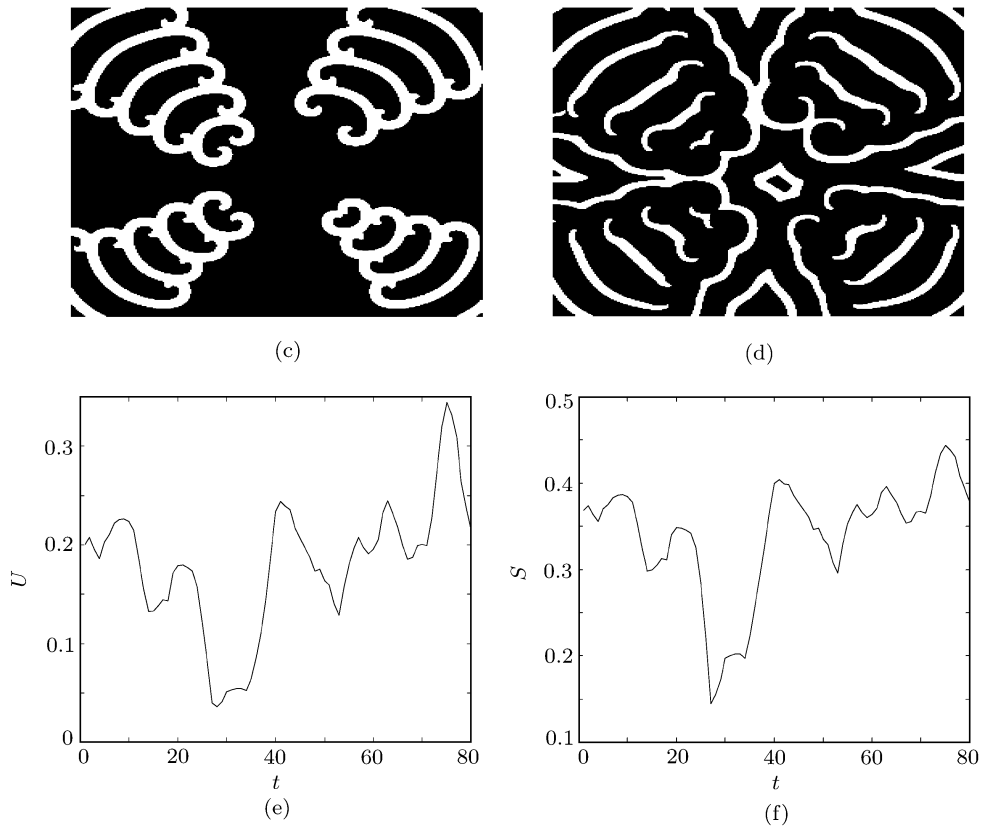
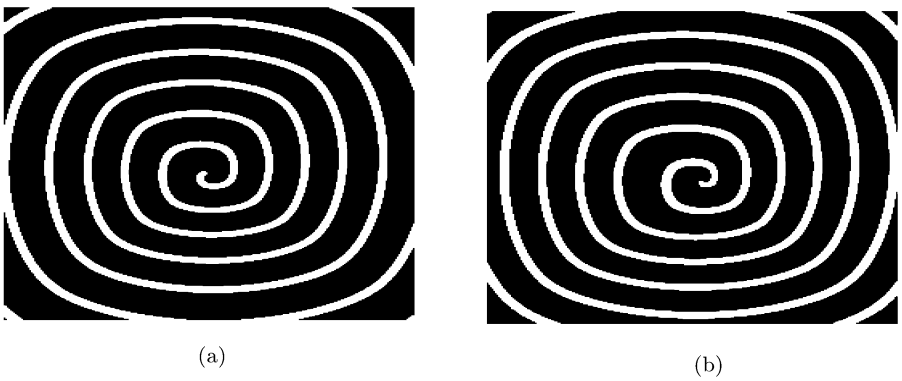


Fig.6. Spiral development sequence with $a = 0.84$, $b_0 = 0.12$, $\varepsilon = 0.015$, $D_u = 1.0$ and $D_v = 0$. Parameter $b = b_0 + \xi(t)$, where $\xi(t)$ is exponentially correlated Gaussian coloured noise. Noise time intensity $D_0 = 0.003481$, noise correlation time $\tau_0 = 100\Delta t$, unit time $\Delta t = 0.02$. (a) snapshot is at time $t = 0$. (b) snapshot is at time $t = 20$. (c) snapshot is at time $t = 40$. (d) snapshot is at time $t = 80$. (e) U is mean value of $u_{i,j}$. (f) S is standard deviation of $u_{i,j}$.

If we decrease noise time intensity D_0 to 0.0004 and set noise correlation time $\tau_0 = 100\Delta t$, spiral wave can not be destroyed. This is verified by Figs.7(a)–7(d). Figure 7(e, f) show that the spiral wave exists all the time and its propagating period is about 40.



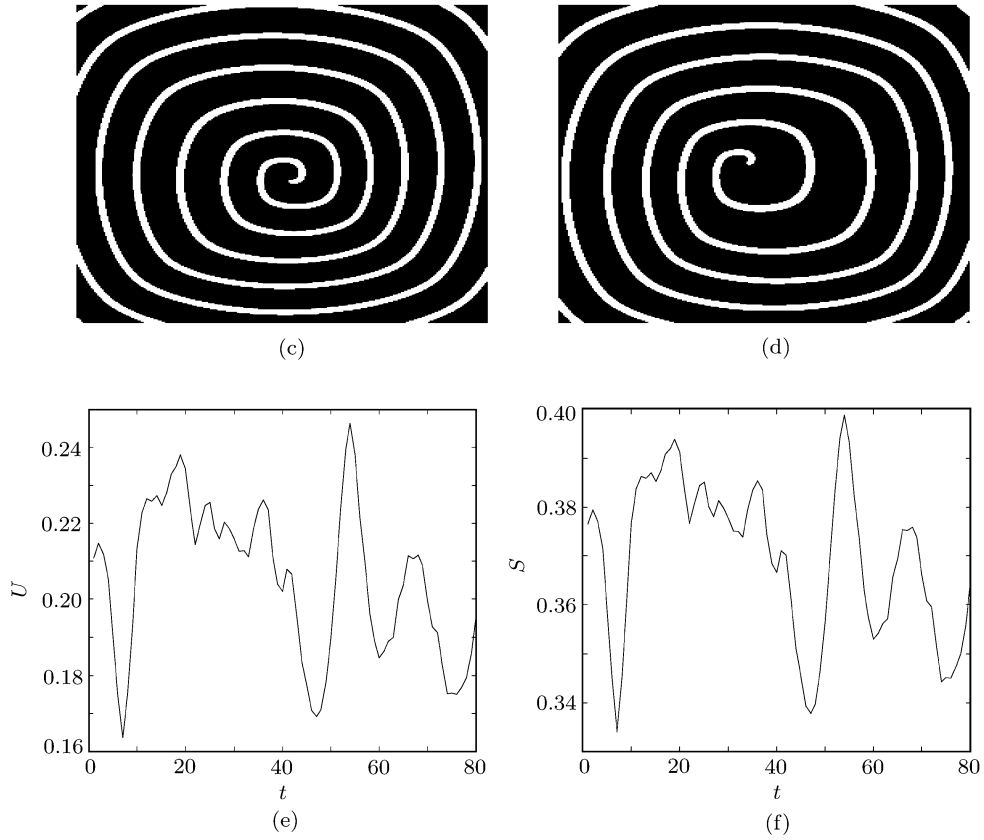


Fig.7. Spiral development sequence with $a = 0.84$, $b_0 = 0.12$, $\varepsilon = 0.015$, $D_u = 1.0$ and $D_v = 0$. Parameter $b = b_0 + \xi(t)$, where $\xi(t)$ is exponentially correlated Gaussian coloured noise. Noise time intensity$D_0 = 0.0004$, noise correlation time $\tau_0 = 100\Delta t$, unit time $\Delta t = 0.02$. (a) snapshot at time $t = 0$, (b) snapshot at time $t = 20$, (c) snapshot at time $t = 40$, (d) snapshot at time $t = 80$. (e) change of U (the mean value of $u_{i,\,j}$. (f) change of S (the standard deviation of $u_{i,\,j}$).

It is obvious that there exists a parameter domain of Gaussian coloured noise in which the spiral wave is unstable. The domain is the region above the borderline of spiral instability with 50% probability (see Fig.8). The probability of spiral instability is higher than 50% at each point in the domain. The two parameters are $T = \tau_0/\Delta t$ and $A = \sqrt{2D_0/\Delta t}$, where unit time $\Delta t = 0.02$. It shows that only enough strong noise can destroy the formation and propagation of spiral waves for a given τ_0. We have mentioned that Gaussian white noise can be looked upon as a special case of the Gaussian coloured noise with correlation time $\tau_0 \to 0$. The results in Fig.8 confirm that the noise time intensity of spiral instability is lowest in the case of Gaussian white noise. That is to say, the increase of noise correlation time means the decrease of the ability to cause spiral instability.

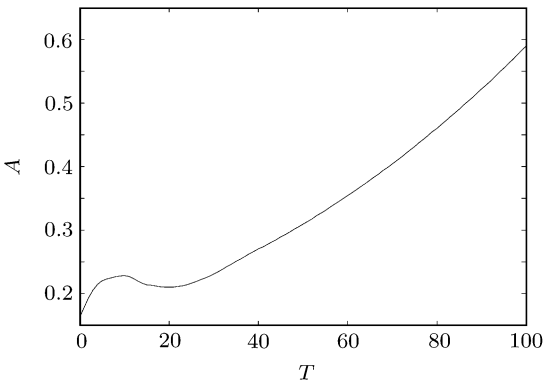


Fig.8. The borderline of spiral instability with 50% probability. Parameter $T = \tau_0/\Delta t$, where $\Delta t = 0.02$ and τ_0 is noise correlation time. Parameter $A = \sqrt{2D_0/\Delta t}$ where D_0 is noise intensity. The domain of spiral instability is the region above the line.

5. Conclusion

We studied spiral formation and spiral instability induced by Gaussian coloured noise in FitzHugh–Nagumo model. It was found that the noise parameters D_0 and τ_0 play important roles in spiral formation and spiral instability.

Noise is usually thought to spoil orderly patterns. In our numerical simulation, although the model is near the excitable fixed point, there is still no propagation of spiral wave. One can obtain spiral waves for some suitable noise parameters D_0 and τ_0 of the exponentially correlated Gaussian coloured noise acting on the parameter b. We present the relation between the probability of spiral formation, D_0 and τ_0 in Fig.4. Too weak or too strong noise will act against the formation and propagation of spiral waves.

Furthermore, when Gaussian coloured noise is imposed on a steady spiral wave, the spiral wave becomes unstable if the noise time intensity D_0 is high enough. The relationship between the probability of spiral instability, D_0 and τ_0 is shown in Fig.8.

Comparing the above results for spiral formation with spiral instability, it is seen that the critical domain of D_0 and τ_0 for the spiral formation with 50% probability induced by Gaussian coloured noise is different from that of spiral instability. D_0 for spiral formation with 50% probability decreases with the increase of τ_0. However, D_0 for spiral instability with 50% probability increases with increasing τ_0. In the case of constant τ_0, D_0 for spiral formation with 50% probability is much smaller than D_0 for spiral instability with 50% probability.

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