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A new hyperchaotic system and its linear feedback control*

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This paper reports a new hyperchaotic system by adding an additional state variable into a three-dimensional chaotic dynamical system, studies some of its basic dynamical properties, such as the hyperchaotic attractor, Lyapunov exponents, bifurcation diagram and the hyperchaotic attractor evolving into periodic, quasi-periodic dynamical behaviours by varying parameter k. Furthermore, effective linear feedback control method is used to suppress hyperchaos to unstable equilibrium, periodic orbits and quasi-periodic orbits. Numerical simulations are presented to show these results.

Keywords: hyperchaos, linear feedback control, Lyapunov exponents, bifurcation
PACC: 0545

1. Introduction

In 1963, Lorenz found a three-dimensional autonomous chaotic system, which produces the well-known Lorenz chaotic attractor.[1] It is due to the first chaotic attractor that the Lorenz system has been extensively studied in the field of chaotic theory and dynamical systems. Recently, there has been increasingly aroused the interest in creating the chaos, particularly, in 1999, Chen and Ueta found a new chaotic system called the Chen system from an engineering feedback control (or, chaotification).[2] Motivated by the consideration of chaotification, great effort has been made to construct the chaotic systems in the past few years. Typical examples are Lü system,[3] unified system,[4] modified Lorenz system,[5] and so on.

Hyperchaos is characterized as a chaotic system with more than one positive exponent, this implies that its dynamics are expended in several different directions simultaneously. It means that hyperchaotic system has more complex dynamical behaviours than chaotic system. It has been studied with increasing interest due to its theoretical and practical applications in technological fields, such as secure communications, lasers, nonlinear circuits, neural networks, generation, control, synchronization, applications of hyperchaos have recently become a central topic in research.[6−15] In recent years, there are various hyperchaotic systems discovered in high-dimensional systems. However, the two most well known hyperchaotic systems are the four-dimensional hyperchaotic Rossler system[16] and the four-dimensional hyperchaotic Chua circuit.[17] Very recently, hyperchaos was found numerically and experimentally by adding a simple state feedback controller.[18−26]

In this paper, a new hyperchaotic system is constructed, and stabilization of the hyperchaotic system is achieved. It is organized as follows. In Section 2, it is constructed based on a three-dimensional system by introducing a nonlinear state feedback controller. In Section 3, the properties and dynamics of the hyperchaotic system are investigated numerically via bifurcation diagram, Lyapunov exponents. In Section 4, simple but effective controllers are designed for stabilizing the hyperchaotic system to unstable equilibrium, periodic and quasi-periodic orbits. Conclusions are given in the final section.

2. Construction of the new hyperchaotic system

More recently, Cai *et al* constructed the three-dimensional autonomous chaotic system.[27] The sys-

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tem is described by

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= bx + cy - xz, \\ \dot{z} &= x^2 - hz, \end{aligned} \tag{1}$$

where a, b, c, h are constants and x, y, z are variables of the system, when $a = 20$, $b = 14$, $c = 10.6$, $h = 2.8$, the Lyapunov exponents of this chaotic system are 2.3554, 0, –14.5561, the Lyapunov dimension of the attractor is 2.1618.

Based on the system (1), a new hyperchaotic system which is judged by two positive Lyapunov exponents can be generated by adding an additional state variable u. The system of four-dimensional autonomous differential equations has the following form:

$$\begin{aligned} \dot{x} &= a(y-x), \\ \dot{y} &= bx + cy - xz + u, \\ \dot{z} &= x^2 - hz, \\ \dot{u} &= -kx, \end{aligned} \tag{2}$$

where k is a parameter to be determined, its value can be varied within a certain range.

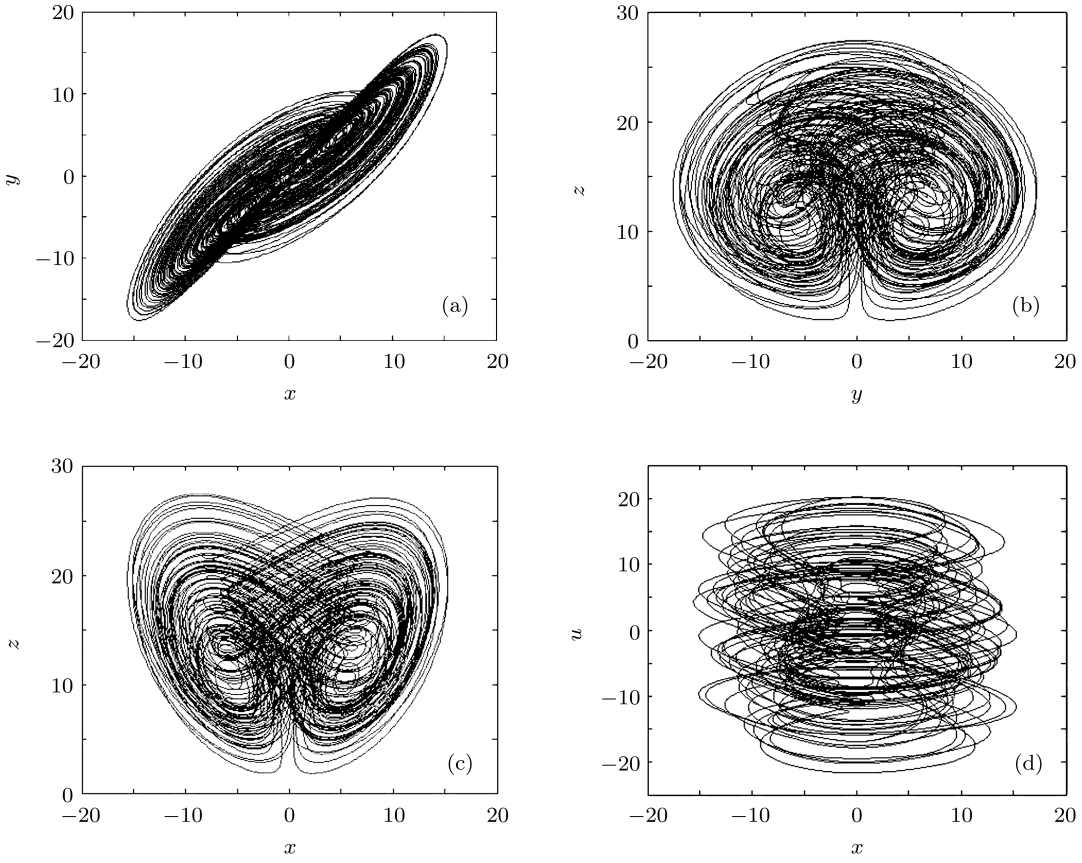
We know that the new system (2) is four-dimensional quadratic autonomous system. To create hyperchaos from system (2) should have two positive Lyapunov exponents along with one zero and one negative Lyapunov exponent.

When $a = 20$, $b = 1$, $c = 10.6$, $h = 2.8$ and $k = 3.7$, the new hyperchaotic system can be obtained from the system (2). Its Lyapunov exponents can be obtained as $\mathrm{LE}_1 = 0.8298$, $\mathrm{LE}_2 = 0.1154$, $\mathrm{LE}_3 = 0$, $\mathrm{LE}_4 = -13.1405$.

The Lyapunov dimension of this system (2) is

$$\begin{aligned} D_{\mathrm{L}} &= j + \frac{1}{|\mathrm{LE}_{j+1}|}\sum_{i=1}^{j}\mathrm{LE_i} \\ &= 3 + \frac{0.8298 + 0.1154}{13.1405} = 3.0719. \end{aligned}$$

Obviously, the Lyapunov dimension of this new hyperchaotic system is also fractional dimension. The new system (2) shows hyperchaotic behaviour, as depicted in Figs.1(a)–1(f).



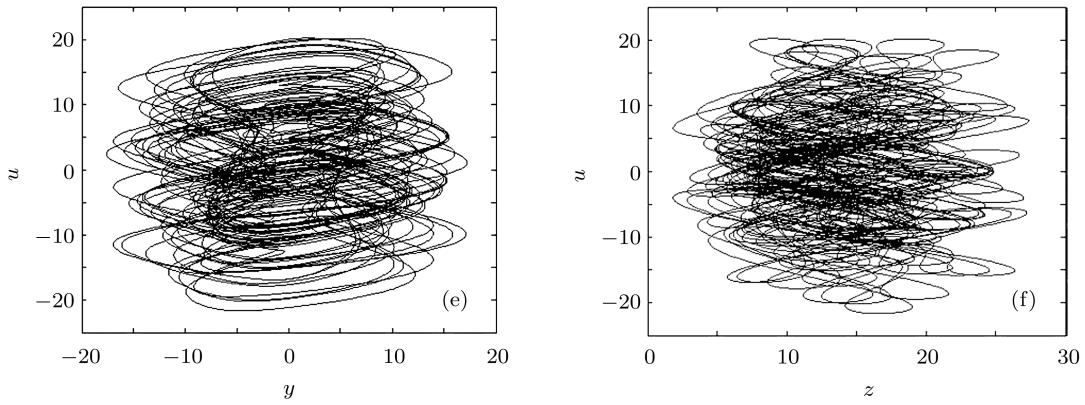


Fig.1. Phase portraits of hyperchaotic system (2), with $a = 20$, $b = 1$, $c = 10.6$, $h = 2.8$ and $k = 3.7$: (a) $x - y$ plane; (b) $y - z$ plane; (c) $x - z$ plane; (d) $x - u$ plane; (e) $y - u$ plane; (f) $z - u$ plane.

3. Basic dynamical behaviours the new hyperchaotic system

In this section, basic properties and complex dynamics of the new system (2) are investigated. Lyapunov exponent, Bifurcation diagram and phase portraits are employed to illustrate how the dynamics of the system (2) alters with the increasing value of the parameter k. The new system (2) has the following basic properties.

3.1. Symmetry

The new system (2) is symmetric with respect to the z-axis, which is invariant under the following coordinate transformations $(x, y, z, u) \to (-x, -y, z, -u)$.

3.2. Dissipation

The divergence of the new four-dimensional system (2) is

$$\begin{aligned}\nabla \dot{V} &= \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} + \frac{\partial \dot{z}}{\partial z} + \frac{\partial \dot{u}}{\partial u} \\ &= -(a - c + h).\end{aligned}$$

Thus, when $a - c + h > 0$, the new system (2) is dissipative. That is to say, a volume element v_0 is apparently contracted by the flow into a volume element $v_0 \mathrm{e}^{-(a-c+h)t}$ in time t. It means that each volume containing the trajectory of this dynamical system (2) shrinks to zero as $t \to \infty$ at an exponential rate $(a - c + h)$. Consequently, all the trajectories of the new system ultimately arrive to an attractor.

3.3. Equilibrium

Obviously, the new system (2) has a unique equilibrium $E_0(0, 0, 0, 0)$ for any parameters.

Linearizing the new system (2) at the equilibrium E_0 yields the Jacobian matrix:

$$\boldsymbol{J} = \begin{pmatrix} -a & a & 0 & 0 \\ b & c & 0 & 1 \\ 0 & 0 & -h & 0 \\ -k & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

and its characteristic equation:

$$(\lambda + h)(\lambda^3 + (a - c)\lambda^2 - a(b + c)\lambda + ak) = 0. \tag{4}$$

From Eq.(4), we could obtain $\lambda_1 = -h < 0$ $(h > 0)$ and the following equation:

$$f(\lambda) = \lambda^3 + (a - c)\lambda^2 - a(b + c)\lambda + ak = 0. \tag{5}$$

According to the Routh–Hurwitz criterion, the real of the roots λ of Eq.(5) are negative if and only if

$$a - c > 0, \quad ak > 0, \quad -a(a - c)(b + c) - ak > 0. \tag{6}$$

Thus, let $a > 0, k > 0$, then $b < \dfrac{k}{c - a} - c$.

Noting that the equilibrium E_0 is unstable under the conditions of $a > 0$, $k > 0$, $a > c$, $b > \dfrac{k}{c - a} - c$, we can possibly generate chaos or hyperchaos in the new system (2).

Obviously, Hopf bifurcation can appear at the point E_0 when $b_{\mathrm{h}} = \dfrac{k}{c - a} - c$.

Therefore, there is an instability equilibrium, that is, there exists an eigenvalue λ, which satisfies $\mathrm{Re}(\lambda) >$

0, only if there is a pair of complex conjugate eigenvalues for the new system (2). Let the two eigenvalues be $\lambda_{2,3} = \pm w\mathrm{i}$, for some real number w, then we have

$$\lambda_2 + \lambda_3 + \lambda_4 = -(a - c).$$

Therefore, $\lambda_4 = -(a - c)$, which is on the margin of stability for the new system (2). Then, we have $f(\lambda_4) = a(b+c)(a-c)+ak = 0$ i.e. $b = b_\mathrm{h} = \dfrac{k}{c-a} - c$.

Hence, when $b = b_\mathrm{h}$, Eq.(5) has three eigenvalues: one is a negative root, the other two are a pair of purely imaginary conjugate roots, i.e. the new system (2) displays a Hopf bifurcation at the point E_0.

For example, let $a = 20$, $c = 10.6$, $h = 2.8$ and $k = 3.7$, then $b_\mathrm{h} = -10.9936$, eigenvalues of Eq.(4) are $\lambda_1 = -2.8000$, $\lambda_{2,3} = \pm 2.8058\mathrm{i}$, $\lambda_4 = -9.4000$, then, system (2) undergoes a Hopf bifurcation at the equilibrium E_0.

When $b = -10.9932 > b_\mathrm{h}$, eigenvalues of Eq.(4) are $\lambda_1 = -2.8000$, $\lambda_{2,3} = 0.0004 \pm 2.8057\mathrm{i}$, $\lambda_4 = -9.4008$, hence, equilibrium E_0 is unstable.

When $b = -10.9938 < b_\mathrm{h}$, eigenvalues of Eq.(4) are $\lambda_1 = -2.8000$, $\lambda_{2,3} = -0.0002 \pm 2.8058\mathrm{i}$, $\lambda_4 = -9.3996$, hence, equilibrium E_0 is stable. The evolution of parameter b_h with a is shown in Fig.2.

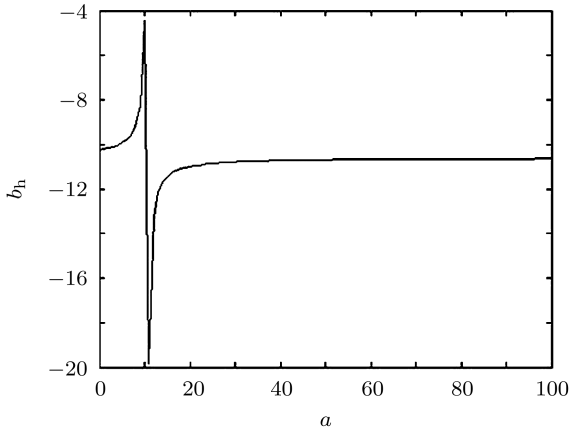


Fig.2. Evolution of parameter b_h with a.

3.4. Bifurcation analysis

The new dynamical system (2) can be characterized with its Lyapunov exponents which are computed by Wolf method.[28] For equilibrium points $\mathrm{LE}_4 < \mathrm{LE}_3 < \mathrm{LE}_2 < \mathrm{LE}_1 < 0$; for periodic orbits, $\mathrm{LE}_1 = 0$, $\mathrm{LE}_4 < \mathrm{LE}_3 < \mathrm{LE}_2 < 0$; for quasi-periodic orbits $\mathrm{LE}_1 = \mathrm{LE}_2 = 0$, $\mathrm{LE}_4 < \mathrm{LE}_3 < 0$; while for chaotic orbits, $\mathrm{LE}_1 > 0$, $\mathrm{LE}_2 = 0$, $\mathrm{LE}_4 < \mathrm{LE}_3 < 0$, $\mathrm{LE}_1 + \mathrm{LE}_3 + \mathrm{LE}_4 < 0$ and for hyperchaotic orbits, $\mathrm{LE}_1 > \mathrm{LE}_2 > 0$, $\mathrm{LE}_3 = 0$, $\mathrm{LE}_4 < 0$, $\mathrm{LE}_1+\mathrm{LE}_2+\mathrm{LE}_4 < 0$.

In order to discover the effect of the parameters on the dynamics of the new four-dimensional dynamical system (2), we fix the parameters $a = 20$, $b = 1$, $c = 10.6$, $h = 2.8$, and let the parameter k vary in the interval [0, 50].

The bifurcation diagram and the three Lyapunov exponents are shown in Figs.3 and 4.

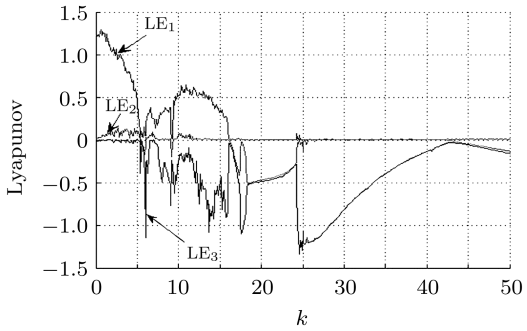


Fig.3. Lyapunov exponents spectrum of the new system (2) versus parameter k.

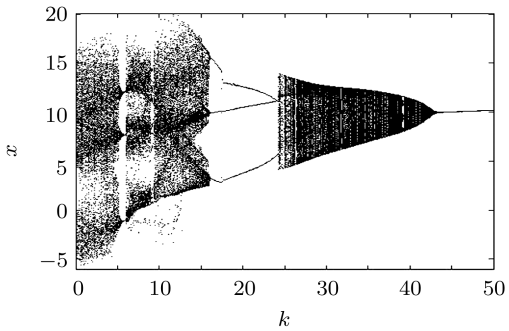


Fig.4. Bifurcation diagram of the new system (2) versus parameter k.

From Figs.3 and 4, we can observe that the Lyapunov exponents spectrum gives results completely consistent with the bifurcation diagram. Figure 3 shows that the system (2) is hyperchaotic for a range of k, and the system can also evolve into chaotic orbits, periodic orbits, and quasi-periodic orbits. With the variation of the parameter k, the dynamical behaviours of the four-dimensional system (2) are as follows:

When $0 < k \leq 5.27$, $6.60 \leq k \leq 7.20$ and $9.85 \leq k \leq 11.59$, the system (2) has two positive Lyapunov exponents, representing that the hyperchaos occurs. When $k = 3.7$, the corresponding phase portraits are shown in Figs.1(a)–1(f).

When $6.06 \leq k \leq 6.59$, $7.21 \leq k \leq 9.84$ and $11.60 \leq k \leq 16.13$ the largest Lyapunov exponent

is positive, implying that the system shows chaotic behaviour, except a very narrow periodic window [9.13, 9.33] in the chaotic region. For example, when $k = 9.2$, the system (2) is a periodic orbit. Figures 5(a)–5(d) displays chaotic attractors for different values of parameter k respectively.

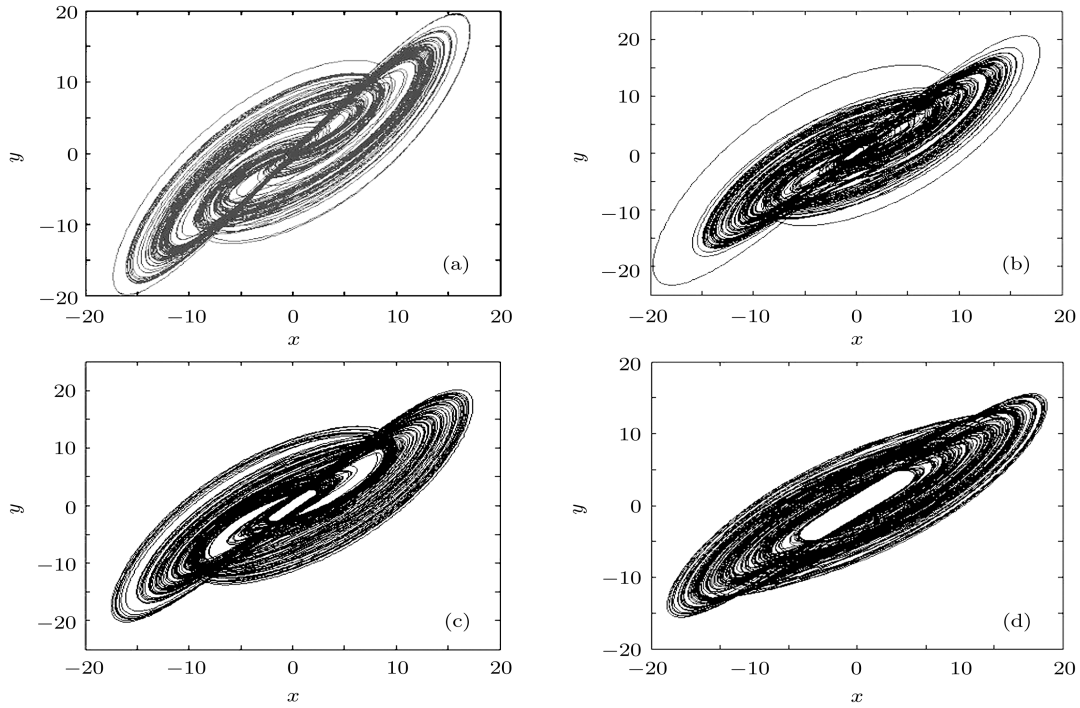


Fig.5. Chaotic phase portraits of the system (2) in the $x - y$ plane for different k: $k = 6.5$; (b) $k = 9.5$; (c) $k = 15$; (d) $k = 24.45$.

When $5.28 \leq k \leq 6.05$, $16.14 \leq k \leq 25.01$, $25.40 \leq k \leq 25.51$ and $43.20 \leq k \leq 50$, the largest Lyapunov exponent almost equals zero, which means that system (2) is periodic. Especially, there are some chaotic orbits in the periodic region. Some periodic phase portraits for special values of k are depicted in Figs.6(a)–6(d).

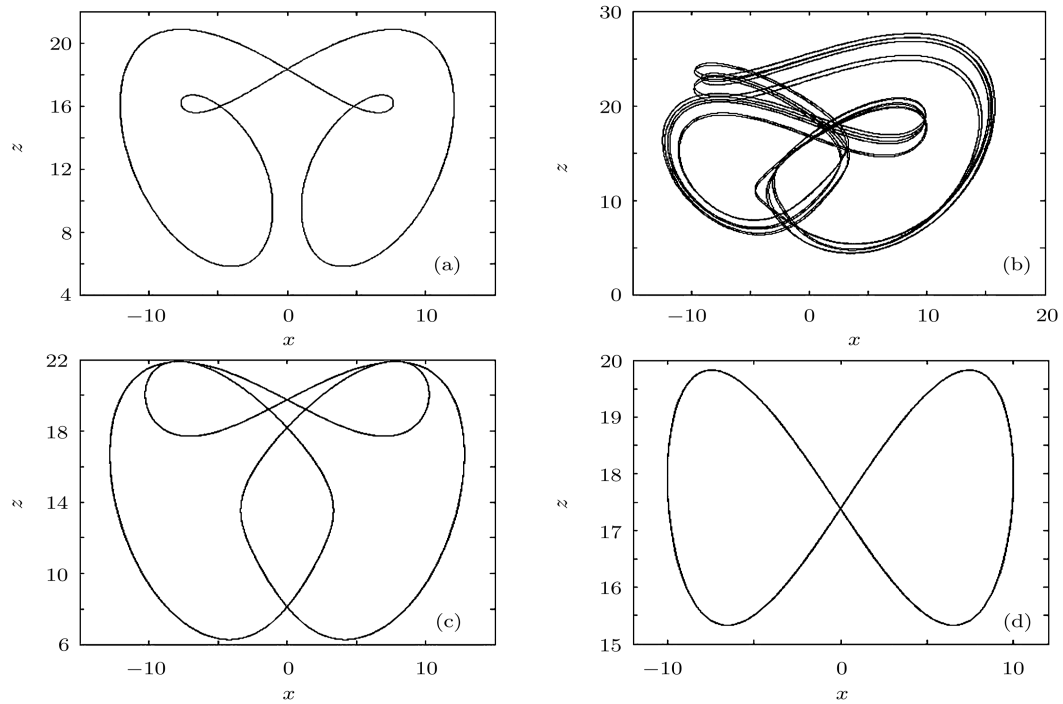


Fig.6. Periodic phase portraits of the system (2) in the $x - z$ plane for different k: (a) $k = 5.9$; (b) $k = 16.15$; (c) $k = 18.8$; (d) $k = 44.6$.

When $25.02 \leq k \leq 25.39$, $25.52 \leq k \leq 43.19$, the first and the second largest Lyapunov exponents are almost zero, which means that the system (2) is quasi-periodic. Some phase portraits of special k are shown in Figs.7(a) and 7(b).

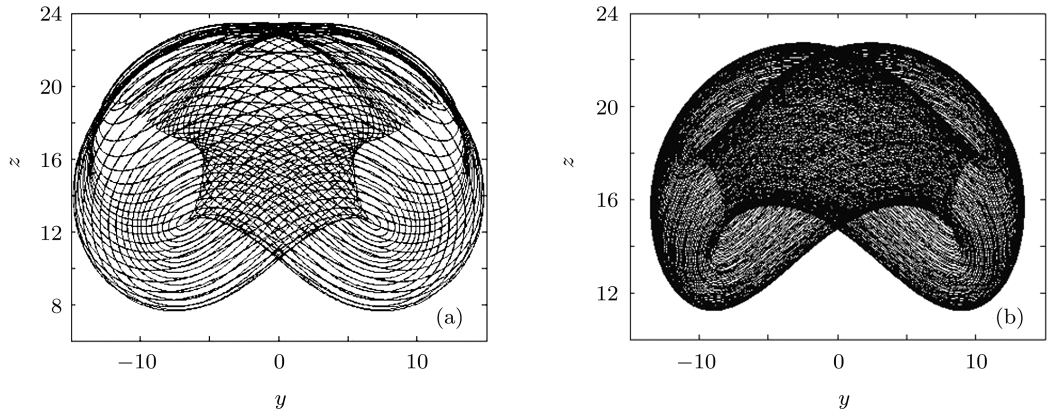


Fig.7. Quasi-periodic phase portraits of the system (2) in the $y-z$ plane for different k: (a) $k = 25.3$; (b) $k = 36.9$.

4. Linear feedback control

In this section, simple but effective linear feedback controllers are designed to control hyperchaos. For simplicity, let the parameters $a = 20$, $b = 1$, $c = 10.6$, $h = 2.8$ and $k = 3.6$, thus, the new system (2) is hyperchaotic system.

4.1. Stabilizing to the equilibrium E_0 (0, 0, 0, 0)

In this subsection, the hyperchaos in dynamical system (2) is controlled to the unique unstable equilibrium via linear feedback control.

For this purpose, we assume that the controlled hyperchaotic system is as follows:

$$\begin{aligned} \dot{x} &= a(y-x) - k_1 x, \\ \dot{y} &= bx + cy - xz + u - k_1 y, \\ \dot{z} &= x^2 - hz, \\ \dot{u} &= -kx - k_1 u, \end{aligned} \tag{7}$$

where k_1 is the control gain.

The Jacobian matrix of the controlled hyperchaotic system (7) is

$$\boldsymbol{J} = \begin{pmatrix} -a-k_1 & a-k_1 & 0 & 0 \\ b-z & c-k_1 & -x & 1 \\ 2x & 0 & -h & 0 \\ -k & 0 & 0 & -k_1 \end{pmatrix}. \tag{8}$$

When the parameters $a = 20$, $b = 1$, $c = 10.6$, $h = 2.8$ and $k = 3.6$, the characteristic equation of the Jacobian matrix (8) is

$$(\lambda + h)(\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3) = 0,$$

in which $c_1 = 3k_1 + 9.4$, $c_2 = 3k_1^2 + 18.8k_1 - 232$ and $c_3 = k_1^3 + 9.4k_1^2 - 232k_1 + 72$.

Obviously, $\lambda = -h < 0$ is one eigenvalue of the Jacobian matrix (8). According to the Ruth–Hurwitz criterion, the controlled system (7) could be asymptotically stable at the equilibrium E_0, as long as the following conditions are satisfied:

(1) $c_i > 0$ $(i = 1, 3)$; (2) $c_1c_2 - c_3 > 0$.

After some calculations, we attain the range of the control gain $k_1 > 11.0342$, which can ensure that the controlled hyperchaotic system (7) will gradually converge to its equilibrium point E_0.

To verify the validity of the stability condition obtained above, choose the control gain $k_1 = 12$. The four characteristic values of the Jacobian matrix (8) are –32.3048, –11.8683, –1.2269 and –2.8000. Consequently, the controlled hyperchaotic system (7) is asymptotically stable at the origin. The time response of the four states of the controlled system (7) is shown in Fig.8. It can be seen from the simulations that the hyperchaotic sate is quickly settled down to the unstable equilibrium E_0.

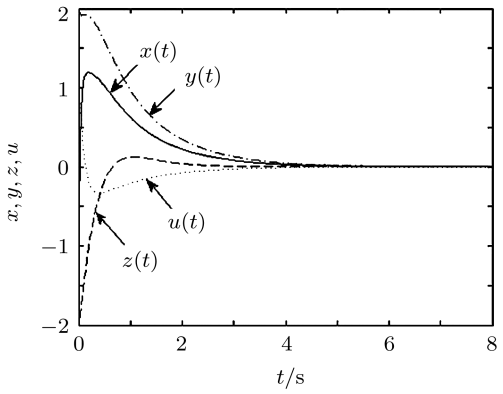


Fig.8. The evolution of feedback controlled system (7) for the control gain $k_1 = 12$.

4.2. Suppression of hyperchaos to periodic and quasi-periodic orbits

In this section, the hyperchaotic system (2) will be controlled to a periodic orbit and quasi-periodic orbit with simple linear feedback.

Consider the controlled hyperchaotic system

$$\begin{aligned} \dot{x} &= a(y - x), \\ \dot{y} &= bx + cy - xz + u + k_1(x + y), \\ \dot{z} &= x^2 - hz, \\ \dot{u} &= -kx, \end{aligned} \tag{9}$$

where $k_1(x+y)$ is the linear feedback, the control gain k_1 will be determined to suppress hyperchaos.

Choosing the control gain $k_1 = -5.68$, we can obtain the Lyapunov exponents by using the Wolf method,[28] the four Lyapunov exponents are 0, −1.2196, −1.2821 and −15.3803. From the Lyapunov exponents above, one can judge that the controller $-5.68(x + y)$ can stabilize the hyperchaotic system (2) to a periodic orbit. The periodic orbit to which the system (2) is stabilized is shown in Fig.9(a). Figure 9(b) shows the time series corresponding to the Fig.9(a).

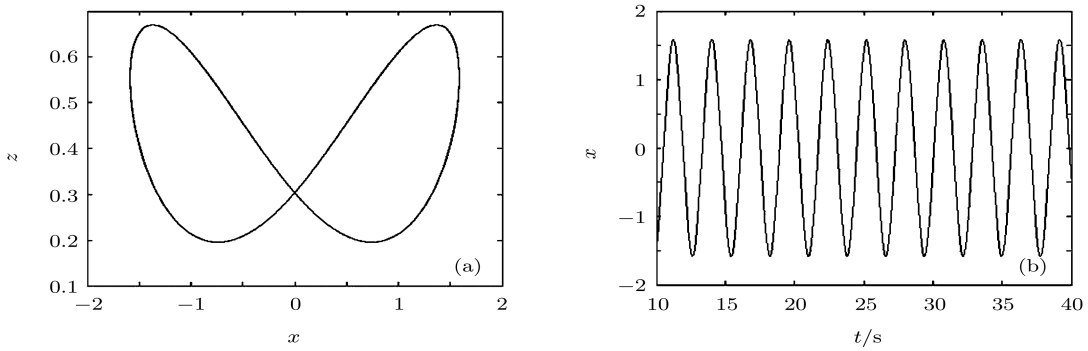


Fig.9. Control hyperchaos to a periodic orbit with the control gain $k_1 = -5.68$.

Choosing the control gain $k_1 = 4.68$, we find that the four Lyapunov exponents of the controlled system (9) are 0, 0, –0.43207 and –7.0966. Thus, we obtain that the hyperchaotic system (2) can be controlled to a quasi-periodic orbit with the gain $k_1 = 4.68$. The quasi-periodic orbit to which the hyperchaos is controlled is shown in Fig.10(a). Figure 10(b) shows the time series corresponding to the Fig.10(a).

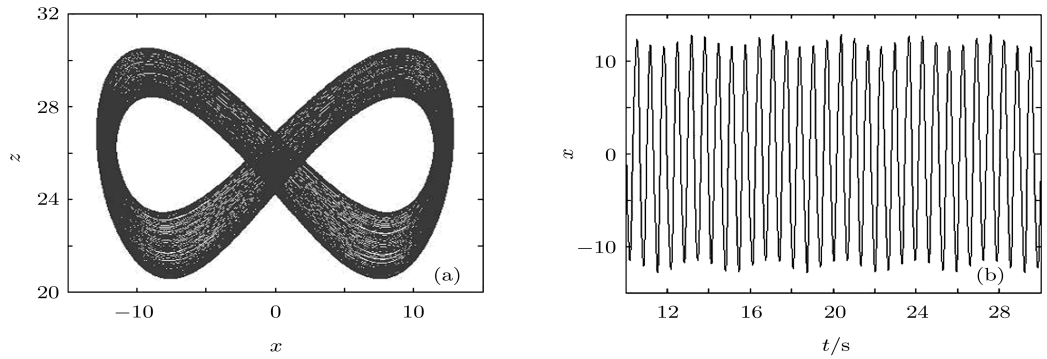


Fig.10. Quasi-periodic orbit of the controlled system (9) with the control gain $k_1 = 4.68$.

5. Conclusion

In this paper, a new hyperchaotic system is introduced. Some basic dynamic behaviours are further studied by calculating its Lyapunov exponent spectrum and bifurcation diagrams. Furthermore, based on the Ruth–Hurwitz criterion and linear feedback control, we stabilize the hyperchaotic system to unstable equilibrium, periodic orbit and quasi-periodic orbit. Numerical simulations are shown to verify and illustrate the effectiveness of these controllers. More detailed and thorough analysis will be provided in the near future.

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