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Multiple attractors and generalized synchronization in delayed Mackey–Glass systems*

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Nonlinear dynamics of the time-delayed Mackey–Glass systems is explored. Coexistent multiple chaotic attractors are found. Attractors with double-scroll structures can be well classified in terms of different return times within one period of the delay time by constructing the Poincaré section. Synchronizations of the drive–response Mackey–Glass oscillators are investigated. The critical coupling strength for the emergence of generalized synchronization against the delay time exhibits the interesting resonant behaviour. We reveal that stronger resonance effect may be observed when different attractors are applied to the drivers, i.e., more resonance peaks can be found.

Keywords: time-delayed Mackey–Glass system, multiple chaotic attractors, return numbers, generalized synchronization

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Since the pioneering finding by Edward Lorenz in 1963,[1] explorations of chaotic dynamics have revealed abundant essential features in diverse nonlinear systems. In the past decades most of these studies have been focused on low-dimensional systems.[2] In recent years, much effort has been devoted to the dynamics in spatiotemporal systems, such as coupled map lattices,[3] coupled chaotic oscillators and partial differential dynamical systems.[4] Contrary to the complicated algorithm of partial differential equations, time-delayed nonlinear equations represent yet a distinctly different class of infinite-dimensional dynamical systems with much simpler algorithm and received much interest due to their ubiquitous physical backgrounds and progressive applications.[5] The understanding of time-delayed dynamical systems is of difficulty because they cannot be simplified by a finite set of autonomous ordinary differential equations. Some efforts have been made by transforming time-delay equations to partial differential equations,[6] but a more thorough insight of time-delayed systems is still lacking.

Synchrony of oscillatory motions can be ubiquitously found in nature and has been exhaustively studied.[7] Different levels of synchronizations were revealed for chaotic motion, such as complete synchronization,[8] generalized synchronization,[9] phase synchronizations,[10] and so on. For generalized synchronization, although there have been numerous discussions,[11,12] many problems still remain unsolved. The synchronization between two coupled time-delayed chaotic oscillators is of great interest and has been discussed recently.[13−15] Pyragas[13] studied the complete synchronization of two one-way coupled Mackey–Glass (MG)[16] systems. They found the increase and saturation behaviour of the coupling threshold by increasing the delay time. Zhan *et al* [17] further considered both complete and generalized synchronizations of the MG systems and found that for generalized synchronizations, the synchronization threshold for different delay times exhibits the interesting parameter resonance effect, i.e., stable synchronization can be achieved at a smaller coupling if the delay time of the driven system is chosen such that it is in resonance with the driving system.

In this paper, we study the dynamics and the synchronization of the one-way coupled MG systems. First we generalize the MG system and discuss its dynamics, and we can find that multiple chaotic attractors can coexist at the same parameter. We further

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classify these attractors by introducing a return number N_r that describes the number of recurrence to the Poincaré surface of section at $x = 0, \frac{d}{dt}x < 0$ in a period of the delay time τ. Dynamical properties of these chaotic motions are studied. Attractors with a larger N_r have fewer positive Lyapunov exponents and a lower Kaplan–Yorke dimension. Next we study the effect of multiple coexisting chaotic attractors on the synchronization. When these different classes of attractors are applied as the drive, the critical coupling strength of generalized synchronization becomes lower for larger N_r, leading to more parameter resonance effect regions.

The equation of motion of the MG system can be written as follows:

$$\dot{x} = \frac{ax_\tau}{1 + x_\tau^b} - cx, \tag{1}$$

where $x_\tau = x(t - \tau)$ denotes the time-delay variable; τ is the delay time. Other parameters are fixed at $a = 2$, $b = 10$, $c = 1$. Some of the dynamical properties of the MG system by varying the delay time τ have been investigated.[5,13,16,17] It has been found that at $\tau < 0.471$, the attractor of the MG system is the stable fixed point $x = 1$. For $0.471 < \tau < 1.33$, stable limit cycle attractors may emerge. The system experiences a period-doubling bifurcation in the interval $1.33 < \tau < 1.68$. When $\tau > 1.68$, the motion becomes chaotic, and the number of positive Lyapunov exponents increases linearly with increasing τ, i.e., the system exhibits the high-dimensional chaotic motion.

Traditionally, due to the physical background for describing blood productions,[16] the dynamics of the MG system is treated in the limited phase space $x > 0$. Mathematically, there is no limit for the x variable, i.e., it can be negative. In this paper we discuss the dynamics in the extended phase space of $-\infty < x < \infty$. In this region, there are three fixed points in MG system: $x = 1$, $x = -1$ and $x = 0$; $x = 1$ and $x = -1$ have the same property of stability. They are stable if $\tau < 0.471$; but $x = 0$ is always unstable for any value of τ. With increasing delay time, the system experiences the same bifurcation process as discussed above. However, different from the traditional case of $x > 0$, in our case we can find the coexistence of multiple dynamical attractors in the chaotic regime. In this regime, if the variable x is restricted to negative values, only the attractor around the unstable fixed point $x = -1$ can be observed. In fact, there exists another kind of chaotic attractor winding around all the three unstable fixed points, in which the variable can jump between positive and negative domains. We draw the phase portrait of the two kinds of attractors projected onto the plane $(x, dx/dt)$ in Fig.1.

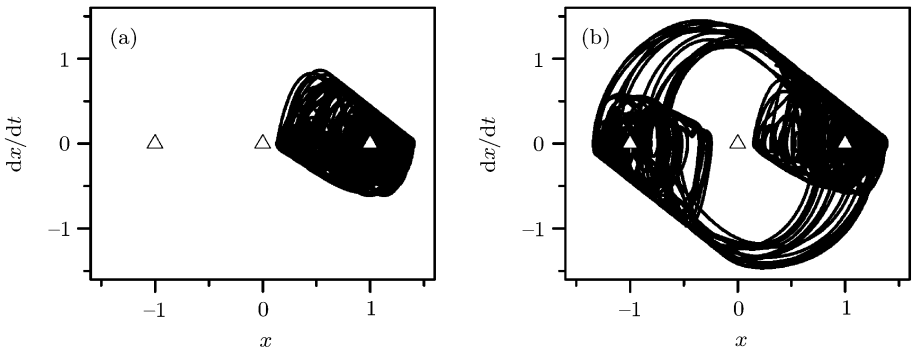


Fig.1. The phase portrait of two kinds of attractors projected onto the plane $(x, dx/dt)$ at $\tau = 100$. (a): The case when the variable x is always positive; (b): The case when $x(t)$ can jump between positive and negative domains. Three triangles denote three unstable fixed points.

Chaotic motion in the extended phase space brings more possibilities. Interestingly, we find that there are still multiple coexisting chaotic attractors for the same parameter τ. This is mainly due to the chaotic motion among the three unstable fixed points. To have an insight into this, one may construct a Poincaré surface of section $x = 0, \frac{d}{dt}x < 0$ to study these different motions. The return times between successive passings of this section can well exhibit the properties of the delay time of MG system. In Table 1, the return times for different delay times τ are shown in the first row. It can be found that the distributions of the return times exhibit some periodicity. The sum of one period approximately equals to

the delay time τ, but the numbers of the return times in one period can be totally different even if the delay time is the same:

$$\sum_{i=1}^{N_r} \tau_i \approx \tau. \tag{2}$$

Here N_r is called the return number, an integer that describes the number of returns in one delay-time period. For instance, for the case of $\tau = 100$ shown in Table 1, $N_r = 5$ in the second column, while $N_r = 4$ in the third column, and $N_r = 3$ in the fourth column. Therefore, one can use the return number N_r to classify these chaotic attractors and describe the features of attractors evolving around the unstable fixed point $x = 0$.

Table 1. Return times between successive crossings of the Poincaré section: $x = 0, \frac{d}{dt}x < 0$.

time-delay	100	100	100	130	200
	30.12	31.62	33.13	37.24	34.28
	18.76	15.25	37.76	29.45	37.31
	26.68	18.48	29.80	10.30	30.10
	16.27	35.38	33.10	24.03	9.30
	8.93	31.48	37.79	29.74	34.72
	30.06	15.36	29.84	37.21	17.31
return	18.75	18.42	33.13	29.50	20.36
	26.71	35.36	37.74	10.16	17.43
times	16.31	31.54	29.75	24.16	34.21
	8.75	15.41	33.13	29.59	37.28
	30.17	18.44	37.83	37.28	30.01
	18.79	35.37	29.79	29.46	9.33
	26.69	31.55	33.16	10.21	34.72
	16.19	15.29	37.74	24.04	17.39
	8.82	18.38	29.85	29.78	20.36
	...	...	...	...	...

For a fixed return number N_r, the summation of the return times is conserved, but τ_i could change in different delay-time period. In Fig.2, we show the time series of return times τ_i for an attractor with $N_r = 4$ returns in a period of the delay time. Here the delay time is chosen to be $\tau = 100$, and τ_i is arranged in a descending order, i.e. $\tau_1 > \tau_2 > \cdots$. It can be observed that τ_i changes with different delay-time periods in an random-walk manner.

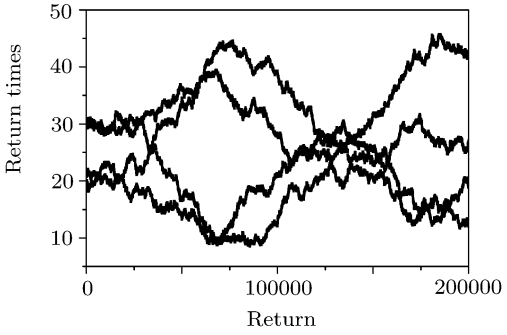


Fig.2. A series of return times when an attractor has four return times in a certain interval approximately equal to the delay time $\tau = 100$.

It is interesting to investigate the dynamical behaviours of these coexisting chaotic attractors. For time-delayed dynamical systems, one usually uses the Lyapunov exponent spectrum to study the dynamics of chaotic motion. The number of positive exponents is found to increase with increasing delay time. For a given delay time, the dynamics of aforementioned coexisting attractors with different return numbers may be different. In Fig.3(a), the Lyapunov exponent spectra $\{\lambda_i\}$ for different return numbers N_r are calculated. Obviously one finds that the distribution of Lyapunov exponents for different N_r are different. In particular, as shown in the inset of Fig.3(a), attractors with a higher return number N_r possess fewer numbers of positive exponents. We can compute the Kaplan–Yorke (Lyapunov) dimensions D_{KY} defined as follows:

$$D_{KY} = M + \frac{1}{|\lambda_{M+1}|}\sum_{j=1}^{M}\lambda_j, \tag{3}$$

where M is an integer satisfying

$$\sum_{j=1}^{M}\lambda_j \geq 0 \quad \text{and} \quad \sum_{j=1}^{M+1}\lambda_j < 0. \tag{4}$$

The Kaplan–Yorke dimension of attractors with different N_r are shown in Fig.3. It can be found that D_{KY} decreases monotonically with increasing return number N_r.

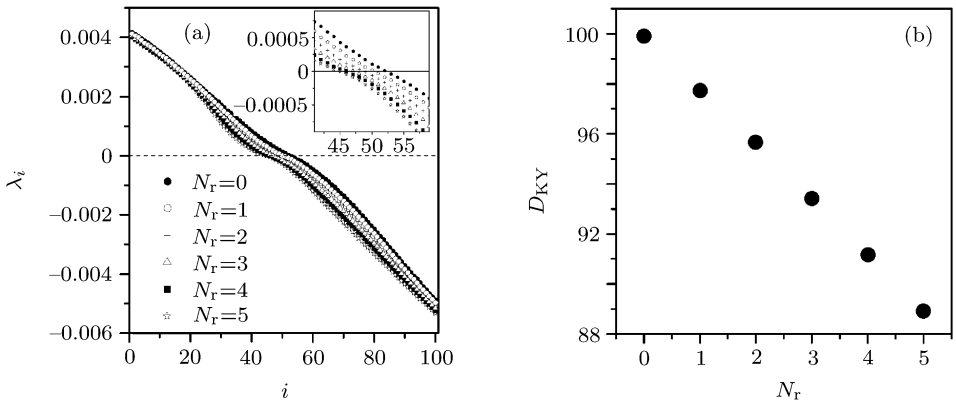


Fig.3. (a) The first one hundred maximum Lyapunov exponents for different classes of attractors. The inset exhibits the details in the vicinity of zero. (b) The Kaplan–Yorke dimensions for different classes of attractors for $\tau = 100$.

Now we study the effect of multiple chaotic attractors on the synchronization. Let us consider the following drive–response MG system:

$$\dot{x} = \frac{ax_{\tau_1}}{1 + x_{\tau_1}^b} - cx, \tag{5}$$

$$\dot{y} = \frac{ay_{\tau_2}}{1 + y_{\tau_2}^b} - cy + \varepsilon(x - y). \tag{6}$$

Here τ_1 and τ_2 are the delay times of the drive and the response systems, respectively; ε denotes the coupling (driving) strength. When $\tau_1 = \tau_2$, both complete synchronization and generalized synchronization are obtained when ε exceeds a threshold value. If $\tau_1 \neq \tau_2$, only generalized synchronization can be achieved. The largest conditional Lyapunov exponent is defined as:

$$\lambda(\varepsilon) = \lim_{t\to\infty} \frac{1}{t} \ln \frac{\left[\int_{-\tau_2}^{0} \Lambda^2(t+\varphi)\,\mathrm{d}\varphi\right]^{1/2}}{\left[\int_{-\tau_2}^{0} \Lambda^2(\varphi)\,\mathrm{d}\varphi\right]^{1/2}}, \tag{7}$$

where $\Lambda(t)$ is a small deviation coming from the linear stability analysis of y. $\lambda(\varepsilon)$ decreases monotonically with increasing ε. When $\lambda(\varepsilon) < 0$ at a critical strength ε_c, stable generalized synchronization could be observed. For fixed $\tau_1 = 100$ and changing τ_2, Zhan *et al* [17] calculated the dependence of the threshold ε_c on τ_2 and found the parameter resonance effect near $\tau_1 = \tau_2$ and $\tau_1 = 2\tau_2$. This is an interesting and significative phenomenon.

We discuss the effect of different coexisting chaotic attractors as drive on generalized synchronization. The delay time of the drive is chosen to be $\tau_1 = 100$. We calculate the critical driving strength ε_c for driving attractors with different N_r. The dependence of ε_c on the delay time of response τ_2 is shown in Fig.4.

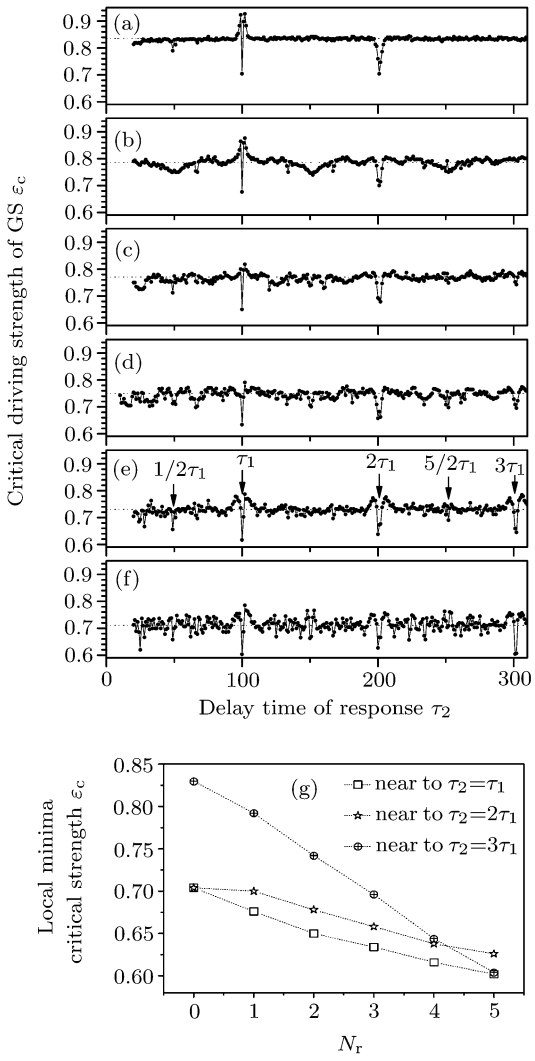


Fig.4. The dependence of the critical driving strength ε_c of generalized synchronization on the delay time of the response τ_2. Drivers have the same delay time $\tau_1 = 100$ but different N_r: (a)$N_r = 0$, (b)$N_r = 1$, (c)$N_r = 2$, (d)$N_r = 3$, (e)$N_r = 4$, and (f)$N_r = 5$. More resonances can be found for a larger N_r, as shown in (e). (g): The local minimum critical strengths ε_c near $\tau_2 = \tau_1,\ 2\tau_1,\ 3\tau_1$ varying against N_r.

When N_r=0 (the case studied by Zhan *et al*), the critical coupling for synchronization is on average $\varepsilon_c \approx 0.84$, except those at $\tau_2/\tau_1 = 1/2$, 1 and 2. This is in good agreement with the result obtained by Zhan *et al.*[17] When $N_r \neq 0$, one can find from Figs.4(b)–4(f) that on average the synchronization threshold is smaller, i.e., a weaker coupling could make the response synchronize with the drive. This result can be well understood intuitively. As seen from Fig.3(b), the Kaplan–Yorke dimension of the chaotic attractors decreases as the return number N_r increases, that is, chaotic motions with higher return numbers N_r exhibit lower complexity. Therefore, one only needs a lower coupling to achieve synchronization. Another significant result is that much more parameter resonances (local minima) near $\tau_1/\tau_2 = m_1/m_2$ can be observed when the return number N_r increases, where m_1 and m_2 are integers, as shown in Fig.4(b)–4(f). We further plot local minima of the critical strength ε_c near $\tau_2 = \tau_1,\ 2\tau_1,\ 3\tau_1$ against the number N_r in Fig.4(f), and find obviously that ε_c decreases with increasing N_r for all resonances.

In conclusion, we have studied the dynamics of the Mackey–Glass systems in the extended phase space. Multiple chaotic attractors for the same parameters are found. These attractors can be well classified by defining a return number N_r around an unstable fixed point $x = 0$, which describes the number of recurrence to the Poincaré surface of section at $x = 0$, and $\frac{d}{dt}x < 0$ in a period of the delay time τ. Dynamical properties of these chaotic motions have been studied. Attractors with a larger N_r have fewer positive Lyapunov exponents and a lower Kaplan–Yorke dimension. We have further studied the effect of multiple coexisting chaotic attractors on the synchronization in the drive–response systems. When the different classes of attractors are applied as the drive, the critical coupling strength of the generalized synchronization becomes lower for larger N_r, leading to more regions with parameter resonance effect. We hope this method of classifying chaotic attractors could make contributions to chaos research.

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