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A new variable coefficient algebraic method and non-travelling wave solutions of nonlinear equations*

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In this paper, a new auxiliary equation method is presented of constructing more new non-travelling wave solutions of nonlinear differential equations in mathematical physics, which is direct and more powerful than projective Riccati equation method. In order to illustrate the validity and the advantages of the method, (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equation is employed and many new double periodic non-travelling wave solutions are obtained. This algorithm can also be applied to other nonlinear differential equations.

Keywords: nonlinear partial differential equations, non-travelling wave solutions, asymmetric Nizhnik–Novikov–Vesselov equation

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1. Introduction

Nonlinear wave phenomena play an important role in the physical world and have been a challenging topic of research for both mathematicians and physicists. There exists an extensive literature dealing with nonlinear wave equations, in which exact explicit solitary wave, kink wave and periodic wave solutions are discussed. To find exact solutions for a given nonlinear wave system, a lot of effective methods have been presented such as tanh-function method, Jacobi elliptic function expansion method, Riccati equation rational expansion method, F-expansion method, projective Riccati equation method and so on.[1−18]

Motivated by the projective Riccati equation method proposed by Conte and Musette[18] in 1992, we present a new direct algebraic method and so obtain many new double periodic wave solutions of partial differential equations which cannot be acquired by using the projective Riccati equation method.

The remainder of this paper is organized as follows. In Section 2, we give the description of the new auxiliary equation method; In Section 3, we apply this method to a (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equation; Finally the conclusions drawn from the present study are given in Section 4.

2. Description of the method

For a given nonlinear partial differential equations with independent variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_m, t)$ and dependent variable u_i,

$$H_i(u_i, u_{it}, u_{ix_1}, u_{ix_2}, u_{ix_3}, u_{itt}, u_{ix_1t}, u_{ix_2t}, u_{ix_1x_1}, u_{ix_2x_2}, u_{ix_3x_3}, \ldots) = 0, \tag{1}$$

we seek its solutions in the form

$$u_i(x) = a_0 + \sum_{i=1}^{m_i} f^{i-1}(\xi)(a_i f(\xi) + b_i g(\xi)) \tag{2}$$

with new variables $f(\xi)$ and $g(\xi)$ satisfy the following relations:

$$\begin{aligned} f'(\xi) &= f(\xi)g(\xi), \\ g'(\xi) &= q + g(\xi)^2 + rf(\xi)^{-2}, \\ g(\xi)^2 &= -\left(q + \frac{r}{2}f(\xi)^{-2} + cf(\xi)^2\right), \end{aligned} \tag{3}$$

where q, r, and c are constants, $f'(\xi) = \dfrac{\mathrm{d}}{\mathrm{d}\xi}f(\xi)$, $g'(\xi) = \dfrac{\mathrm{d}}{\mathrm{d}\xi}g(\xi)$, and ξ, a_0, a_i, b_i $(i = 1, 2, \ldots, m_i)$ are the functions of x which are to be determined later.

To determine u_i explicitly, we take the following steps.

Step 1 Determine m_i by balancing the nonlinear term and highest order derivative term in Eq.(1). If m_i is not a positive integer, we firstly make the transformation $u_i(x_1, x_2, \ldots, x_m, t) = v_i(x_1, x_2, \ldots, x_m, t)^{m_i}$.

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Step 2 Substituting Eq.(2) into Eq.(1) along with Eqs.(3) , then equating all coefficients of $f(\xi)^i g(\xi)^j (i,j = 0,1,\dots)$ of the resulting system's numerator to zero, we can deduce a set of over-determined partial differential equations with respect to $a_i,\ b_j\ (i = 0,1,2,\dots,m_i, j = 1,2,\dots,m_i)$ and ξ.

Step 3 Solving the over-determined partial differential equations with the aid of the symbolic computation system Maple, we can obtain the explicit expressions for $a_i,\ b_j\ (i = 0,1,2,\dots,m_i, j = 1,2,\dots,m_i)$ and ξ.

Step 4 Substituting these results obtained in step 3 into Eq.(2) with the corresponding solutions $f(\xi)$ and $g(\xi)$ of Eqs.(3), we can obtain the exact solutions of Eq.(1).

In the rest of this section, we will analyse the solutions of Eq.(3). From Eqs.(3), one has

$$g(\xi) = \frac{f'(\xi)}{f(\xi)}. \tag{4}$$

Substituting Eq.(4) into Eq.(3) leads to

$$-f''(\xi)f(\xi) + 2f'(\xi)^2 + qf(\xi)^2 + r = 0. \tag{5}$$

In order to solve Eq.(5),we introduce the following transformation:

$$f(\xi) = \frac{1}{w(\xi)}, \tag{6}$$

then

$$\frac{f'(\xi)}{f(\xi)} = -\frac{w'(\xi)}{w(\xi)}, \quad g(\xi) = -\frac{w'(\xi)}{w(\xi)}, \tag{7}$$

and

$$w''(\xi) + qw(\xi) + rw(\xi)^3 = 0, \tag{8}$$

hence,

$$(w'(\xi))^2 + qw(\xi)^2 + \frac{r}{2}w(\xi)^4 + c = 0. \tag{9}$$

It is found that elliptic equation (8) or (9) admits the following several types of solutions:

Type 1 As $q = 1 + m^2$, $r = -2m^2$, and $c = -1$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{sn}(\xi, m). \tag{10}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_1(\xi) = \frac{1}{\mathrm{sn}(\xi, m)},$$

and

$$g_1(\xi) = -\frac{\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}. \tag{11}$$

Type 2 As $q = 1-2m^2$, $r = 2m^2$, and $c = m^2-1$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{cn}(\xi, m). \tag{12}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_2(\xi) = \frac{1}{\mathrm{cn}(\xi, m)},$$

and

$$g_2(\xi) = \frac{\mathrm{sn}(\xi, m)\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{13}$$

Type 3 As $q = -2 + m^2$, $r = 2$, and $c = 1 - m^2$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{dn}(\xi, m). \tag{14}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_3(\xi) = \frac{1}{\mathrm{dn}(\xi, m)},$$

and

$$g_3(\xi) = \frac{m^2\mathrm{sn}(\xi, m)\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}. \tag{15}$$

Type 4 As $q = 1 + m^2$, $r = -2$, and $c = -m^2$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{ns}(\xi, m) = \frac{1}{\mathrm{sn}(\xi, m)}. \tag{16}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_4(\xi) = \mathrm{sn}(\xi, m),$$

and

$$g_4(\xi) = \frac{\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}. \tag{17}$$

Type 5 As $q = 1 - 2m^2$, $r = -2 + 2m^2$, and $c = m^2$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{nc}(\xi, m) = \frac{1}{\mathrm{cn}(\xi, m)}. \tag{18}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_5(\xi) = \mathrm{cn}(\xi, m),$$

and

$$g_5(\xi) = -\frac{\mathrm{sn}(\xi, m)\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{19}$$

Type 6 As $q = -2+m^2$, $r = 2-2m^2$, and $c = 1$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{nd}(\xi, m) = \frac{1}{\mathrm{dn}(\xi, m)}. \tag{20}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_6(\xi) = \mathrm{dn}(\xi, m),$$

and

$$g_6(\xi) = -m^2 \frac{\mathrm{sn}(\xi, m)\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}. \tag{21}$$

Type 7 As $q = -2 + m^2$, $r = -2 + 2m^2$, and $c = -1$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{sc}(\xi, m) = \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{22}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_7(\xi) = \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)},$$

and

$$g_7(\xi) = -\frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)\mathrm{cn}(\xi, m)}. \tag{23}$$

Type 8 As $q = 1 - 2m^2$, $r = 2m^2 - 2m^4$, and $c = -1$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{sd}(\xi, m) = \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}. \tag{24}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_8(\xi) = \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)},$$

and

$$g_8(\xi) = -\frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)\mathrm{dn}(\xi, m)}. \tag{25}$$

Type 9 As $q = -2 + m^2$, $r = -2$, $c = -1 + m^2$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{cs}(\xi, m) = \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}. \tag{26}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_9(\xi) = \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)},$$

and

$$g_9(\xi) = \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)\mathrm{cn}(\xi, m)}. \tag{27}$$

Type 10 As $q = 1 + m^2$, $r = -2m^2$, and $c = -1$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{cd}(\xi, m) = \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)}. \tag{28}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{10}(\xi) = \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)},$$

and

$$g_{10}(\xi) = (1 - m^2) \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m)}. \tag{29}$$

Type 11 As $q = 1 - 2m^2$, $r = -2$, and $c = m^2 - m^4$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{ds}(\xi, m) = \frac{\mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}. \tag{30}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{11}(\xi) = \frac{\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)},$$

and

$$g_{11}(\xi)) = \frac{\mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)\mathrm{dn}(\xi, m)}. \tag{31}$$

Type 12 As $q = 1 + m^2$, $r = -2$, and $c = -m^2$, then the solution of Eq.(8) takes the form

$$w(\xi) = \mathrm{dc}(\xi, m) = \frac{\mathrm{dn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{32}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{12}(\xi) = \frac{\mathrm{cn}(\xi, m)}{\mathrm{dn}(\xi, m)},$$

and

$$g_{12}(\xi) = (-1 + m^2) \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m)}. \tag{33}$$

Type 13 As $q = \dfrac{-1 + 2m^2}{2}$, $r = -\dfrac{1}{2}$, and $c = -\dfrac{1}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m) \pm 1}. \tag{34}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{13}(\xi) = \frac{\mathrm{cn}(\xi, m) \pm 1}{\mathrm{sn}(\xi, m)},$$

and

$$g_{13}(\xi, m) = \mp \mathrm{ds}(\xi, m). \tag{35}$$

Type 14 As $q = -\dfrac{m^2 + 1}{2}$, $r = \dfrac{1 - m^2}{2}$, and $c = \dfrac{1 - m^2}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{\pm 1 + m\mathrm{sn}(\xi, m)}{\mathrm{dn}(\xi, m)}. \tag{36}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{14}(\xi) = \frac{\mathrm{dn}(\xi, m)}{\pm 1 + m\mathrm{sn}(\xi, m)},$$

and

$$g_{14}(\xi) = \mp m\mathrm{cd}(\xi, m). \tag{37}$$

Type 15 As $q = \dfrac{-m^2+2}{2}$, $r = -\dfrac{m^2}{2}$, and $c = -\dfrac{m^2}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{1}{\mathrm{sn}(\xi, m) \pm \mathrm{icn}(\xi, m)}. \tag{38}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{15}(\xi) = \mathrm{sn}(\xi, m) + \mathrm{icn}(\xi, m),$$

and

$$g_{15}(\xi) = -\mathrm{idn}(\xi, m). \tag{39}$$

Type 16 As $q = -\dfrac{1+m^2}{2}$, $r = \dfrac{1-2m^2+m^4}{2}$, and $c = \dfrac{1}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{1}{m\mathrm{cn}(\xi, m) \pm \mathrm{dn}(\xi, m)}. \tag{40}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{16}(\xi) = m\mathrm{cn}(\xi, m) + \mathrm{dn}(\xi, m),$$

and

$$g_{16}(\xi) = -m\mathrm{sn}(\xi, m). \tag{41}$$

Type 17 As $q = -\dfrac{1+m^2}{2}$, $r = -\dfrac{1-2m^2+m^4}{2}$, and $c = -\dfrac{1}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m) \pm \mathrm{dn}(\xi, m)}. \tag{42}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{17}(\xi) = \frac{\mathrm{cn}(\xi, m) \pm \mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)},$$

and

$$g_{17}(\xi) = \mp \mathrm{ns}(\xi, m). \tag{43}$$

Type 18 As $q = m^2 - 6m + 1$, $r = -2$, and $c = 4m - 8m^2 + 4m^3$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{-1 + m\mathrm{sn}(\xi, m)^2}{\mathrm{sn}(\xi, m)}. \tag{44}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{18}(\xi) = \frac{\mathrm{sn}(\xi, m)}{-1 + m\mathrm{sn}(\xi, m)^2},$$

and

$$g_{18}(\xi) = -\mathrm{cs}(\xi, m)\mathrm{dn}(\xi, m)\frac{m\mathrm{sn}(\xi, m)^2 + 1}{m\mathrm{sn}(\xi, m)^2 - 1}. \tag{45}$$

Type 19 As $q = m^2 + 6m + 1$, $r = -2$, and $c = -4m - 8m^2 - 4m^3$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{1 + m\mathrm{sn}(\xi, m)^2}{\mathrm{sn}(\xi, m)}. \tag{46}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{19}(\xi) = \frac{\mathrm{sn}(\xi, m)}{1 + m\mathrm{sn}(\xi, m)^2},$$

and

$$g_{19}(\xi) = -\mathrm{cs}(\xi, m)\mathrm{dn}(\xi, m)\frac{m\mathrm{sn}(\xi, m)^2 - 1}{m\mathrm{sn}(\xi, m)^2 + 1}. \tag{47}$$

Type 20 As $q = -\dfrac{1+m^2}{2}$, $r = \dfrac{-1+m^2}{2}$, and $c = \dfrac{m^2-1}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{\pm 1 + \mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m)}. \tag{48}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{20}(\xi) = \frac{\mathrm{cn}(\xi, m)}{\pm 1 + \mathrm{sn}(\xi, m)},$$

and

$$g_{20}(\xi) = \mp \mathrm{dc}(\xi, m). \tag{49}$$

Type 21 As $q = \dfrac{2-m^2}{2}$, $r = -\dfrac{m^4}{2}$, and $c = -\dfrac{1}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{\mathrm{sn}(\xi, m)}{\pm 1 + \mathrm{dn}(\xi, m)}. \tag{50}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{21}(\xi) = \frac{\pm 1 + \mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)},$$

and

$$g_{21}(\xi) = \mp \mathrm{cs}(\xi, m). \tag{51}$$

Type 22 As $q = \dfrac{-1+2m^2}{2}$, $r = -\dfrac{1}{2}$, and $c = -\dfrac{1}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{1 \pm \mathrm{cn}(\xi, m)}{\mathrm{sn}(\xi, m)}. \tag{52}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{22}(\xi) = \frac{\mathrm{sn}(\xi, m)}{1 \pm \mathrm{cn}(\xi, m)},$$

and

$$g_{22}(\xi) = \pm \mathrm{ds}(\xi, m). \tag{53}$$

Type 23 As $q = -\dfrac{1+m^2}{2}$, $r = -\dfrac{1}{2}$, and $c = -\dfrac{m^4-2m^2+1}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{\mathrm{cn}(\xi, m) \pm \mathrm{dn}(\xi, m)}{\mathrm{sn}(\xi, m)}. \tag{54}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{23}(\xi) = \frac{\mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m) \pm \mathrm{dn}(\xi, m)},$$

and

$$g_{23}(\xi) = \pm \mathrm{ns}(\xi, m). \tag{55}$$

Type 24 As $q = \dfrac{-1+2m^2}{2}$, $r = -\dfrac{1}{2(3m^2+1)}$, and $c = -\dfrac{3m^2+1}{4}$, then the solution of Eq.(8) takes the form

$$w(\xi) = \frac{\mathrm{cn}(\xi, m) + 2}{\sqrt{\dfrac{-3}{3m^2+1}}\mathrm{dn}(\xi, m) + \mathrm{sn}(\xi, m)}. \tag{56}$$

Hence, considering Eqs.(6) and (7), we have the following results:

$$f_{24}(\xi) = \frac{\sqrt{\dfrac{-3}{3m^2+1}}\mathrm{dn}(\xi, m) + \mathrm{sn}(\xi, m)}{\mathrm{cn}(\xi, m) + 2},$$

and

$$g_{24}(\xi) = \frac{\mathrm{dn}(\xi, m) + 2\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m) - (m^2-1)\sqrt{\dfrac{-3}{3m^2+1}}\mathrm{sn}(\xi, m) + 2\sqrt{\dfrac{-3}{3m^2+1}}m^2\mathrm{cn}(\xi, m)\mathrm{sn}(\xi, m)}{(2 + \mathrm{cn}(\xi, m))\left(\mathrm{sn}(\xi, m) + \sqrt{\dfrac{-3}{3m^2+1}}\mathrm{dn}(\xi, m)\right)}, \tag{57}$$

where $\mathrm{sn}(\xi, m)$, $\mathrm{cn}(\xi, m)$ and $\mathrm{dn}(\xi, m)$ are Jacobi elliptic sine function, Jacobi elliptic cosine function, Jacobi elliptic function of third kind respectively, and m denotes the modulus of Jacobi elliptic functions. When $m \to 1$, Jacobi functions degenerate to the hyperbolic functions. When $m \to 0$, Jacobi functions degenerate to the triangular functions. For detailed information about Jacobi elliptic functions see Refs.[19, 20].

3. The non-travelling wave solutions of (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equation

Now we consider the asymmetric Nizhnik–Novikov–Vesselov (ANNV) equation,

$$H_t - H_{xxx} - 3(HG)_x = 0,$$

and

$$H_x - G_y = 0. \tag{58}$$

The ANNV equation is also named the (2+1)-dimensional Korteweg-de-Vries (KdV) equation or Boiti–Leon–Manna–Pempinelli equation obtained by Boiti *et al* based on the idea of the weak Lax pair.[21] Lou and Hu[22] point out that Eq.(58) is an asymmetric part of the Nizhnik–Novikov–Vesselov (NNV) equation. Vekslerchik[23] derived the Bäcklund transformation of Eq.(58). Interested readers are advised to see related achievements[21−24] for more details about this system.

Now we apply the above method to Eq.(58). By balancing the highest nonlinear terms and the highest order partial derivative terms in Eq.(58), we suppose that Eq.(58) has the following formal non-travelling wave solution:

$$H(\xi) = a_0 + a_1 f(\xi) + a_2 g(\xi) + a_3 f(\xi)^2 + a_4 f(\xi) g(\xi),$$

and

$$G(\xi) = b_0 + b_1 f(\xi) + b_2 g(\xi) + b_3 f(\xi)^2 + b_4 f(\xi) g(\xi), \tag{59}$$

where $\xi = p(x) + m(y,t)$, $p(x)$ is the function of x, a_i, b_i $(i = 0,1,2,3,4)$ and $m(y,t)$ are the functions of (y,t) and the new variables $f(\xi)$ and $g(\xi)$ satisfy Eq.(3).

With the aid of Maple, substituting Eq.(59) along with Eq.(3) into Eq.(58) and equating the coefficients of these terms $f(\xi)^i g(\xi)^j (i = 0,1,\ldots, j = 0,1,\ldots)$ to zero yield a set of over-determined partial differential equations with respect to $a_i(y,t), b_i(y,t)$ $(i = 0,1,2,3,4)$, $p(x)$ and $m(y,t)$. By use of the Maple soft package 'PDEtools', solving the over-determined partial differential equations, we obtain the following results:

Case 3.1

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0,$$
$$a_3 = 2C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right), \quad a_4 = 0,$$
$$b_0 = F_1(t), \quad b_1 = 0, \quad b_2 = 0,$$
$$b_3 = 2cC_1^2, \quad b_4 = 0, \quad p(x) = C_1 x + C_2,$$

and

$$m(y,t) = \int 3C_1 F_1(t)\mathrm{d}t - 4C_1^3 qt + F_2(y); \tag{60}$$

Case 3.2

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right),$$
$$a_4 = (-c)^{\frac{1}{2}} C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right),$$
$$b_0 = F_1(t), \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = cC_1^2,$$
$$b_4 = (-c)^{\frac{1}{2}} C_1^2,$$
$$p(x) = C_1 x + C_2,$$

and

$$m(y,t) = \int 3C_1 F_1(t)\mathrm{d}t - C_1^3 qt + F_2(y); \tag{61}$$

Case 3.3

$$a_0 = 0, \quad a_1 = 0, \quad a_2 = 0, \quad a_3 = cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right),$$
$$a_4 = -(-c)^{\frac{1}{2}} C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right),$$
$$b_0 = F_1(t), \quad b_1 = 0, \quad b_2 = 0,$$
$$b_3 = cC_1^2, \quad b_4 = -(-c)^{\frac{1}{2}} C_1^2,$$
$$p(x) = C_1 x + C_2,$$

and

$$m(y,t) = \int 3C_1 F_1(t)\mathrm{d}t - C_1^3 qt + F_2(y); \tag{62}$$

Case 3.4

$$a_0 = \frac{(C_3 + 4C_1^3 q)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{3C_1^2}, \quad a_1 = 0, \quad a_2 = 0,$$
$$a_3 = 2cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right), \quad a_4 = 0,$$
$$b_0 = F_1(t), \quad b_1 = 0, \quad b_2 = 0,$$
$$b_3 = 2cC_1^2, \quad b_4 = 0,$$
$$p(x) = C_1 x + C_2,$$

and

$$m(y,t) = \int 3C_1 F_1(t)\mathrm{d}t + C_3 t + F_3(y); \tag{63}$$

Case 3.5

$$a_0 = \frac{(C_3 + C_1^3 q)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{3C_1^2}, \quad a_1 = 0, \quad a_2 = 0,$$
$$a_3 = cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right),$$
$$a_4 = -(-c)^{\frac{1}{2}} C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right),$$
$$b_0 = F_1(t), \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = cC_1^2,$$
$$b_4 = -(-c)^{\frac{1}{2}} C_1^2, \quad p(x) = C_1 x + C_2,$$

and

$$m(y,t) = \int 3C_1 F_1(t)\mathrm{d}t + C_3 t + F_3(y); \tag{64}$$

Case 3.6

$$a_0 = \frac{(C_3 + C_1^3 q)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{3C_1^2}, \quad a_1 = 0, \quad a_2 = 0,$$
$$a_3 = cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right),$$
$$a_4 = (-c)^{\frac{1}{2}} C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right),$$
$$b_0 = F_1(t), \quad b_1 = 0, \quad b_2 = 0, \quad b_3 = cC_1^2,$$
$$b_4 = (-c)^{\frac{1}{2}} C_1^2, \quad p(x) = C_1 x + C_2,$$

$$m(y,t)=\int 3C_1F_1(t)\mathrm{d}t+C_3t+F_3(y), \tag{65}$$

where $C_1\neq 0$, q, r, c, C_2 and C_3 are arbitrary constants; $F_1(t)$ is an arbitrary function of t; $F_2(y)$ and $F_3(y)$ are arbitrary functions of y respectively.

Substituting Cases 3.1–3.6 into Eqs.(59), we have six kinds of formal solutions of Eq.(58) respectively:

Family 1

$$H_1(\xi)=2C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)f(\xi)^2,$$

and

$$G_1(\xi)=F_1(t)+2cC_1^2f(\xi)^2, \tag{66}$$

where $\xi=C_1x+F_2(y)+\int 3C_1F_1(t)\mathrm{d}t-4C_1^3qt+C_2$, with C_1 and C_2 are arbitrary nonzero constants; $F_1(t)$ and $F_2(y)$ being arbitrary functions of t and y respectively.

Family 2

$$\begin{aligned}H_2(\xi)=&cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)f(\xi)^2\\&+(-c)^{\frac{1}{2}}C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)f(\xi)g(\xi),\end{aligned}$$

and

$$\begin{aligned}G_2(\xi)=&F_1(t)+cC_1^2f(\xi)^2\\&+(-c)^{\frac{1}{2}}C_1^2f(\xi)g(\xi),\end{aligned} \tag{67}$$

where $\xi=C_1x+F_2(y)+\int 3C_1F_1(t)\mathrm{d}t-C_1^3qt+C_2$, with C_1 and C_2 are arbitrary nonzero constants; $F_1(t)$ and $F_2(y)$ are arbitrary functions of t and y respectively.

Family 3

$$\begin{aligned}H_3(\xi)=&cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)f(\xi)^2\\&-(-c)^{\frac{1}{2}}C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_2(y)\right)f(\xi)g(\xi),\end{aligned}$$

and

$$\begin{aligned}G_3(\xi)=&F_1(t)+cC_1^2f(\xi)^2\\&-(-c)^{\frac{1}{2}}C_1^2f(\xi)g(\xi),\end{aligned} \tag{68}$$

where $\xi=C_1x+F_2(y)+\int 3C_1F_1(t)\mathrm{d}t-C_1^3qt+C_2$, with C_1 and C_2 are arbitrary nonzero constants; $F_1(t)$ and $F_2(y)$ are arbitrary functions of t and y respectively.

Family 4

$$\begin{aligned}H_4(\xi)=&\frac{(C_3+4C_1^3q)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{3C_1^2}\\&+2cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)f(\xi)^2,\end{aligned}$$

and

$$G_4(\xi)=F_1(t)+2cC_1^2f(\xi)^2, \tag{69}$$

where $\xi=C_1x+F_3(y)+\int 3C_1F_1(t)\mathrm{d}t+C_3t+C_2$, with $C_1\neq 0$; C_2 and C_3 being arbitrary constants; $F_1(t)$ and $F_3(y)$ are arbitrary functions of t and y respectively.

Family 5

$$\begin{aligned}H_5(\xi)=&\frac{(C_3+C_1^3q)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{3C_1^2}\\&+cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)f(\xi)^2\\&-(-c)^{\frac{1}{2}}C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)f(\xi)g(\xi),\end{aligned}$$

and

$$\begin{aligned}G_5(\xi)=&F_1(t)+cC_1^2f(\xi)^2\\&-(-c)^{\frac{1}{2}}C_1^2f(\xi)g(\xi),\end{aligned} \tag{70}$$

where $\xi=C_1x+F_3(y)+\int 3C_1F_1(t)\mathrm{d}t+C_3t+C_2$, with $C_1\neq 0$, C_2 and C_3 are arbitrary constants; $F_1(t)$ and $F_3(y)$ are arbitrary functions of t and y respectively.

Family 6

$$\begin{aligned}H_6(\xi)=&\frac{(C_3+C_1^3q)\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{3C_1^2}\\&+cC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)f(\xi)^2\\&+(-c)^{\frac{1}{2}}C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)f(\xi)g(\xi),\end{aligned}$$

and

$$\begin{aligned}G_6(\xi)=&F_1(t)+cC_1^2f(\xi)^2\\&+(-c)^{\frac{1}{2}}C_1^2f(\xi)g(\xi),\end{aligned} \tag{71}$$

where $\xi=C_1x+C_2+\int 3C_1F_1(t)\mathrm{d}t+C_3t+F_3(y)$, with $C_1\neq 0$, C_2 and C_3 are arbitrary constants; $F_1(t)$ and $F_3(y)$ are arbitrary functions of t and y respectively.

Families 1–6 are different from Case 1 in Ref.[25], and they can not be obtained by using the method in Refs.[25, 26]. From expressions (66)–(71) and Types 1–24, we can derive many new explicit non-travelling wave solutions of the ANNV equations. Since the solutions of ANNV equation we have obtained here are so many, it is very tedious to write all the possible solutions of ANNV equation. For simplicity, we only enumerate some of them as example to illustrate the effectiveness of our method.

If we select $q = 1 + m^2$, $r = -2$, and $c = -m^2$, according to Eq.(71) and Type 4, we obtain double periodic non-travelling solution of Eq.(58) as

$$\begin{aligned} H_1(\xi) =& \frac{(C_3 + C_1^3(1+m^2))\left(\dfrac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{3C_1^2} \\ & - mC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)(m\mathrm{sn}(\xi,m)^2 \\ & \pm \mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m)), \end{aligned}$$

and

$$\begin{aligned} G_1(\xi) =& F_1(t) + mC_1^2(m\mathrm{sn}(\xi,m)^2 \\ & \pm \mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m)). \end{aligned} \tag{72}$$

For a better understanding, we plot H_1, which shows the dynamics of Jacobi elliptic wave solutions. A particular case is to take $F_3(y) = \sin(y)$, and $F_1(t) = \mathrm{e}^t$, the "+" in each sign "±", see Fig.1. Jacobi elliptic wave H moves continuously to the left ,which is reflected in Fig.2.

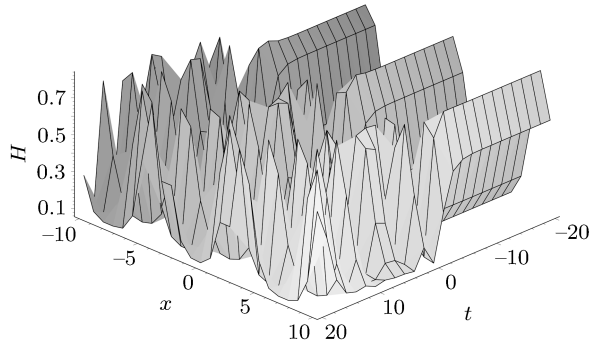


Fig.1. The dynamics of Jacobi elliptic wave solutions given by H_1 for $C_1 = 1$, $C_2 = 0$, $C_3 = -1$, $m = 7/10$ and $y = 1$.

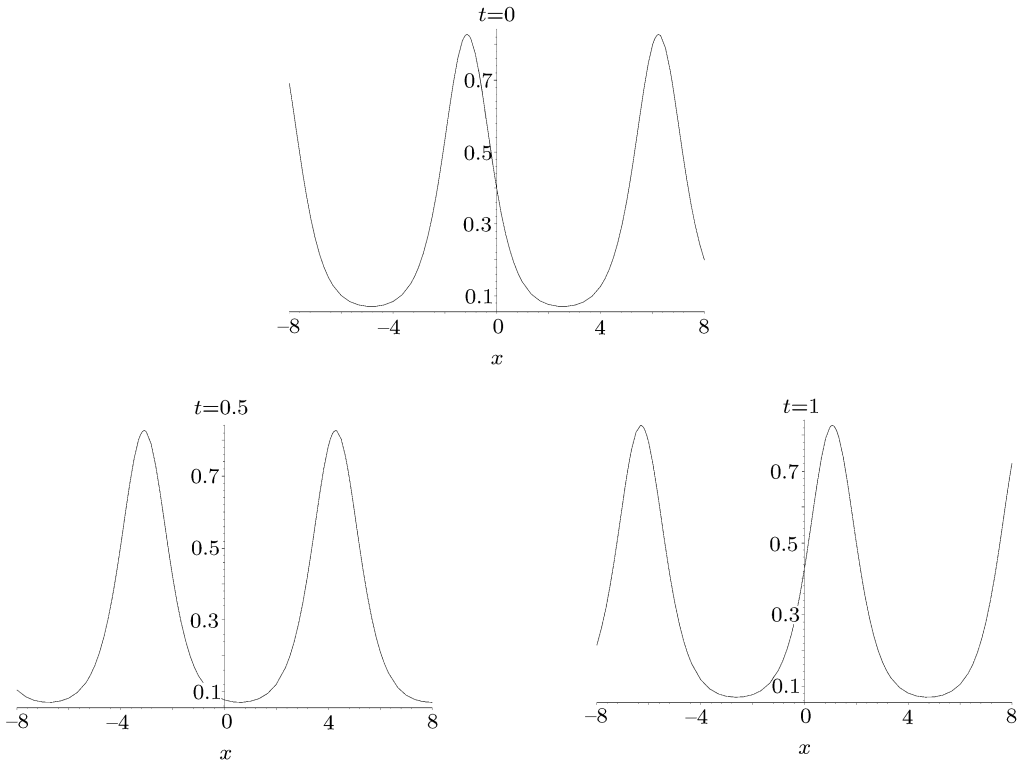


Fig.2. Shape and motion of Jacobi elliptic wave solutions given by H_1 for $C_1 = 1$, $C_2 = 0$, $C_3 = -1$, $m = 7/10$ and $y = 1$.

If we select $q = -\dfrac{m^2+1}{2}$, $r = \dfrac{1-m^2}{2}$, and $c = \dfrac{1-m^2}{4}$, according to Eq.(71) and Type 14, we obtain double periodic non-travelling solution of Eq.(58) as

$$\begin{aligned} H_2(\xi) =& \frac{(2C_3 - C_1^3(m^2+1))\left(\dfrac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{6C_1^2} + \frac{1-m^2}{4}C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)\frac{\mathrm{dn}(\xi,m)^2}{(\pm 1 + m\mathrm{sn}(\xi,m))^2} \\ & \mp \frac{\sqrt{-1+m^2}}{2}mC_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)\frac{\mathrm{cn}(\xi,m)}{\pm 1 + m\mathrm{sn}(\xi,m)}, \end{aligned}$$

and

$$G_2(\xi) = F_1(t) + \frac{1-m^2}{4}C_1^2\frac{\mathrm{dn}(\xi,m)^2}{(\pm 1 + m\mathrm{sn}(\xi,m))^2} \mp \frac{\sqrt{-1+m^2}}{2}mC_1^2\frac{\mathrm{cn}(\xi,m)}{\pm 1 + m\mathrm{sn}(\xi,m)}. \quad (73)$$

For a better understanding, we plot H_2, which shows the dynamics of Jacobi elliptic wave solutions. As a particular case we take $F_3(y) = \tanh(y)$, and $F_1(t) = \mathrm{e}^t$, the '+' in each sign "±", see Fig.3. The Jacobi elliptic wave H moves continuously to the left, which is reflected in Fig.4.

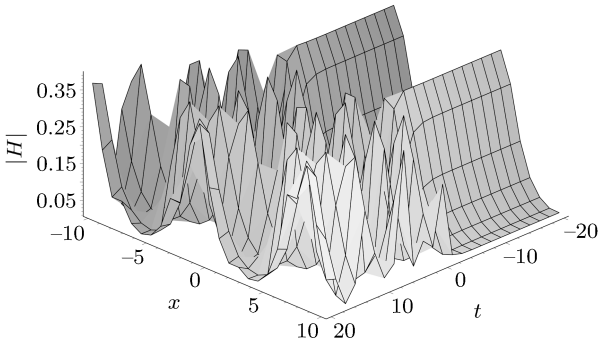


Fig.3. The dynamics of Jacobi elliptic wave solutions given by $|H_2|$ for $C_1 = 1$, $C_2 = 0$, $C_3 = -1$, $m = 7/10$ and $y = 1$.

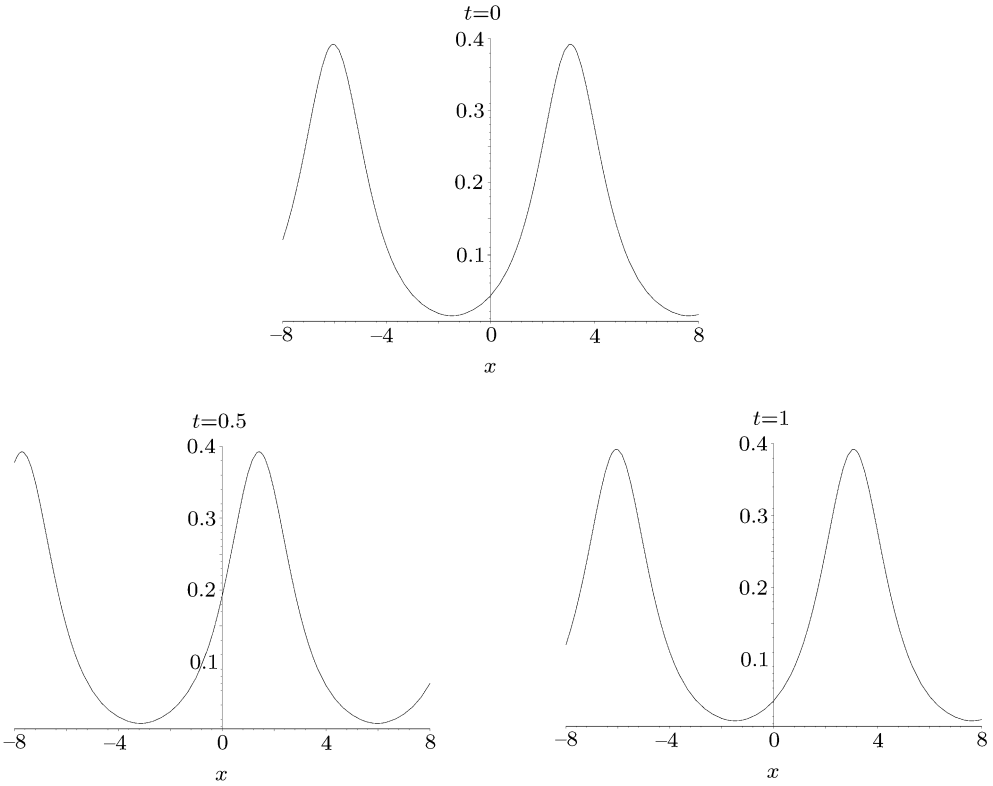


Fig.4. Shape and motion of Jacobi elliptic wave solutions given by $|H_2|$ for $C_1 = 1$, $C_2 = 0$, $C_3 = -1$, $m = 7/10$ and $y = 1$.

If we select $q = m^2 - 6m + 1$, $r = -2$, and $c = 4m - 8m^2 + 4m^3$, according to Eq.(71) and Type 18, we obtain double periodic non-travelling solution of Eq.(58) as

$$H_3(\xi) = \frac{(C_3 + C_1^3(m^2-6m+1))\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)}{3C_1^2} + (4m - 8m^2 + 4m^3)C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)\frac{\mathrm{sn}(\xi,m)^2}{(-1+m\mathrm{sn}(\xi,m)^2)^2} - \sqrt{-4m+8m^2-4m^3}C_1\left(\frac{\mathrm{d}}{\mathrm{d}y}F_3(y)\right)\frac{\mathrm{cn}(\xi,m)\mathrm{dn}(\xi,m)}{m\mathrm{sn}(\xi,m)^2+1},$$

and

$$\begin{aligned} G_3(\xi) =& F_1(t) + (4m - 8m^2 + 4m^3) \\ & \times C_1^2 \frac{\mathrm{sn}(\xi, m)^2}{(-1 + m\mathrm{sn}(\xi, m)^2)^2} \\ & - \sqrt{-4m + 8m^2 - 4m^3} \\ & \times C_1^2 \frac{\mathrm{cn}(\xi, m)\mathrm{dn}(\xi, m)}{m\mathrm{sn}(\xi, m)^2 + 1}. \end{aligned} \tag{74}$$

For a better understanding, we plot H_3, which shows the dynamics of Jacobi elliptic wave solutions. As a particular case we take $F_3(y) = \tanh(y)$, and $F_1(t) = \mathrm{e}^t$, the '+' in each sign '±', see Fig.5. The Jacobi elliptic wave H moves continuously to the left, which is reflected in Fig.6.

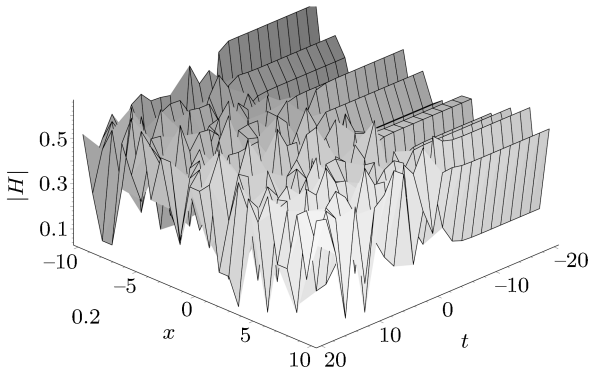


Fig.5. The dynamics of Jacobi elliptic wave solutions given by $|H_3|$ for $C_1 = 1$, $C_2 = 0$, $C_3 = -1$, $m = 7/10$ and $y = 1$.

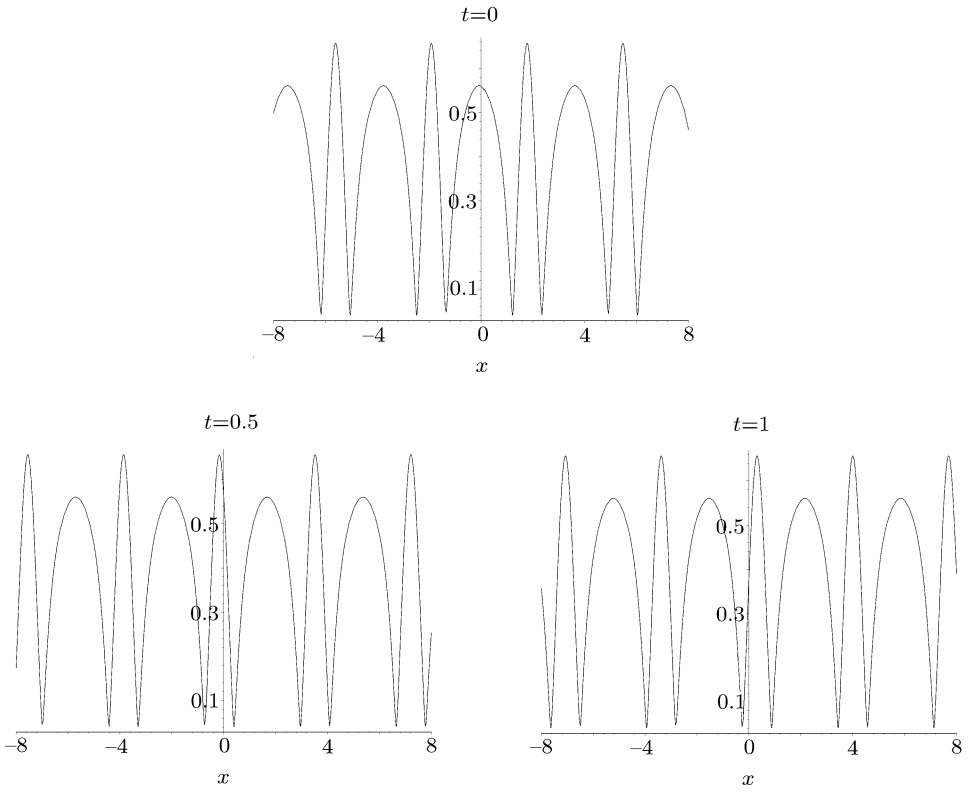


Fig.6. Shape and motion of Jacobi elliptic wave solutions given by $|H_3|$ for $C_1 = 1$, $C_2 = 0$, $C_3 = -1$, $m = 7/10$ and $y = 1$.

Similarly, we can illustrate the characteristics of other Jacobi elliptic wave solutions.

4. Conclusion

We have proposed a new variable coefficient algebraic method and derived many families of exact non-travelling wave solutions of nonlinear partial differential equations in a unified way. The (2+1)-dimensional asymmetric Nizhnik–Novikov–Vesselov equation is chosen to illustrate the method so that many families of new formal Jacobi elliptic function solutions are obtained. When $k \to 1$, some of these obtained solutions degenerate into solitary wave solutions, and when $k \to 0$, some of these obtained solutions degenerate into triangular periodic solutions. In this paper, we have successfully recovered some previously known solutions in Refs.[25–59]. More importantly, for some equations we found more general forms of new solutions with no extra effort. We believe that our method can also be used to find new solutions for other nonlinear differential equations in mathematical physics.

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