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Non-isospectral integrable couplings of Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy with self-consistent sources*

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A hierarchy of non-isospectral Ablowitz–Kaup–Newell–Segur (AKNS) equations with self-consistent sources is derived. As a general reduction case, a hierarchy of non-isospectral nonlinear Schrödinger equations (NLSE) with self-consistent sources is obtained. Moreover, a new non-isospectral integrable coupling of the AKNS soliton hierarchy with self-consistent sources is constructed by using the Kronecker product.

Keywords: equations hierarchy, self-consistent sources, integrable couplings
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1. Introduction

It is an important and interesting topic to search for the new integrable systems in soliton theory.[1−4] In particular, some methods to construct the integrable hierarchies with self-consistent sources have been developed.[5] However, these methods are not designed to construct the non-isospectral integrable hierarchies with self-consistent sources. Hence, in this paper we will establish a non-isospectral integrable equations with self-consistent sources by using the compatibility condition of Lax pairs.

Recently, the integrable couplings have been receiving growing attention. A few ways to establish integrable couplings are presented by using perturbations,[6−8] enlarging spectral problems,[9,10] semidirect sums of Lie algebras,[11,12] and creating new Lie algebras.[13−19] Moreover, the Hamiltonian forms of the integrable nonlinear equations can be constructed by the bi-trace identity,[20] which is a beautiful result. Here we will give a new method to establish the integrable coupling equations with self-consistent sources by using the Kronecker product.

In this paper, we present a non-isospectral integrable coupling of the Ablowitz–Kaup–Newell–Segur (AKNS) hierarchy with self-consistent sources, which is a novel result. The paper is organized as follows. In Section 2, the hierarchy of non-isospectral AKNS equations with self-consistent sources is constructed. In Section 3, by reducing, we work out the non-isospectral NLSE with self-consistent sources. In Section 4, a non-isospectral integrable coupling of the resulting AKNS soliton hierarchy with self-consistent sources is derived by using the Kronecker product. Finally, some conclusions are given in Section 5.

2. Non-isospectral AKNS equation hierarchy with self-consistent sources

In this section, we construct a non-isospectral zero curvature equation. First, consider the following non-isospectral linear problem:

$$\begin{aligned} \boldsymbol{\psi}_{nx} &= \boldsymbol{U}_n \boldsymbol{\psi}_n, \\ \boldsymbol{\psi}_{nt} &= \boldsymbol{V}_n \boldsymbol{\psi}_n, \end{aligned} \tag{1}$$

where $\boldsymbol{U}_n = \boldsymbol{e}_0 + \boldsymbol{u}_1\boldsymbol{e}_1 + \ldots + \boldsymbol{u}_p\boldsymbol{e}_p$, $\boldsymbol{u}_i(x,t) = \boldsymbol{u}_i (i = 1, 2, \ldots, p)$, and $\psi_n = \psi(x,t)$ are field variables defined on $x \in R$, and $t \in R$; $\boldsymbol{e}_i = \boldsymbol{e}_i(\lambda) \in G$ (G denotes a finite-dimensional Lie algebra over C). Then, the spectral parameter $\lambda = \lambda(t,x)$ satisfies a non-isospectral condition in the form of

$$\lambda_t = \beta(\lambda), \tag{2}$$

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where $\beta(\lambda)$ is a function to be specified. From spectral problem (1), the compatibility condition of Eq.(1) gives

$$\frac{\partial \boldsymbol{U}_n}{\partial t} + \frac{\partial \boldsymbol{U}_n}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \boldsymbol{V}_{nx} + [\boldsymbol{U}_n, \boldsymbol{V}_n] = 0, \tag{3}$$

which is a non-isospectral zero curvature equation. When $\dfrac{\partial \lambda}{\partial t} = 0$, Eq.(3) gives rise to the well-known zero curvature equation

$$\boldsymbol{U}_{nt} - \boldsymbol{V}_{nx} + [\boldsymbol{U}_n, \boldsymbol{V}_n] = 0.$$

Next, we consider the following AKNS matrix spectral problem, i.e.

$$\boldsymbol{\phi}_x = \boldsymbol{U}(u,\lambda)\boldsymbol{\phi}, \quad \text{and} \quad \boldsymbol{U}(u,\lambda) = \begin{pmatrix} -\lambda & q \\ r & \lambda \end{pmatrix}, \tag{4}$$

where λ is a spectral parameter. Because $\boldsymbol{U}$ has multiple eigenvalues, the spectral problem (4) is degenerate.

To establish non-isospectral AKNS soliton hierarchy, we first solve the adjoint equation

$$\boldsymbol{W}_x = [\boldsymbol{U}, \boldsymbol{W}] \tag{5}$$

of spectral problem (4) through the generalized Tu scheme.[21] Assume that a solution $\boldsymbol{W}$ is given by

$$\boldsymbol{W} = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \tag{6}$$

then we will have

$$[\boldsymbol{U}, \boldsymbol{W}] = \begin{pmatrix} qc - rb & -2\lambda b - 2qa \\ 2\lambda c + 2ra & rb - qc \end{pmatrix}. \tag{7}$$

Now, we seek a formal solution of the type

$$\begin{aligned} \boldsymbol{W} &= \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \sum_{k=0}^{\infty} \boldsymbol{W}_k \lambda^{-k} \\ &= \sum_{k=0}^{\infty} \begin{pmatrix} a_k & b_k \\ c_k & -a_k \end{pmatrix} \lambda^{-k}. \end{aligned} \tag{8}$$

And substitute Eq.(8) into Eqs.(5) and (7), then we will have the following recursion relation:

$$\begin{aligned} &a_0 = -1, \quad b_0 = c_0 = a_1 = 0, \\ &a_{nx} = qc_n - rb_n, \\ &b_{n+1} = -\frac{1}{2}b_{nx} - qa_n, \\ &c_{n+1} = \frac{1}{2}c_{nx} - ra_n, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &b_1 = q, \ c_1 = r, \ a_2 = \frac{1}{2}qr, \\ &b_2 = -\frac{1}{2}q_x, \ c_2 = \frac{1}{2}r_x. \end{aligned} \tag{10}$$

From Eq.(9), the following recursion relation can be obtained for b_{n+1} and c_{n+1}:

$$\begin{pmatrix} c_{n+1} \\ b_{n+1} \end{pmatrix} = \boldsymbol{\Psi} \begin{pmatrix} c_n \\ b_n \end{pmatrix}, n \geq 1, \tag{11}$$

where the 2×2 matrix operator $\boldsymbol{\Psi}$ is given by

$$\boldsymbol{\Psi} = \frac{-1}{2} \begin{pmatrix} \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \\ -2q\partial^{-1}q & -\partial + 2q\partial^{-1}r \end{pmatrix}. \tag{12}$$

As usual, for an arbitrary integer $n \geq 0$, we take

$$\boldsymbol{V}^{(n)} = (\lambda^n \boldsymbol{W})_+ = \sum_{j=0}^{n} \boldsymbol{W}_j \lambda^{(n-j)}, \tag{13}$$

and introduce a time evolution law for the eigenfunction ϕ as follow:

$$\phi_{t_n} = \boldsymbol{V}^{(n)}\phi = \boldsymbol{V}^{(n)}(u, u_x, \ldots; \lambda)\phi. \tag{14}$$

According to the zero curvature equation $\boldsymbol{U}_{t_n} - \boldsymbol{V}_x^{(n)} + [\boldsymbol{U}, \boldsymbol{V}^{(n)}] = 0$, we can obtain a system of evolution equations

$$\boldsymbol{u}_{t_n} = \begin{pmatrix} q \\ r \end{pmatrix}_{t_n} = \boldsymbol{K}_n = \begin{pmatrix} -2b_{n+1} \\ 2c_{n+1} \end{pmatrix}. \tag{15}$$

The first nonlinear system of soliton hierarchy (15) reads

$$\begin{aligned} q_{t_2} &= -\frac{1}{2}(q_{xx} - 2q^2 r), \\ r_{t_2} &= \frac{1}{2}(r_{xx} - 2qr^2), \end{aligned} \tag{16}$$

which is the nonlinear AKNS equations.

Considering the following non-isospectral case, and according to the non-isospectral zero curvature equation

$$\frac{\partial \boldsymbol{U}_n}{\partial t} + \frac{\partial \boldsymbol{U}_n}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \boldsymbol{V}_{nx} + [\boldsymbol{U}_n, \boldsymbol{V}_n] = 0,$$

we have

$$-\lambda_t - a_{nx} + qc_n - rb_n = 0, \tag{17a}$$

$$q_t - b_{nx} - 2\lambda b_n - 2qa_n = 0, \tag{17b}$$

and

$$r_t - c_{nx} + 2\lambda c_n + 2ra_n = 0. \tag{17c}$$

From (17a), we have

$$a_n = \partial^{-1}qc_n - \partial^{-1}rb_n - \lambda_t x + a_0, \tag{18}$$

where $\partial = \dfrac{\partial}{\partial x}$, $\partial\partial^{-1} = 1$, and a_0 is an integral constant.

Substituting Eq.(18) into Eqs.(17b) and (17c), we obtain a non-isospectral equation hierarchy

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = 2\lambda \begin{pmatrix} b_n \\ -c_n \end{pmatrix} + 2\lambda_t \begin{pmatrix} -xq \\ xr \end{pmatrix} + \begin{pmatrix} b_{nx} + 2q\partial^{-1}qc_n - 2q\partial^{-1}rb_n \\ c_{nx} - 2r\partial^{-1}qc_n + 2r\partial^{-1}rb_n \end{pmatrix} + \begin{pmatrix} 2qa_0 \\ -2ra_0 \end{pmatrix}. \tag{19}$$

If we assume that $\lambda_t = 0$, $a_0 = -2^{n-1}\lambda^n$, and $b_0 = c_0 = 0$, Eq.(19) becomes

$$\begin{pmatrix} q \\ r \end{pmatrix}_t = \begin{pmatrix} -\partial + 2q\partial^{-1}r & 2q\partial^{-1}q \\ -2r\partial^{-1}r & \partial - 2r\partial^{-1}q \end{pmatrix} \begin{pmatrix} -b_n \\ c_n \end{pmatrix} - \lambda \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} -b_n \\ c_n \end{pmatrix} + 2^n\lambda^n \begin{pmatrix} -q \\ r \end{pmatrix}. \tag{20}$$

Next, we will construct the hierarchy of non-isospectral AKNS equations with self-consistent sources. Consider auxiliary linear problems

$$\begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_x = \boldsymbol{U}(u, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}_t = \boldsymbol{V}(u, \lambda_j) \begin{pmatrix} \psi_{1j} \\ \psi_{2j} \end{pmatrix}. \tag{21}$$

From Eq.(21), it is known that (up to a constant factor)

$$\frac{\delta\lambda_j}{\delta u_i} = \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Psi}_j \frac{\partial \boldsymbol{U}(u, \lambda_j)}{\partial u_i}\right), \quad \boldsymbol{\Psi}_j = \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}, \ j = 1, \ldots, N,$$

and

$$\sum_{j=1}^{N} \frac{\delta\lambda_j}{\delta u_i} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \langle \boldsymbol{\Lambda}\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2 \rangle + \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \langle \boldsymbol{\Lambda}\boldsymbol{\Phi}_2, \boldsymbol{\Phi}_2 \rangle - \frac{1}{2} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \langle \boldsymbol{\Lambda}\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1 \rangle, i = 1, \ldots, s,$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $\boldsymbol{\Phi}_1 = (\phi_{11}, \ldots, \phi_{1N})^{\mathrm{T}}$, $\boldsymbol{\Phi}_2 = (\phi_{21}, \ldots, \phi_{2N})^{\mathrm{T}}$, and $\langle \cdot, \cdot \rangle$ is the inner product in R^N.

Based on the results in Refs.[5, 22], we have the following equation:

$$\frac{\delta H_k}{\delta u} + \sum_{j=1}^{N} \alpha_j \frac{\delta\lambda_j}{\delta u} = 0,$$

where α_j are constants, $\dfrac{\delta H_k}{\delta u}$ determine a finite-dimensional invariant set for the flow $u_{t_n} = J\dfrac{\delta H_{n+1}}{\delta u}$, $n = 1, 2, \ldots$. So, the equation hierarchy of non-isospectral AKNS equations with self-consistent sources is presented as follows:

$$\begin{aligned} \begin{pmatrix} q \\ r \end{pmatrix}_t = {} & 2\lambda \begin{pmatrix} b_n \\ -c_n \end{pmatrix} + 2\lambda_t \begin{pmatrix} -xq \\ xr \end{pmatrix} \\ & + \begin{pmatrix} b_{nx} + 2q\partial^{-1}qc_n - 2q\partial^{-1}rb_n \\ c_{nx} - 2r\partial^{-1}qc_n + 2r\partial^{-1}rb_n \end{pmatrix} \\ & + \begin{pmatrix} 2qa_0 \\ -2ra_0 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \langle \boldsymbol{\Phi}_2, \boldsymbol{\Phi}_2 \rangle \\ -\langle \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1 \rangle \end{pmatrix}, \end{aligned} \tag{22}$$

and

$$\phi_{1j,x} = -\lambda_j\phi_{1j} + q\phi_{2j}, \quad \phi_{2j,x} = r\phi_{1j} + \lambda_j\phi_{2j},$$

with

$$\zeta_j = \lambda_j, \quad \mathrm{Im}\zeta_j = \mathrm{Im}\lambda_j > 0, \ j = 1, \ldots, N_1,$$

$$\bar{\zeta}_j = \lambda_j, \quad \mathrm{Im}\bar{\zeta}_j = \mathrm{Im}\lambda_j < 0, \quad j = N_1 + 1, \ldots, N,$$

and

$$\lambda_j \neq \lambda_m > 0, \ j \neq m, \ j, m = 1, \ldots, N.$$

Considering the term of λ^n at $\lambda_t = 0$, $a_0 = -2^{n-1}\lambda^n$, and $b_0 = c_0 = 0$, we have

$$\begin{aligned} \begin{pmatrix} q \\ r \end{pmatrix}_t = {} & \begin{pmatrix} -\partial + 2q\partial^{-1}r & 2q\partial^{-1}q \\ -2r\partial^{-1}r & \partial - 2r\partial^{-1}q \end{pmatrix} \begin{pmatrix} -b_n \\ c_n \end{pmatrix} \\ & + 2^n \begin{pmatrix} -q \\ r \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \langle \boldsymbol{\Phi}_2, \boldsymbol{\Phi}_2 \rangle \\ -\langle \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1 \rangle \end{pmatrix}. \end{aligned} \tag{23}$$

When $n = 2$ in Eq.(23), we obtain

$$\begin{aligned} q_{,t_2} &= -\frac{1}{2}(q_{xx} - 2q^2r) - 4q - \frac{1}{2}\langle \boldsymbol{\Phi}_2, \boldsymbol{\Phi}_2 \rangle, \\ r_{t_2} &= \frac{1}{2}(r_{xx} - 2r^2q) + 4r + \frac{1}{2}\langle \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1 \rangle. \end{aligned} \tag{24}$$

If $\lambda_t = 2^{n-1}\lambda^n$, and $a_0 = 0$, Eq.(21) becomes

$$\begin{aligned} \begin{pmatrix} q \\ r \end{pmatrix}_t = {} & \begin{pmatrix} -\partial + 2q\partial^{-1}r & 2q\partial^{-1}q \\ -2r\partial^{-1}r & \partial - 2r\partial^{-1}q \end{pmatrix} \begin{pmatrix} -b_n \\ c_n \end{pmatrix} \\ & + 2^n \begin{pmatrix} -qx \\ rx \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \langle \boldsymbol{\Phi}_2, \boldsymbol{\Phi}_2 \rangle \\ -\langle \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1 \rangle \end{pmatrix}. \end{aligned} \tag{25}$$

For $n = 2$ we have

$$\begin{aligned} q_{,t_2} &= -\frac{1}{2}(q_{xx} - 2q^2r) - 4qx - \frac{1}{2}\langle \boldsymbol{\Phi}_2, \boldsymbol{\Phi}_2 \rangle, \\ r_{t_2} &= \frac{1}{2}(r_{xx} - 2r^2q) + 4rx + \frac{1}{2}\langle \boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1 \rangle. \end{aligned} \tag{26}$$

So we obtain the non-isospectral AKNS equations with self-consistent sources. In the present papers,

they are not designed specially for constructing non-isospectral integrable hierarchies with self-consistent sources.

3. Non-isospectral derivative NLSE hierarchy with self-consistent sources

Consider a reduced AKNS matrix spectral problem for $r=-q^*$, given as

$$\boldsymbol{\phi}_x=\boldsymbol{U}(u,\lambda)\phi,\quad \text{and}\ \ \boldsymbol{U}(u,\lambda)=\begin{pmatrix}-\lambda & q\\ -q^* & \lambda\end{pmatrix}. \tag{27}$$

To derive an associated soliton hierarchy, we seek a formal solution of the type

$$\begin{aligned}\boldsymbol{W}&=\begin{pmatrix}a & b\\ c & d\end{pmatrix}=\sum_{k=0}^{\infty}\boldsymbol{W}_k\lambda^{-k}\\ &=\sum_{k=0}^{\infty}\begin{pmatrix}a_k & b_k\\ c_k & d_k\end{pmatrix}\lambda^{-k}.\end{aligned} \tag{28}$$

We first solve the adjoint equation

$$\boldsymbol{W}_x=[\boldsymbol{U},\boldsymbol{W}] \tag{29}$$

of the spectral problem (27) through the generalized Tu scheme. Then we have

$$[\boldsymbol{U},\boldsymbol{W}]=\begin{pmatrix}qc_n+q^*b_n & -2\lambda b_n-2qa_n\\ 2\lambda c_n-2q^*a_n & -q^*b_n-qc_n\end{pmatrix}. \tag{30}$$

Therefore, condition (30) becomes the following recursion relation:

$$a_0=-2, b_0=c_0=0, \tag{31}$$

and

$$\begin{aligned}&b_1=2q, c_1=-2q^*, a_1=0,\\ &a_2=-qq^*, b_2=-q_x, c_2=-q_x^*.\end{aligned} \tag{32}$$

We can obtain the following recursion relation:

$$\begin{pmatrix}c_{n+1}\\ b_{n+1}\end{pmatrix}=\boldsymbol{\Psi}\begin{pmatrix}c_n\\ b_n\end{pmatrix}, n\geq 1, \tag{33}$$

where the 2×2 matrix operator $\boldsymbol{\Psi}$ is given by

$$\boldsymbol{\Psi}=\frac{1}{2}\begin{pmatrix}\partial+2q^*\partial^{-1}q & 2q^*\partial^{-1}q^*\\ -2q\partial^{-1}q & -\partial-2q\partial^{-1}q^*\end{pmatrix}. \tag{34}$$

The hierarchy of the derivative nonlinear Schrödinger equations hierarchy reads

$$q_{t_n}=-2b_{2n+1}=-2\frac{\delta H_{2n+1}}{\delta q}, \tag{35}$$

with

$$H_{2n+1}=\frac{2a_{2n+2}}{2n+1}. \tag{36}$$

In this soliton hierarchy (35), the first nonlinear system reads

$$q_{t_2}=-q_{xx}-2q^2q^*, \tag{37}$$

which is an NLSE equation.

According to non-isospectral zero curvature equation (3), we have

$$\lambda_t+a_{nx}-qc_n-b_nq^*=0, \tag{38a}$$

$$q_t-b_{nx}-2\lambda b_n-2qa_n=0, \tag{38b}$$

and

$$-q_t^*-c_{nx}+2\lambda c_n-2q^*a_n=0. \tag{38c}$$

From Eq.(38a), we have

$$a_n=\partial^{-1}qc_n+\partial^{-1}b_nq^*-\lambda_t x+a_0. \tag{39}$$

Substituting Eq.(39) into Eqs.(38b) and (38c) and based on the results in Refs.[5, 22], we have an NLSE hierarchy with self-consistent sources as follows:

$$\begin{aligned}q_t=\ &2\lambda b_n+2q\partial^{-1}qc_n+2q\partial^{-1}q^*b_n-2q\lambda_t x\\ &+2qa_0-2(\langle\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\rangle-\langle\boldsymbol{\Phi}_2^*,\boldsymbol{\Phi}_2^*\rangle),\end{aligned}$$

and

$$\begin{aligned}\phi_{1j,x}&=-\lambda_j\phi_{1j}+q\phi_{2j},\\ \phi_{2j,x}&=-q^*\phi_{1j}+\lambda_j\phi_{2j},\end{aligned} \tag{40}$$

where $\boldsymbol{\theta}=\mathrm{diag}(\lambda_1,\ldots,\lambda_N)$,

$$\zeta_j=\lambda_j,\quad \mathrm{Im}\zeta_j=\mathrm{Im}\lambda_j>0, j=1,\ldots,N_1,$$

and

$$\bar{\zeta}_j=\lambda_j,\quad \mathrm{Im}\bar{\zeta}_j=\mathrm{Im}\lambda_j<0, j=N_1+1,\ldots,N.$$

The zero curvature representation for the NLSE hierarchy with self-consistent source is given by Eq.(27) with the auxiliary linear problems[5]

$$\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}_x=\begin{pmatrix}-\lambda & q\\ -q^* & \lambda\end{pmatrix}\begin{pmatrix}\boldsymbol{\psi}_1\\ \boldsymbol{\psi}_2\end{pmatrix},$$

$$\boldsymbol{\psi}_{1,t_n} = A^{(2n)}\psi_1 + B^{(2n)}\psi_2 = \sum_{k=0}^{2n}(a_k\lambda^{2n-k}\psi_1 + b_k\lambda^{2n-k}\psi_2) + \eta\psi_1 + \sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{2j}(\phi_{2j}\psi_1 - \phi_{1j}\psi_2) + \sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j^*}\phi_{2j}^*(\phi_{1j}^*\psi_1 + \phi_{2j}^*\psi_2),$$

and

$$\boldsymbol{\psi}_{2,t_n} = C^{(n)}\psi_1 + D^{(n)}\psi_2 = \sum_{k=0}^{2n}(c_k\lambda^{2n-k}\psi_1 - a_k\lambda^{2n-k}\psi_2) + \eta\psi_2 + \sum_{j=1}^{N}\frac{1}{\lambda-\lambda_j}\phi_{2j}(\phi_{2j}\psi_1 - \phi_{1j}\psi_2) - \sum_{j=1}^{N}\frac{1}{\lambda+\lambda_j^*}\phi_{1j}^*(\phi_{1j}^*\psi_1 + \phi_{2j}^*\psi_2).$$

If we assume that $\lambda_t = 0$, $a_0 = -(2)^{n-1}\lambda^n$, and $b_0 = c_0 = 0$, Eq.(40) becomes

$$q_t = \partial b_n + 2q\partial^{-1}qc_n + 2q\partial^{-1}q^*b_n - 2^n q - 2(\langle\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\rangle - \langle\boldsymbol{\Phi}_2^*,\boldsymbol{\Phi}_2^*\rangle). \quad (41)$$

When $n = 2$, Eq.(41) gives an NLSE equation with self-consistent sources

$$q_{t_2} = -q_{xx} - 2q^2q^* - 4q - 2(\langle\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\rangle - \langle\boldsymbol{\Phi}_2^*,\boldsymbol{\Phi}_2^*\rangle). \quad (42)$$

If $\lambda_t = (2)^{n-1}\lambda^n$, and $a_0 = 0$, Eq.(40) becomes

$$q_t = \partial b_n + 2q\partial^{-1}qc_n + 2q\partial^{-1}q^*b_n - 2^n qx - 2(\langle\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\rangle - \langle\boldsymbol{\Phi}_2^*,\boldsymbol{\Phi}_2^*\rangle).$$

For $n = 2$, we have

$$q_{t_2} = -q_{xx} - 2q^2q^* - 4qx - 2(\langle\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\rangle - \langle\boldsymbol{\Phi}_2^*,\boldsymbol{\Phi}_2^*\rangle), \quad (43)$$

which is a hierarchy of non-isospectral Schrödinger equations with self-consistent sources.

4. Integrable couplings of the AKNS equations hierarchy with self-consistent sources

By using the theory on continuous integrable couplings,[13] some integrable coupling systems have been obtained, such as AKNS hierarchy, TD hierarchy, etc., in Refs.[14–19]. In this section, we will construct a novel integrable coupling system of non-isospectral AKNS equation hierarchy with self-consistent sources by using the Kronecker product.

Consider the following non-isospectral problem:

$$\boldsymbol{\psi}_x = \boldsymbol{U}_3\boldsymbol{\psi}, \quad \text{and} \quad \boldsymbol{\psi}_t = \boldsymbol{V}_3\boldsymbol{\psi},$$

then we will give a new pair, $\boldsymbol{U}_3$ and $\boldsymbol{V}_3$, to produce an integrable coupling by using the Kronecker product.

The new Lax pair, $\boldsymbol{U}_3$ and $\boldsymbol{V}_3$, is presented as follows:

$$\boldsymbol{U}_3 = I_N \otimes \boldsymbol{U}_1 + \boldsymbol{T}_N \otimes \boldsymbol{U}_2,$$

and

$$\boldsymbol{V}_3 = \boldsymbol{I}_N \otimes \boldsymbol{V}_1 + \boldsymbol{T}_N \otimes \boldsymbol{V}_2, \quad (44)$$

where the matrices $\boldsymbol{I}_N$ and $\boldsymbol{T}_N$ have the same ranks, $\boldsymbol{I}_N$ is a unit matrix, and $\boldsymbol{T}_N$ is a 2-nilpotent matrix. We can obtain

$$[\boldsymbol{U}_3,\boldsymbol{V}_3] = \boldsymbol{I}_N^2 \otimes \boldsymbol{U}_1\boldsymbol{V}_1 - \boldsymbol{I}_N^2 \otimes \boldsymbol{V}_1\boldsymbol{U}_1 + \boldsymbol{I}_N\boldsymbol{T}_N \otimes \boldsymbol{U}_1\boldsymbol{V}_2 - \boldsymbol{I}_N\boldsymbol{T}_N \otimes \boldsymbol{V}_1\boldsymbol{U}_2 + \boldsymbol{T}_N\boldsymbol{I}_N \otimes \boldsymbol{U}_2\boldsymbol{V}_1 - \boldsymbol{T}_N\boldsymbol{I}_N \otimes \boldsymbol{V}_2\boldsymbol{U}_1 + \boldsymbol{T}_N^2 \otimes \boldsymbol{U}_2\boldsymbol{V}_2 - \boldsymbol{T}_N^2 \otimes \boldsymbol{V}_2\boldsymbol{U}_2,$$

with $\boldsymbol{I}_N^2 = \boldsymbol{I}_N$, $\boldsymbol{I}_N\boldsymbol{T}_N = \boldsymbol{T}_N\boldsymbol{I}_N = \boldsymbol{T}_N$, and $\boldsymbol{T}_N^2 = 0$. And we have

$$[\boldsymbol{U}_3,\boldsymbol{V}_3] = \boldsymbol{I}_N \otimes \boldsymbol{U}_1\boldsymbol{V}_1 - \boldsymbol{I}_N \otimes \boldsymbol{V}_1\boldsymbol{U}_1 + \boldsymbol{T}_N \otimes \boldsymbol{U}_1\boldsymbol{V}_2 - \boldsymbol{T}_N \otimes \boldsymbol{V}_1\boldsymbol{U}_2 + \boldsymbol{T}_N \otimes \boldsymbol{U}_2\boldsymbol{V}_1 - \boldsymbol{T}_N \otimes \boldsymbol{V}_2\boldsymbol{U}_1$$

or

$$[\boldsymbol{U}_3,\boldsymbol{V}_3] = \boldsymbol{I}_N \otimes [\boldsymbol{U}_1,\boldsymbol{V}_1] + \boldsymbol{T}_N \otimes [\boldsymbol{U}_1,\boldsymbol{V}_2] + \boldsymbol{T}_N \otimes [\boldsymbol{U}_2,\boldsymbol{V}_1]. \quad (45)$$

Substituting

$$\begin{aligned}\boldsymbol{U}_{3t} &= \boldsymbol{I}_N \otimes \boldsymbol{U}_{1t} + \boldsymbol{I}_N \otimes \frac{\partial \boldsymbol{U}_1}{\partial \lambda}\frac{\partial \lambda}{\partial t} + \boldsymbol{T}_N \otimes \boldsymbol{U}_{2t} \\ &\quad + \boldsymbol{T}_N \otimes \frac{\partial \boldsymbol{U}_2}{\partial \lambda}\frac{\partial \lambda}{\partial t}, \\ \boldsymbol{V}_{3x} &= \boldsymbol{I}_N \otimes \boldsymbol{V}_{1x} + \boldsymbol{T}_N \otimes \boldsymbol{V}_{2x}\end{aligned}$$

and Eq.(45) into the zero curvature equation

$$\boldsymbol{U}_{3t} + \frac{\partial \boldsymbol{U}_3}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \boldsymbol{V}_{3x} + [\boldsymbol{U}_3, \boldsymbol{V}_3] = 0,$$

new zero curvature equations are given as

$$\begin{aligned}&\boldsymbol{U}_{1t} + \frac{\partial \boldsymbol{U}_1}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \boldsymbol{V}_{1x} + [\boldsymbol{U}_1, \boldsymbol{V}_1] = 0, \\ &\boldsymbol{U}_{2t} + \frac{\partial \boldsymbol{U}_2}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \boldsymbol{V}_{2x} + [\boldsymbol{U}_1, \boldsymbol{V}_2] + [\boldsymbol{U}_2, \boldsymbol{V}_1] = 0. \quad (46)\end{aligned}$$

Therefore, we obtain a new form of non-isospectral zero curvature equations (46). The first equation of Eq.(46) here exactly presents Eq.(3), and the second equation of Eq.(46) provides a coupling system for Eq.(3). In a word, the Kronecker product provides a great choice of candidates of non-isospectral integrable couplings for Eq.(46).

Considering

$$\boldsymbol{\Gamma} = \boldsymbol{V}_3 + \bar{\boldsymbol{\Delta}}, \quad \text{and} \quad \bar{\boldsymbol{\Delta}} = \boldsymbol{I}_N \otimes \boldsymbol{\Delta}_1 + \boldsymbol{T}_N \otimes \boldsymbol{\Delta}_2,$$

from the zero curvature equation $\boldsymbol{U}_{3t} + \frac{\partial \boldsymbol{U}_3}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \boldsymbol{\Gamma}_x + [\boldsymbol{U}, \boldsymbol{\Gamma}] = 0$, we obtain a new integrable coupling system

$$\begin{aligned}&\boldsymbol{U}_{1t} + \frac{\partial \boldsymbol{U}_1}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \boldsymbol{V}_{1x} + [\boldsymbol{U}_1, \boldsymbol{V}_1] \\ &- \boldsymbol{\Delta}_{1x} + [\boldsymbol{U}_1, \boldsymbol{\Delta}_1] = 0, \\ &\boldsymbol{U}_{2t} + \frac{\partial \boldsymbol{U}_2}{\partial \lambda}\frac{\partial \lambda}{\partial t} - \boldsymbol{V}_{2x} + [\boldsymbol{U}_1, \boldsymbol{V}_2] \\ &+ [\boldsymbol{U}_2, \boldsymbol{V}_1] - \boldsymbol{\Delta}_{2x} + [\boldsymbol{U}_1, \boldsymbol{\Delta}_2] + [\boldsymbol{U}_2, \boldsymbol{\Delta}_1] = 0. \quad (47)\end{aligned}$$

To illustrate our method, we consider the following non-isospectral problem:

$$\begin{aligned}\boldsymbol{U}_1 &= \begin{pmatrix} \lambda & q \\ r & -\lambda \end{pmatrix}, \quad \boldsymbol{U}_2 = \begin{pmatrix} 1 & u_1 \\ u_2 & 1 \end{pmatrix}, \\ \boldsymbol{V}_1 &= \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad \boldsymbol{V}_2 = \begin{pmatrix} d & e \\ f & g \end{pmatrix}, \end{aligned} \quad (48)$$

where q and r are potentials; u_1 and u_2 are new dependent variables and $\bar{\boldsymbol{u}} = (q, r, u_1, u_2)^{\mathrm{T}}$; $\boldsymbol{U}_1, \boldsymbol{V}_1, \boldsymbol{U}_2$, and $\boldsymbol{V}_2$ are matrices of 2×2; $\boldsymbol{I}_N = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$; $\boldsymbol{T}_N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$; $a = \sum\limits_{k=0}^{\infty} a_k \lambda^k$; $b = \sum\limits_{k=0}^{\infty} b_k \lambda^k$; $c = \sum\limits_{k=0}^{\infty} c_k \lambda^k$; $d = \sum\limits_{k=0}^{\infty} d_k \lambda^k$; $e = \sum\limits_{k=0}^{\infty} e_k \lambda^k$; $f = \sum\limits_{k=0}^{\infty} f_k \lambda^k$; $g = \sum\limits_{k=0}^{\infty} g_k \lambda^k$.

The corresponding enlarged stationary zero curvature equation is obtained as

$$\begin{aligned}&\boldsymbol{V}_x = [\boldsymbol{U}, \boldsymbol{V}], \\ &\boldsymbol{V}_{2x} = [\boldsymbol{U}_1, \boldsymbol{V}_2] + [\boldsymbol{U}_2, \boldsymbol{V}_1]. \end{aligned} \quad (49)$$

Obviously, we have

$$[\boldsymbol{U}_1, \boldsymbol{V}_2] = \begin{pmatrix} qf_n - re_n & 2\lambda e_n + qg_n - qd_n \\ rd_n - rg_n - 2\lambda f_n & re_n - qf_n \end{pmatrix}, \quad (50a)$$

and

$$[\boldsymbol{U}_2, \boldsymbol{V}_1] = \begin{pmatrix} u_1 c_n - u_2 b_n & -2u_1 a_n \\ 2u_2 a_n & u_2 b_n - u_1 c_n \end{pmatrix}. \quad (50b)$$

Thus, from Eq.(50) we obtain

$$\begin{aligned}d_{nx} &= qf_n - re_n + u_1 c_n - b_n u_2, \\ e_{nx} &= 2e_{n+1} + qg_n - qd_n - 2u_1 a_n, \\ f_{nx} &= rd_n - rg_n - 2f_{n+1} + 2u_2 a_n, \\ g_{nx} &= re_n - qf_n + u_2 b_n - u_1 c_n. \end{aligned} \quad (51)$$

Set $d_0 = e_0 = f_0 = g_0 = 0$, then we will see that all sets of functions d_n, e_n, f_n, and g_n are uniquely determined. In particular, the first few sets are

$$\begin{aligned}&d_1 = 0, \quad e_1 = -u_1, \quad f_1 = -u_2, \quad g_1 = 0, \\ &d_2 = \frac{1}{2}(qu_2 + ru_1), \quad e_2 = -\frac{1}{2}u_{1x}, \\ &f_2 = \frac{1}{2}u_{2x}, \quad g_2 = -\frac{1}{2}(qu_2 + ru_1).\end{aligned}$$

According to non-isospectral zero curvature equation (47), we have

$$\begin{aligned}d_n = -g_n &= \partial^{-1} q f_n - \partial^{-1} r e_n \\ &\quad + \partial^{-1} u_1 c_n - \partial^{-1} b_n u_2 + d_0, \end{aligned} \quad (52a)$$

$$u_{1t} = e_{nx} - 2e_{n+1} - qg_n + qd_n + 2u_1 a_n, \quad (52b)$$

and

$$u_{2t} = f_{nx} - rd_n + rg_n + 2f_{n+1} - 2u_2 a_n. \quad (52c)$$

Substituting Eqs.(18) and (52a) into Eqs.(52b) and (52c) gives rise to a novel integrable coupling system of the non-isospectral equations hierarchy

$$\begin{pmatrix} q \\ r \\ u_1 \\ u_2 \end{pmatrix}_t = \left(\begin{array}{c|c} J_1 & 0 \\ \hline 0 & J_2 \end{array}\right) \begin{pmatrix} b_{n+1} \\ c_{n+1} \\ f_{n+1} \\ e_{n+1} \end{pmatrix} + 2\lambda_t \begin{pmatrix} -xq \\ xr \\ -xu_1 \\ xu_2 \end{pmatrix} + \begin{pmatrix} 2qa_0 \\ -2ra_0 \\ 2u_1a_0 + 2qd_0 \\ -2u_2a_0 - 2rd_0 \end{pmatrix}$$
$$+ \begin{pmatrix} -\partial + q\partial^{-1}r & -2q\partial^{-1}q & 0 & 0 \\ 2r\partial^{-1}r & -\partial - 2r\partial^{-1}q & 0 & 0 \\ -2q\partial^{-1}u_2 - 2u_1\partial^{-1}r & 2q\partial^{-1}u_1 + 2u_1\partial^{-1}q & 2q\partial^{-1}q & \partial - 2q\partial^{-1}r \\ 2r\partial^{-1}u_2 + 2u_2\partial^{-1}r & -2r\partial^{-1}u_1 - 2u_2\partial^{-1}q & \partial - 2r\partial^{-1}q & 2r\partial^{-1}r \end{pmatrix} \begin{pmatrix} b_n \\ c_n \\ f_n \\ e_n \end{pmatrix}, \quad (53)$$

with

$$\boldsymbol{J}_1 = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}, \quad \text{and} \quad \boldsymbol{J}_2 = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix},$$

where $\boldsymbol{J}_1$ and $\boldsymbol{J}_2$ are Hamiltonian operators.

According to Eq.(51), we have the following reduced relation:

$$\begin{pmatrix} b_{n+1} \\ c_{n+1} \\ f_{n+1} \\ e_{n+1} \end{pmatrix} = \boldsymbol{L} \begin{pmatrix} b_n \\ c_n \\ f_n \\ e_n \end{pmatrix}, \boldsymbol{L} = \left(\begin{array}{c|c} \theta & 0 \\ \hline L_1 & L_2 \end{array}\right), \quad (54)$$

$$\boldsymbol{\theta} = \frac{1}{2} \begin{pmatrix} -\partial + q\partial^{-1}r & -2q\partial^{-1}q \\ 2r\partial^{-1}r & -\partial - 2r\partial^{-1}q \end{pmatrix},$$

$$\boldsymbol{L}_1 = \begin{pmatrix} L_{31} & L_{32} \\ L_{41} & L_{42} \end{pmatrix}, \quad \text{and} \quad \boldsymbol{L}_2 = \begin{pmatrix} L_{33} & L_{34} \\ L_{43} & L_{44} \end{pmatrix},$$

with

$$\begin{aligned} L_{31} &= -r\partial^{-1}u_2 - 2u_2\partial^{-1}r, \\ L_{32} &= r\partial^{-1}u_1 + 2u_2\partial^{-1}q, \\ L_{33} &= -\frac{1}{2}\partial + r\partial^{-1}q, \\ L_{34} &= -r\partial^{-1}r, \\ L_{41} &= -q\partial^{-1}u_2 - 2u_1\partial^{-1}r, \\ L_{42} &= q\partial^{-1}u_1 + 2u_1\partial^{-1}q, \end{aligned}$$

and

$$L_{43} = q\partial^{-1}q,$$

$$L_{44} = \frac{1}{2}\partial - q\partial^{-1}r.$$

Next, we will construct an integrable coupling of non-isospectral AKNS equations hierarchy with self-consistent sources. Consider auxiliary linear problems

$$\begin{pmatrix} \boldsymbol{\psi}_{1j} \\ \boldsymbol{\psi}_{2j} \\ \boldsymbol{\psi}_{3j} \\ \boldsymbol{\psi}_{4j} \end{pmatrix}_x = \boldsymbol{U}_3(u, \lambda_j) \begin{pmatrix} \boldsymbol{\psi}_{1j} \\ \boldsymbol{\psi}_{2j} \\ \boldsymbol{\psi}_{3j} \\ \boldsymbol{\psi}_{4j} \end{pmatrix},$$

and

$$\begin{pmatrix} \boldsymbol{\psi}_{1j} \\ \boldsymbol{\psi}_{2j} \\ \boldsymbol{\psi}_{3j} \\ \boldsymbol{\psi}_{4j} \end{pmatrix}_t = \boldsymbol{V}_3(u, \lambda_j) \begin{pmatrix} \boldsymbol{\psi}_{1j} \\ \boldsymbol{\psi}_{2j} \\ \boldsymbol{\psi}_{3j} \\ \boldsymbol{\psi}_{4j} \end{pmatrix}. \quad (55)$$

For Eq.(55), it is known that (up to a constant factor)

$$\frac{\delta \lambda_j}{\delta u_i} = \frac{1}{2}\mathrm{Tr}\left(\boldsymbol{\Psi}_j \frac{\partial \boldsymbol{U}(u, \lambda_j)}{\partial u_i}\right),$$

$$\boldsymbol{\Psi}_j = \begin{pmatrix} \phi_{1j}\phi_{2j} & -\phi_{1j}^2 & \phi_{3j}\phi_{4j} & -\phi_{3j}^2 \\ \phi_{2j}^2 & -\phi_{1j}\phi_{2j} & \phi_{4j}^2 & -\phi_{3j}\phi_{4j} \\ 0 & 0 & \phi_{1j}\phi_{2j} & -\phi_{1j}^2 \\ 0 & 0 & \phi_{2j}^2 & -\phi_{1j}\phi_{2j} \end{pmatrix}, \quad j = 1, \ldots, N, \quad (56)$$

and

$$\sum_{j=1}^{N} \frac{\delta \lambda_j}{\delta u_i} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \langle \boldsymbol{\Lambda}\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_2 \rangle + \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \langle \boldsymbol{\Lambda}\boldsymbol{\Phi}_2, \boldsymbol{\Phi}_2 \rangle - \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \langle \boldsymbol{\Lambda}\boldsymbol{\Phi}_1, \boldsymbol{\Phi}_1 \rangle$$
$$+ \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \langle \boldsymbol{\Lambda}\boldsymbol{\Phi}_4, \boldsymbol{\Phi}_4 \rangle - \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \langle \boldsymbol{\Lambda}\boldsymbol{\Phi}_3, \boldsymbol{\Phi}_3 \rangle, \quad i = 1, \ldots, s, \quad (57)$$

where $\boldsymbol{\Lambda} = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$, $\boldsymbol{\Phi}_1 = (\phi_{11}, \ldots, \phi_{1N})^{\mathrm{T}}$, $\boldsymbol{\Phi}_2 = (\phi_{21}, \ldots, \phi_{2N})^{\mathrm{T}}$, $\boldsymbol{\Phi}_3 = (\phi_{31}, \ldots, \phi_{3N})^{\mathrm{T}}$, $\boldsymbol{\Phi}_4 = (\phi_{41}, \ldots, \phi_{4N})^{\mathrm{T}}$, and $\langle \cdot, \cdot \rangle$ is the inner product in R^N.

Based on the results in Refs.[5, 22], we show the following equation:

$$\frac{\delta H_k}{\delta u}+\sum_{j=1}^{N}\alpha_j\frac{\delta\lambda_j}{\delta u}=0,$$

where α_j are constants, $\dfrac{\delta H_k}{\delta u}$ determine a finite-dimensional invariant set for the flow $u_{t_n}=J\dfrac{\delta H_{n+1}}{\delta u}, n=1,2,\ldots$. So, a novel integrable coupling system of the non-isospectral equations hierarchy with self-consistent sources is presented as follows:

$$\begin{aligned}\begin{pmatrix}q\\r\\u_1\\u_2\end{pmatrix}_t=&\begin{pmatrix}-\partial+q\partial^{-1}r & -2q\partial^{-1}q & 0 & 0\\ 2r\partial^{-1}r & -\partial-2r\partial^{-1}q & 0 & 0\\ -2q\partial^{-1}u_2-2u_1\partial^{-1}r & 2q\partial^{-1}u_1+2u_1\partial^{-1}q & 2q\partial^{-1}q & \partial-2q\partial^{-1}r\\ 2r\partial^{-1}u_2+2u_2\partial^{-1}r & -2r\partial^{-1}u_1-2u_2\partial^{-1}q & \partial-2r\partial^{-1}q & 2r\partial^{-1}r\end{pmatrix}\begin{pmatrix}b_n\\c_n\\f_n\\e_n\end{pmatrix}\\&+\left(\begin{array}{c|c}J_1 & 0\\ \hline 0 & J_2\end{array}\right)\begin{pmatrix}b_{n+1}\\c_{n+1}\\f_{n+1}\\e_{n+1}\end{pmatrix}+2\lambda_t\begin{pmatrix}-xq\\xr\\-xu_1\\xu_2\end{pmatrix}+\begin{pmatrix}2qa_0\\-2ra_0\\2u_1a_0+2qd_0\\-2u_2a_0-2rd_0\end{pmatrix}-\frac{1}{2}\begin{pmatrix}\langle\boldsymbol{\Phi}_2,\boldsymbol{\Phi}_2\rangle\\-\langle\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\rangle\\\langle\boldsymbol{\Phi}_4,\boldsymbol{\Phi}_4\rangle\\-\langle\boldsymbol{\Phi}_3,\boldsymbol{\Phi}_3\rangle\end{pmatrix},\end{aligned}\tag{58}$$

If $\lambda_t=2^{n-1}\lambda^n$, $a_0=-2^{n-1}\lambda^n$, and $d_0=0$, Eq.(58) becomes

$$\begin{aligned}\begin{pmatrix}q\\r\\u_1\\u_2\end{pmatrix}_t=&\begin{pmatrix}-\partial+q\partial^{-1}r & -2q\partial^{-1}q & 0 & 0\\ 2r\partial^{-1}r & -\partial-2r\partial^{-1}q & 0 & 0\\ -2q\partial^{-1}u_2-2u_1\partial^{-1}r & 2q\partial^{-1}u_1+2u_1\partial^{-1}q & 2q\partial^{-1}q & \partial-2q\partial^{-1}r\\ 2r\partial^{-1}u_2+2u_2\partial^{-1}r & -2r\partial^{-1}u_1-2u_2\partial^{-1}q & \partial-2r\partial^{-1}q & 2r\partial^{-1}r\end{pmatrix}\begin{pmatrix}b_n\\c_n\\f_n\\e_n\end{pmatrix}\\&+2^n\lambda^n\begin{pmatrix}-xq\\xr\\-xu_1\\xu_2\end{pmatrix}+2^n\begin{pmatrix}-q\\r\\-u_1\\u_2\end{pmatrix}-\frac{1}{2}\begin{pmatrix}\langle\boldsymbol{\Phi}_2,\boldsymbol{\Phi}_2\rangle\\-\langle\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\rangle\\\langle\boldsymbol{\Phi}_4,\boldsymbol{\Phi}_4\rangle\\-\langle\boldsymbol{\Phi}_3,\boldsymbol{\Phi}_3\rangle\end{pmatrix}.\end{aligned}\tag{59}$$

By using Eqs.(51) and (52), for $n=2$ we have the first system in Eq.(59)

$$\begin{aligned}q_{t_2}&=-\frac{1}{2}(q_{xx}-2q^2r)-4qx-4q-\frac{1}{2}\langle\boldsymbol{\Phi}_2,\boldsymbol{\Phi}_2\rangle,\\ r_{t_2}&=\frac{1}{2}(r_{xx}-2r^2q)+4rx+4r+\frac{1}{2}\langle\boldsymbol{\Phi}_1,\boldsymbol{\Phi}_1\rangle,\\ u_{1t_2}&=\frac{1}{2}u_{1xx}+2qru_1+q^2u_2-4u_1x-4u_1-\frac{1}{2}\langle\boldsymbol{\Phi}_4,\boldsymbol{\Phi}_4\rangle,\\ u_{2t_2}&=-\frac{1}{2}u_{2xx}+2qru_2+r^2u_1+4u_2x+4u_2+\frac{1}{2}\langle\boldsymbol{\Phi}_3,\boldsymbol{\Phi}_3\rangle.\end{aligned}\tag{60}$$

We obtain an integrable coupling system of the non-isospectral AKNS equations hierarchy with self-consistent sources. When $u_1=u_2=0$, it is a non-isospectral AKNS equation hierarchy with self-consistent sources. So system (58) is a novel soliton equation hierarchy.

5. Conclusions

In this paper, we derive a non-isospectral AKNS equation hierarchy with self-consistent sources. Moreover, by means of the Kronecker product, a non-isospectral integrable coupling of the corresponding AKNS hierarchy with self-consistent sources is constructed. The method can be applied to the other soliton hierarchies.

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