Vol 17 No 11, November 2008
1674-1056/2008/17(11)/3942-11

Chinese Physics B

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Symplectic-energy-first integrators of discrete mechanico-electrical dynamical systems*

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(Received 6 February 2008; revised manuscript received 8 April 2008)

A discrete total variation calculus with variable time steps is presented for mechanico-electrical systems where there exist non-potential and dissipative forces. By using this discrete variation calculus, the symplectic-energy-first integrators for mechanico-electrical systems are derived. To do this, the time step adaptation is employed. The discrete variational principle and the Euler–Lagrange equation are derived for the systems. By using this discrete algorithm it is shown that mechanico-electrical systems are not symplectic and their energies are not conserved unless they are Lagrange mechanico-electrical systems. A practical example is presented to illustrate these results.

Keywords: total variation, symplectic-energy–momentum integrator, mechanico-electrical system
PACC: 0320

1. Introduction

In the 1980's, Lee proposed energy-preserving discrete mechanics with variable time steps by taking discrete time as dynamical variable.[1−3] On the other hand, a version of discrete Lagrangian mechanics has been developed and variational integrators that preserve discrete symplectic two-form have been obtained.[4−8] Recently, Fu *et al* have given discrete variation principle and first integrals of the mechanico-electrical systems[9] and Zhang and Chen *et al* have obtained many results on discrete variation principles and first integrals of dynamical systems.[10−13] However, variational integrators obtained in this way have fixed time steps and consequently they are not energy preserving in general.

It is well known that the energy-preserving discrete mechanics and variational integrator are more preferable sine solution of the Euler–Lagrange equations for a conservative continuous systems are not only symplectic but also energy-preserving. In order to attain this goal, some discrete mechanics and symplectic variation integral need studying. Recently, Kane–Marsden–Ortiz and Chen–Guo–Wu *et al* have employed appropriate time steps to conserve a defined discrete energy and developed that they called symplectic-energy-momentum preserving variational integrators in Refs.[14, 15].

The purpose of this paper is to generalize these approaches and apply the relationships among discrete full variation, Lee's discrete mechanics method and Kane *et al*'s and Chen *et al*'s integrators to mechanico-electrical systems with non-potential and non-dissipative forces. We will present a discrete total variation calculus with variable time steps and a discrete mechanico-electrical system that is discretely neither symplectic nor energy preserving. In fact, this discrete variation calculus and discrete mechanico-electrical system is a generalization of Lee's discrete mechanics in symplectic property.

The remainder of this paper is organized as follows. In Section 2, we briefly review the Lagrange–Maxwell equations of mechanico-electrical systems. In Section 3, we present a discrete variation principle and discrete Euler–Lagrange equations of the systems with Lee's discrete algorithm. In Section 4, we obtain the

*Project supported by the National Natural Science Foundation of China (Grant Nos 10672143 and 60575055) and the State Key Laboratory of Scientific and Engineering Computing, Chinese Academy of Sciences and the Natural Science Foundation of Henan Province Government, China (Grant No 0511022200).
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first integral of discrete mechanico-electrical systems with an infinitesimal transformation group. In Section 5, we derive the discrete energy of the systems with constant and variable time steps. In Section 6, we study time step solvability and optimization technique. In Section 7, we prove the symplectic nature of discrete algorithms with a discrete total variation calculus, which leads to the fact that the discrete algorithms for the systems are not symplectic in nature unless the systems are Lagrange mechanico-electrical systems. In Section 8, we give an example on the carriage of magnetic train (aerotrain). The conclusions drawn from the present study are given in Section 9.

2. Lagrange–Maxwell equation of mechanico-electrical systems

Now it is well known how to establish the equation of motion of a mechanico-electrical system and so we briefly review it here (also see, for example, Ref.[16]).

Mechanico-electrical system is the system in which a mechanical process and an electromagnetic process are coupled. The mechanical part, constituted by N particles, is described by n generalized coordinates q_s $(s = 1, 2, \ldots, n)$, and the electrodynamical part, which is composed of m electric circuits, is described by the model of electrics. Consider a mechanico-electrical system, which is composed of m circuits consisting of linear conductors and capacitors and let i_k , u_k , $e_k(\dot{e}_k = i_k)$, R_k, and C_k denote the current, the electronic potential, the charge in the capacitor, the resistance, and the capacitance respectively, then the Lagrangian of the mechanico-electrical system will be

$$L = T(\boldsymbol{q}, \dot{q}, \boldsymbol{e}, \dot{\boldsymbol{e}}) - V(\boldsymbol{q}) + W_m(\boldsymbol{q}, \dot{\boldsymbol{e}}) - W_{\mathrm{e}}(\boldsymbol{q}, \boldsymbol{e}), \quad (1)$$

where T and V are respectively the kinetic energy and the potential energy, and the electric field energy and the magnetic field energy of the mth circuit are respectively determined by

$$W_{\mathrm{e}} = \frac{1}{2}\frac{e_k^2}{C_k},$$

and

$$W_{\mathrm{m}} = \frac{1}{2}L_{kr}i_k i_r \quad (k, r = 1, \ldots, m), \quad (2)$$

where $L_{kr}(k \neq r)$ is the mutual inductance between the kth circuit and the rth circuit, L_{kk} is the self-inductance of the kth circuit, and repeated subscripts refer to the sum in this paper.

The motion of system is governed by the Lagrange–Maxwell equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{e}_k} - \frac{\partial L}{\partial e_k} + \frac{\partial F}{\partial \dot{e}_k} = u_k,$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s} = Q_s'' \quad (k = 1, \ldots, m; s = 1, \ldots, n), \quad (3)$$

where Q_s is the generalized non-potential force. Equation (3) consists of $n + m$ 2-order ordinary differential equations with respect to n generalized coordinates q_s (s =1,2,...,n) and m generalized electric quantities e_k (k =1,2,...,m). In Eq.(3), the dissipative function F of the system is

$$F = F_{\mathrm{e}}(\dot{\boldsymbol{e}}) + F_{\mathrm{m}}(\boldsymbol{q}, \dot{\boldsymbol{q}}), \quad (4)$$

where the electric dissipative function is given by

$$F_{\mathrm{e}} = \frac{1}{2}R_k i_k^2 = \frac{1}{2}R_k \dot{e}_k^2 \quad (k = 1, \ldots, m), \quad (5)$$

where F_{m} is the dissipative function of the viscous frictional damping force, and Q_s'' is the non-potential general force.

We now take $q_s(s = 1, \ldots, n, n+1, \ldots, n+m)$ to denote the generalized coordinates, in which $q_s(s = 1, \ldots, n)$ denote the components of space coordinates, $q_s(s = n+1, \ldots, n+m)$ denote the components of electrics, then Eq.(3) can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} + \frac{\partial F}{\partial \dot{q}_s} = Q_s \quad (s = 1, \ldots, n, n+1, \ldots, n+m), \quad (6)$$

where $Q_s(s = 1, \ldots, n)$ denote non-potential generic forces, and $Q_s(s = n+1, \ldots, n+m)$ denote generic electrokinetic potentials.

When the mechanico-electrical system satisfies the condition $Q_s - \dfrac{\partial F}{\partial \dot{q}_s} = 0$, Eq.(6) is reduced to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = 0 \quad (s = 1, \ldots, n, \ldots, n+m). \quad (7)$$

Equation (7) is called the Lagrange equation of mechanico-electrical system.

3. Discrete variational principle and Euler–Lagrange equations of Lagrange–Maxwell mechanico-electrical systems

Variational schemes for the Vesdov discretization technique are now well known and so we apply this

method to a mechanico-electrical system here.

These schemes are known to be automatically symplectic (often to second-order accuracy) integrations that retain conserved quantities (such as linear and angular momentum) associated with symmetries provided that the discrete Lagrangian has these symmetries.

Given configuration space Q, a discrete Lagrangian is a map

$$L_d : Q \times Q \to R.$$

The time step information will be contained in L_d, $Q_{s,d}$, and F_d, we regard L_d, $Q_{s,d}$, and F_d as a function of the nearby points (q_k, q_{k+1}).

To illustrate the procedure, consider a mechanico-electrical system with a position vector $\boldsymbol{x}$ and a Lagrangian $L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t)$. Then the virtual path is expressed as

$$\tilde{x}_n = (1-\alpha)x_n + \alpha x_{n+1}, \tag{8}$$

where α is an interpolation parameter and $0 \leq \alpha \leq 1$. With the choice of virtual path the time derivative becomes a difference operator. Specifically, it is

$$\tilde{v}_n = \frac{\mathrm{d}}{\mathrm{d}\tilde{t}}\tilde{x}_n = \frac{x_{n+1} - x_n}{h}, \tag{9}$$

where we define

$$\tilde{t}_n = (1-\alpha)t_n + \alpha t_{n+1} = t_n + \alpha h \tag{10}$$

with $h \in R_+$ being the time step. Using the given forms of $L(\boldsymbol{q}, \dot{\boldsymbol{q}})$, $Q_s(\boldsymbol{q}, \dot{\boldsymbol{q}})$ and $F(\boldsymbol{q}, \dot{\boldsymbol{q}})$, we regard L_d, Q_s and F as a function of 2 nearly points (q_k, q_{k+1}), expressed as

$$\begin{aligned} L_d(q_k, q_{k+1}) =& T_d\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right) \\ & - V((1-\alpha)q_k + \alpha q_{k+1}) \\ & + W_n\left((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right) \\ & - W_{\mathrm{e}}((1-\alpha)q_k + \alpha q_{k+1}), \end{aligned} \tag{11}$$

$$\begin{aligned} F_d =& F_{\mathrm{e}\,d}\left(\frac{q_{k+1} - q_k}{h}\right) + F_{\mathrm{m}\,d}\left((1-\alpha)q_k \right. \\ & \left. + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}\right), \end{aligned} \tag{12}$$

$$Q_{s\,d} = Q_{s\,d}((1-\alpha)q_k + \alpha q_{k+1}, \frac{q_{k+1} - q_k}{h}). \tag{13}$$

For a positive integer $N + M$, the action sum is the map $S_d : Q^{N+M} \to R$ defined by

$$S_d = \sum_{k=0}^{N+M-1} L_d(q_k, q_{k+1}), \tag{14}$$

where $q_k \in Q$ and k is a non-negative integer. The action sum is the discrete analogue of the action integral in mechanics.

The (continuous) integral Lagrange–d'Alembert principle is

$$\begin{aligned} & \delta \int L(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))\mathrm{d}t + \int \left[Q_s(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t)) \right. \\ & \left. - \frac{\partial F}{\partial \dot{q}_s}(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))\right]\delta q_s \mathrm{d}t = 0 \\ & (s = 1, \ldots, n, n+1, \ldots, n+m). \end{aligned} \tag{15}$$

We define the discrete Lagrange–d'Alembert principle as

$$\begin{aligned} & \delta \sum L_d(q_k, q_{k+1}) + \sum [Q^-_{s\,d}(q_k, q_{k+1}) \\ & - \left.\frac{\partial F}{\partial \dot{q}_s}\right|^-_d (q_k, q_{k+1})]\delta q_k + \sum \left[Q^+_{s\,d}(q_k, q_{k+1}) \right. \\ & \left. - \left.\frac{\partial F}{\partial \dot{q}_s}\right|^+_d (q_k, q_{k+1})\right]\delta q_{k+1} = 0, \end{aligned} \tag{16}$$

where L_d is the discrete Lagrangian, $Q^-_{s\,d} - \left.\dfrac{\partial F}{\partial \dot{q}_s}\right|^-_d$ and $Q^+_{s\,d} - \left.\dfrac{\partial F}{\partial \dot{q}_s}\right|^+_d$ are the left and the right discrete no-potential and dissipative forces.

The discrete variational principle states that the evolution equations extremize the action sum given fixed end points q_0, q_{N+M}. Extremizing S_d over $q_1, \ldots, q_{N+M}$ leads to the discrete Euler–Lagrange (DEL) equations:

$$\begin{aligned} & D_1 L_d(q_{k+1}, q_{k+2}) + D_2 L_d(q_k, q_{k+1}) \\ & + \left[Q^-_{s\,d}(q_{k+1}, q_{k+2}) - \left.\frac{\partial F}{\partial \dot{q}_s}\right|^-_d (q_{k+1}, q_{k+2})\right] \\ & + \left[Q^+_{s\,d}(q_k, q_{k+1}) - \left.\frac{\partial F}{\partial \dot{q}_s}\right|^+_d (q_k, q_{k+1})\right] = 0 \end{aligned} \tag{17}$$

for all $k = 1, \ldots, N + M - 1$, where $D_1 L$ and $D_2 L$ denote the derivative of L with respect to its first and second slots.

We can write the equation in terms of a discrete algorithm

$$\Phi : Q \times Q \to Q \times Q, \tag{18}$$

defined implicitly by

$$D_1L_d\circ\Phi+D_2L_d+\left[Q_{s\,d}^{-}\circ\Phi-\frac{\partial F}{\partial\dot{q}_s}\Big|_d^{-}\circ\Phi\right]+\left[Q_{s\,d}^{+}-\frac{\partial F}{\partial q_s}\Big|_d^{+}\right]=0, \tag{19}$$

i.e.

$$\Phi(q_k,q_{k+1})=(q_{k+1},q_{k+2}). \tag{20}$$

If, for each fixed $q\in Q$, the map $D_1L_d(\boldsymbol{q},\dot{\boldsymbol{q}}):T_qQ\to T_d^*Q$ is invertible, then $D_1L_d:Q\times Q\to T^*Q$ is locally invertible in the neighbourhood of the diagonal and so the algorithm Φ is well defined for small time steps.

4. First integral of discrete Lagrange–Maxwell mechanico-electrical systems

For discrete Lagrange–Maxwell mechanico-electrical systems, a motion for constructing constants of motion is given as follows.

Consider an infinitesimal transformation

$$q_s'=q_s+\delta q_s=q_s+\varepsilon u_s, \tag{21}$$

where ε is the infinitesimal parameter and u_s is a function of q_k and q_{k+1}, then we will have

$$\begin{aligned}\delta L_d(q_k,q_{k+1})=&\frac{\partial L_d}{\partial q_k}\delta q_k+\frac{\partial L_d}{\partial q_{k+1}}\delta q_{k+1}\\=&\varepsilon\frac{\partial L_d(q_k,q_{k+1})}{\partial q_k}u_k+\varepsilon\frac{\partial L_d(q_k,q_{k+1})}{\partial q_{k+1}}u_{k+1}\\=&\varepsilon\frac{\partial L_d(q_k,q_{k+1})}{\partial q_k}u_k-\varepsilon\frac{\partial L_d(q_{k+1},q_{k+2})}{\partial q_{k+1}}u_{k+1}-\varepsilon\Big[Q_{s\,d}^{-}(q_{k+1},q_{k+2})\\&-\frac{\partial F}{\partial\dot{q}_s}\Big|_d^{-}(q_{k+1},q_{k+2})\Big]u_{k+1}-\varepsilon\left[Q_{s\,d}^{+}(q_k,q_{k+1})-\frac{\partial F}{\partial\dot{q}_s}\Big|_d^{+}(q_k,q_{k+1})\right]u_{k+1}\end{aligned} \tag{22}$$

on the true trajectories. Now suppose there exist a sequence $v\ (q_k,\ q_{k+1})$, and use transformation (21) then δL_d will be able to be written as

$$\begin{aligned}\delta L_d=&\varepsilon v_{k+1}-\varepsilon v_k-\left\{\left[Q_{s\,d}^{-}(q_{k+1},q_{k+2})-\frac{\partial F}{\partial\dot{q}_s}\Big|_d^{-}(q_{k+1},q_{k+2})\right]u_{k+1}\right.\\&\left.+\left[Q_{s\,d}^{+}(q_k,q_{k+1})-\frac{\partial F}{\partial\dot{q}_s}\Big|_d^{+}(q_k,q_{k+1})\right]u_k\right\}.\end{aligned} \tag{23}$$

Then we can immediately deduce the quantity from Eq.(22) as

$$I=\frac{\partial L_d(q_k,q_{k+1})}{\partial q_k}u_k+v_k(q_k,q_{k+1}), \tag{24}$$

which is conserved over the classical trajectories. Such a constant of motion is referred to as a first integrator of Lagrange–Maxwell mechanico-electrical systems.

5. The variational energy algorithm of the mechanical-electrical systems

We consider a discrete mechanico-electrical system that is (possibly) time dependent and has an associated time step h_1 that may be coupled to the current choice of points $(q_1,\ q_2)$. We denote the discrete Lagrangian as $L_{h_1}(q_1,q_2):=L_d(q_1,q_2,h_1)$, the dissipative function as $F_{h_1}(q_1,q_2):=F_d(q_1,q_2,h_1)$, and the non-potential generic force as $Q_{s\,h_1}:=Q_{s\,d}(q_1,q_2,h_1)$.

This setup differs from the usual discrete Lagrangian procedures including the time step information h_k that is coupled to the current configuration data $(q_k,\ q_{k+1})$.

We can find q_2, h_1 together, by solving an equation similar to the discrete Euler–Lagrange equation for q_2, while we seek for h_1 using the connection between the $E_d(q_0,q_1,h_0)$ and $E_d(q_1,q_2,h_1)$, where E_d is the discrete energy function and determined below.

We first present the discrete action, the dissipa-

tive and non-potential forces of a mechanico-electrical system. One choice of discrete action is obtained by just applying the following approximation to the action integral for the first two sets of points, i.e. (q_0, q_1, h_0) and (q_1, q_2, h_1):

$$h_0 L_d(q_0, q_1, h_0) + h_1 L_d(q_1, q_2, h_1), \tag{25}$$

and the dissipative and non-potential forces

$$\begin{aligned} &h_0\left[Q^+_{s\,d}(q_0, q_1, h_0) - \frac{\partial F}{\partial \dot{q}_s}\bigg|^+_d (q_0, q_1, h_0)\right] \\ &+h_1\left[Q^-_{s\,d}(q_1, q_2, h_1) - \frac{\partial F}{\partial \dot{q}_s}\bigg|^-_d (q_1, q_2, h_1)\right]. \end{aligned} \tag{26}$$

We further give the discrete algorithm of the mechanico-electrical system according to the same discrete variation principle as before, but the parametrization is based on the time step information as expressed below:

$$\begin{aligned} &h_0 D_2 L_d(q_0, q_1, h_0) + h_1 D_1 L_d(q_1, q_2, h_1) \\ &+h_0\left[Q^+_{s\,d}(q_0, q_1, h_0) - \frac{\partial F}{\partial \dot{q}_s}\bigg|^+_d (q_0, q_1, h_0)\right] \\ &+h_1\left[Q^-_{s\,d}(q_1, q_2, h_1) - \frac{\partial F}{\partial \dot{q}_s}\bigg|^-_d (q_1, q_2, h_1)\right] = 0, \end{aligned} \tag{27}$$

with time steps h_0 and h_1 fixed. We will later derive the symplectic relation by considering variations of solutions, just as done in the continuous case.

5.1. The discrete energy of mechanico-electrical systems

We define the discrete energy for the point (q_0, q_1, h_0) as

$$\begin{aligned} &E_d(q_0, q_1, h_0) \\ =&-h_0 D_3 L_d(q_0, q_1, h_0) - L_d(q_0, q_1, h_0) \\ =&-\frac{\partial}{\partial h_0}[h_0 L_d(q_0, q_1, h_0)]. \end{aligned} \tag{28}$$

The virtual work done by a non-potential and dissipative force along the path from (q_0, q_1, h_0) to (q_1, q_2, h_1) is expressed as

$$\begin{aligned} &\sum_{k=0}^{N-1} (Q(q_k, q_{k+1}, \frac{q_{k+1}-q_k}{h_k}) - \frac{\partial F}{\partial \dot{q}}(q_k, q_{k+1}, \frac{q_{k+1}-q_k}{h_k}))\frac{q_{k+1}-q_k}{h_k}(t_{k+1}-t_k) \\ =&h_0\left[Q^+_d(q_0, q_1, \frac{q_1-q_0}{h_0}) - \frac{\partial F}{\partial \dot{q}}\bigg|^+_d \left(q_0, q_1, \frac{q_1-q_0}{h_0}\right)\right]\frac{(q_1-q_0)}{h_0} \\ &+h_1\left[Q^-_d(q_1, q_2, \frac{q_2-q_1}{h_1}) - \frac{\partial F}{\partial \dot{q}}\bigg|^+_d \left(q_1, q_2, \frac{q_2-q_1}{h_1}\right)\right]\frac{(q_2-q_1)}{h_1}. \end{aligned} \tag{29}$$

The relation between the energy $E_d(q_0, q_1, h_0)$ and the energy $E_d(q_1, q_2, h_1)$ of mechanico-electrical system is

$$\begin{aligned} &E_d(q_0, q_1, h_0) - \sum_{k=0}^{N-1}\left(Q\left(q_k, q_{k+1}, \frac{q_{k+1}-q_k}{h_k}\right) - \frac{\partial F}{\partial \dot{q}}\left(q_k, q_{k+1}, \frac{q_{k+1}-q_k}{h_k}\right)\right)\frac{q_{k+1}-q_k}{h_k}(t_{k+1}-t_k) \\ =&E_d(q_1, q_2, h_1) \end{aligned} \tag{30}$$

or, expressed with the algorithmic notation as follows:

$$\begin{aligned} E_k :=&E_d(q_k, q_{k+1}, h_k) - h_k\left[Q^+_d(q_k, q_{k+1}, \frac{q_{k+1}-q_k}{h_k}) - \frac{\partial F}{\partial \dot{q}}\bigg|^+_d \left(q_k, q_{k+1}, \frac{q_{k+1}-q_k}{h_k}\right)\right]\frac{q_{k+1}-q_k}{h_k} : \\ =&E_{k+1}(q_{k+1}, q_{k+2}, h_{k+1}) + h_{k+1}\left[Q^-_d\left(q_{k+1}, q_{k+2}, \frac{q_{k+2}-q_{k+1}}{h_{k+1}}\right)\right. \\ &\left.-\frac{\partial F}{\partial \dot{q}}\bigg|^-_d \left(q_{k+1}, q_{k+2}, \frac{q_{k+2}-q_{k+1}}{h_{k+1}}\right)\right]\frac{q_{k+2}-q_{k+1}}{h_{k+1}} := E_{k+1}. \end{aligned} \tag{31}$$

6. Time step solvability and optimization method

An alternative strategy to deal with how to take the time steps near turning points, which is adapted in this paper, is to adopt the following optimization technique. Given h_0, q_0, and q_1 we need to find h_1 and q_2

such that q_2 is determined by the generalized DEL equations

$$\begin{aligned} g(q_0,q_1,q_2,h_0,h_1) :=& h_0 D_2 L_d(q_0,q_1,h_0) + h_1 D_1 L_d(q_1,q_2,h_1) \\ &+ h_0 \left[Q^+_{s\,d}(q_0,q_1,h_0) - \frac{\partial F}{\partial \dot{q}_s}\bigg|^+_d (q_0,q_1,h_0) \right] \\ &+ h_1 \left[Q^-_{s\,d}(q_1,q_2,h_1) - \frac{\partial F}{\partial \dot{q}_s}\bigg|^-_d (q_1,q_2,h_1) \right] = 0, \end{aligned} \tag{32}$$

and the energy condition

$$\begin{aligned} f(q_0,q_1,q_2,h_0,h_1) :=& E_d(q_1,q_2,h_1) - E_d(q_0,q_1,h_0) \\ &+ h_1 \left[Q^-_d(q_1,q_2,\frac{q_2-q_1}{h_1}) - \frac{\partial F}{\partial \dot{q}}\bigg|^+_d \left(q_1,q_2,\frac{q_2-q_1}{h_1}\right) \right] \frac{(q_2-q_1)}{h_1} \\ &- h_0 \left[Q^+_d(q_0,q_1,\frac{q_1-q_0}{h_0}) - \frac{\partial F}{\partial \dot{q}}\bigg|^+_d \left(q_0,q_1,\frac{q_1-q_0}{h_0}\right) \right] \frac{(q_1-q_0)}{h_0} = 0. \end{aligned} \tag{33}$$

The basic equations to be solved are thus

$$f(q_0,q_1,q_2,h_0,h_1) = 0, \tag{34}$$

and

$$g(q_0,q_1,q_2,h_0,h_1) = 0 \tag{35}$$

for variables q_2 and h_1 as a function of q_0, q_1, h_0. The technique we use is to minimize the quantity

$$\begin{aligned} B =& [f(q_0,q_1,q_2,h_0,h_1)]^2 \\ &+ [g(q_0,q_1,q_2,h_0,h_1)]^2 \end{aligned} \tag{36}$$

over variables h_1 and q_2, with the other variables given, and subject to the constraint $h_1 > 0$. As indicated above, this constraint means, in practice, that the computed kinetic energy is positive. Of course, this manipulation is then iterated, and our algorithm is defined as a map

$$(q_{k-1}, q_k, h_{k-1}) \mapsto (q_k, q_{k+1}, h_1). \tag{37}$$

7. Symplectic nature of the algorithm of mechanico-electrical systems

We now show the sense in which the algorithm above is non-symplectic. To determine the symplectic nature of the mapping $\bar{\Phi}$, we consider the action sum

$$S = h_0 L_d(q_0,q_1,h_0) + h_1 L_d(q_1,q_2,h_1)$$

and take its full differential as a function of all the variables, keeping in mind that h_1 and q_2 are functions of (q_0 , q_1, h_0). Using the definition of the discrete energy, we have

$$\begin{aligned} \mathrm{d}S =& h_0 D_1 L_d(q_0,q_1,h_0)\mathrm{d}q_0 \\ &+ h_0 D_2 L_d(q_0,q_1,h_0)\mathrm{d}q_1 \\ &+ h_1 D_1 L_d(q_1,q_2,h_1)\mathrm{d}q_1 \\ &+ h_1 D_2 L_d(q_1,q_2,h_1)\mathrm{d}q_2 \\ &+ L_d(q_0,q_1,h_0)\mathrm{d}h_0 \\ &+ h_0 D_3 L_d(q_0,q_1,h_0)\mathrm{d}h_0 \\ &+ L_d(q_1,q_2,h_1)\mathrm{d}h_1 \\ &+ h_1 D_3 L_d(q_1,q_2,h_1)\mathrm{d}h_1 \\ =& h_0 D_1 L_d(q_0,q_1,h_0)\mathrm{d}q_0 \\ &+ h_0 D_2 L_d(q_0,q_1,h_0)\mathrm{d}q_1 \\ &+ h_1 D_1 L_d(q_1,q_2,h_1)\mathrm{d}q_1 \\ &+ h_1 D_2 L_d(q_1,q_2,h_1)\mathrm{d}q_2 \\ &- E_d(q_0,q_1,h_0)\mathrm{d}h_0 \\ &- E_d(q_1,q_2,h_1)\mathrm{d}h_1. \end{aligned} \tag{38}$$

According to the DEL equations, the above equation can be simplified into

$$\begin{aligned}\mathrm{d}S =&h_0D_1L_d(q_0,q_1,h_0)\mathrm{d}q_0+h_1D_2L_d(q_1,q_2,h_1)\mathrm{d}q_2\\&-E_d(q_0,q_1,h_0)\mathrm{d}h_0-E_d(q_1,q_2,h_1)\mathrm{d}h_1-\Big\{h_0\Big[Q^+_{s\,d}(q_0,q_1,h_0)-\frac{\partial F}{\partial\dot q_s}\Big|^+_d(q_0,q_1,h_0)\Big]\\&+h_1\Big[Q^-_{s\,d}(q_1,q_2,h_1)-\frac{\partial F}{\partial\dot q_s}\Big|^-_d(q_1,q_2,h_1)\Big]\Big\}\mathrm{d}q_1.\end{aligned}\tag{39}$$

In view of the equations defining the algorithm, we can write this as

$$\begin{aligned}\mathrm{d}S =&\Theta^-_L+\bar\Phi^*\Theta^+_L-\Big\{h_0\Big[Q^+_{s\,d}(q_0,q_1,h_0)-\frac{\partial F}{\partial\dot q_s}\Big|^+_d(q_0,q_1,h_0)\Big]\\&+h_1\Big[Q^-_{s\,d}(q_1,q_2,h_1)-\frac{\partial F}{\partial\dot q_s}\Big|^-_d(q_1,q_2,h_1)\Big]\Big\}\mathrm{d}q_1,\end{aligned}\tag{40}$$

where the one-form Θ^-_L and Θ^+_L are defined by

$$\Theta^-_L(q_0,q_1,h_0)=h_0D_1L_d(q_0,q_1,h_1)\mathrm{d}q_0-E_d(q_0,q_1,h_1)\mathrm{d}h_0\tag{41}$$

and

$$\Theta^+_L(q_0,q_1,h_0)=h_0D_2L_d(q_0,q_1,h_0)\mathrm{d}q_1-E_d(q_0,q_1,h_0)\mathrm{d}h_0.\tag{42}$$

Now notice that according to the definition of E_d, we have

$$\Theta^-_L(q_0,q_1,h_0)+\Theta^+_L(q_0,q_1,h_0)=d[h_0L_d]-E_d\mathrm{d}h_0.\tag{43}$$

Substituting Eq.(43) into Eq.(40) gives

$$\begin{aligned}\mathrm{d}S =&\mathrm{d}[h_0L_d]-\theta^+_d+\bar\Phi^*\Theta^+_L-\Big\{h_0\Big[Q^+_{s\,d}(q_0,q_1,h_0)-\frac{\partial F}{\partial\dot q_s}\Big|^+_d(q_0,q_1,h_0)\Big]\\&+h_1\Big[Q^-_{s\,d}(q_1,q_2,h_1)-\frac{\partial F}{\partial\dot q_s}\Big|^-_d(q_1,q_2,h_1)\Big]\Big\}\mathrm{d}q_1,\end{aligned}\tag{44}$$

where θ^+_d is the discrete analogue of the canonical one form, $p_i\mathrm{d}q^i$, namely

$$\theta^+_d=\Theta^+_d+E_d\mathrm{d}h_0=h_0D_2L_d(q_0,q_1,h_0)\mathrm{d}q_1.\tag{45}$$

Taking the differential of Eq.(44), using d^2=0 and according to the fact that pull-back commutes with the differential we can obtain our identity, namely

$$\begin{aligned}\bar\Phi^*\Omega_d =&\omega_d-d\Big\{h_0\Big[Q^+_{s\,d}(q_0,q_1,h_0)-\frac{\partial F}{\partial\dot q_s}\Big|^+_d(q_0,q_1,h_0)\Big]\mathrm{d}q_1\\&+h_0\Big[Q^+_{s\,d}(q_0,q_1,h_0)-\frac{\partial F}{\partial\dot q_s}\Big|^+_d(q_0,q_1,h_0)\Big]\mathrm{d}q_0\Big\},\end{aligned}\tag{46}$$

where

$$\Omega_d=-d\Theta^+_d,\tag{47}$$

and

$$\omega_d=-d\theta^+_d.\tag{48}$$

Note that, for nonzero forcing, the discrete Φ does not preserve the symplectic two-form. This can be seen by calculating $\mathrm{d}S$ in this section, and realizing that it contains a term with an integral of the non-potential generic and dissipative force which does not vanish except when $Q_s-\partial F/\partial\dot q_s$=0. Now the calculation similar to that given above shows that the discrete symplectic form will be preserved for a Lagrange mechanico-electrical system in which $Q_s-\partial F/\partial\dot q_s$=0.

8. Example

In this section, an example is studied for a mechanico-electrical system. We will perform the constant time step and adaptive time step algorithms, which will show the orbit in space and variation in energy as a function of time.

We discuss the plumb motion of the carriage of magnetic aerotrain. In the Fig.1, an electrical magnet is installed under the carriage of magnetic aerotrain, and so there appears an attraction force between electrical magnet and rail. It is balanceable between the suspension force and weight of carriage, when both the electric current of the electrical magnet loop and the spacing between the electrical magnet and the rail are set to be normative values.

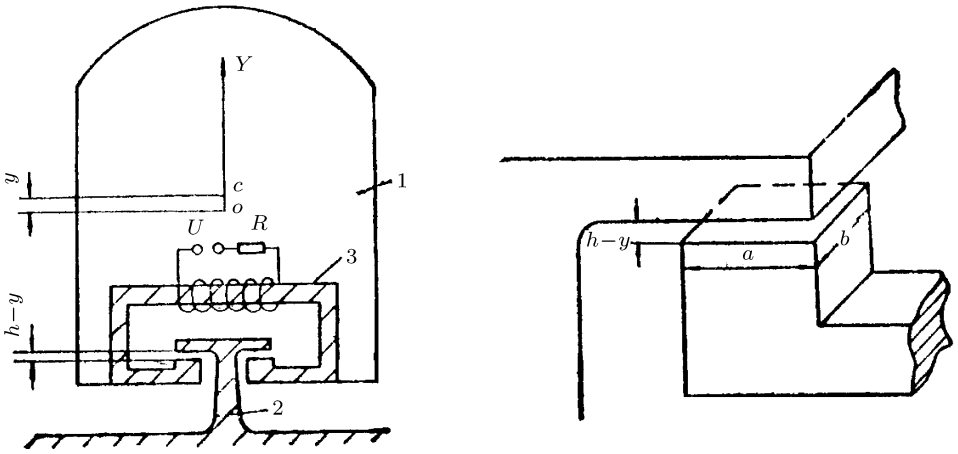


Fig.1. A sketch of plumb motion of the carriage of magnetism aerotrain.

We consider the beeline motion of the carriage along plumb y-axis, and take point o as the coordinate origin on which the centre of mass is superposed. Let y denote the displacement of carriage, e the electric quantity in the loop, U the voltage across the two end of the loop, then the kinetic energy of carriage will be expressed as

$$T = \frac{1}{2}m\dot{y}^2,$$

and the potential energy as

$$V = mgy.$$

Assume that the spacing between the electric magnet and rail is $l-y$, then the magnetic field energy in the spacing will be given as

$$W_{\mathrm{m}} = \frac{\mu_0 SN^2}{4(l-y)}\dot{e}^2,$$

where S denotes the section area of magnet, μ_0 represents the magnet electric coefficient of spacing, N refers to the number of wire turns.

The continuous Lagrangian of the mechanico-electrical system is written as

$$L = \frac{1}{2}m\dot{y}^2 - mgy + \frac{1}{4}\frac{\mu_0 SN^2}{(l-y)}\dot{e}^2 \tag{49}$$

and dissipative function is expressed as

$$F = \frac{1}{2}b\dot{y}^2 + \frac{1}{2}R\,\dot{e}^2, \tag{50}$$

where b is viscosity resistance coefficient of motion, R is ohm resistance in this electric circuit.

Use this notation

$$q_0 = (y_0, e_0), \quad \text{and} \quad q_1 = (y_1, e_1),$$

then the corresponding discrete Lagrangian (with $\gamma = \frac{1}{2}$) will be given as

$$\begin{aligned} &L_d(y_0, e_0, y_1, e_1, h_0) \\ =&\frac{1}{2}m\frac{(y_1-y_0)^2}{h^2} - mg\left(\frac{y_0+y_1}{2}\right) \\ &+\frac{1}{4}\frac{\mu_0 SN^2}{\left(l-\dfrac{y_0+y_1}{2}\right)}\frac{(e_1-e_0)^2}{h^2}, \end{aligned} \tag{51}$$

where y_0, y_1, e_0, e_1, and h are positive real numbers, and corresponding discrete dissipation forces will be expressed as

$$\left.\frac{\partial F}{\partial \dot{y}}\right|_d = b\frac{(y_1-y_0)}{h},$$

and

$$\left.\frac{\partial F}{\partial \dot{e}}\right|_d = R\frac{(e_1-e_0)}{h}. \tag{52}$$

The corresponding energy, according to formula (28) is given as

$$\begin{aligned} &E_h(y_0,y_1,e_0,e_1,h)\\ =&\frac{1}{2}m\frac{(y_1-y_0)^2}{h^2}+\frac{1}{4}\frac{\mu_0 SN^2}{\left(l-\dfrac{y_0+y_1}{2}\right)}\frac{(e_1-e_0)^2}{h^2}\\ &+mg\left(\frac{y_0+y_1}{2}\right). \end{aligned} \tag{53}$$

8.1. Constant time step algorithm

Given h, y_0, y_1, e_0, and e_1, we need to find y_2 and e_2 such that y_2 and e_2 are determined from the discrete Lagrange–Maxwell equations

$$\begin{aligned} &hD_3L_d(y_0,e_0,y_1,e_1,h_0)+hD_1L_d(y_1,e_1,y_2,e_2,h)\\ &+\left[-\frac{\partial F}{\partial \dot{y}}(\frac{y_2-y_1}{h})-\frac{\partial F}{\partial \dot{y}}(\frac{y_1-y_0}{h})\right]=0, \end{aligned} \tag{54}$$

and

$$\begin{aligned} &hD_4L_d(y_0,e_0,y_1,e_1,h_0)+hD_2L_d(y_1,e_1,y_2,e_2,h)\\ &+h\left[U-\frac{\partial F}{\partial \dot{e}}\left(\frac{e_2-e_1}{h}\right)+U-\frac{\partial F}{\partial \dot{e}}\left(\frac{e_1-e_0}{h}\right)\right]=0. \end{aligned} \tag{55}$$

As $h\neq 0$, Eqs.(54) and (55) lead to

$$\begin{aligned} &m\frac{(y_2-2y_1+y_1)}{h^2}+mg\\ &-\frac{1}{8}\frac{\mu_0 SN^2}{\left(l-\dfrac{y_0+y_1}{2}\right)^2}\frac{(e_1-e_0)^2}{h^2}\\ &-\frac{1}{8}\frac{\mu_0 SN^2}{\left(l-\dfrac{y_2+y_1}{2}\right)^2}\frac{(e_2-e_1)^2}{h^2}\\ &+b\frac{(y_2-y_0)}{h}=0, \end{aligned} \tag{56}$$

$$\begin{aligned} &\frac{1}{2}\frac{\mu_0 SN^2}{l-\dfrac{y_1+y_2}{2}}\frac{e_2-e_1}{h^2}-\frac{1}{2}\frac{\mu_0 SN^2}{l-\dfrac{y_0+y_1}{2}}\frac{e_1-e_0}{h^2}\\ &-2U+R\frac{e_2-e_0}{h}=0, \end{aligned} \tag{57}$$

which are to be solved for y_2 and e_2. Keeping h fixed, this is the variational integrator that we use for the constant time step algorithm.

8.2. Adaptive time step algorithm

Given h_0, y_0, e_0, y_1, e_1, we need to find h_1, y_2, and e_2 such that y_2, and e_2 are determined by the discrete Lagrange–Maxwell equations

$$\begin{aligned} &h_0D_3L_d(y_0,e_0,y_1,e_1,h_0)+h_1D_1L_d(y_1,e_1,y_2,e_2,h_1)\\ &-h_0\left.\frac{\partial F}{\partial \dot{y}}\right|_d^+\left(\frac{y_1-y_0}{h_0}\right)-h_1\left.\frac{\partial F}{\partial \dot{y}}\right|_d^-\left(\frac{y_2-y_1}{h_1}\right)=0, \end{aligned} \tag{58}$$

$$\begin{aligned} &h_0D_4L_d(y_0,e_0,y_1,e_1,h_0)+h_1D_2L_d(y_1,e_1,y_2,e_2,h_1)\\ &+h_0\left[U-\left.\frac{\partial F}{\partial \dot{y}}\right|_d^-\left(\frac{e_1-e_0}{h_0}\right)\right]+h_1\left[U-\left.\frac{\partial F}{\partial \dot{y}}\right|_d^-(\frac{e_2-e_1}{h_1})\right]=0, \end{aligned} \tag{59}$$

and the energy condition

$$\begin{aligned} &E(y_0,e_0,y_1,e_1,h_0)+h_0\left[-\left.\frac{\partial F}{\partial \dot{y}}\right|_d^+\right]\left(\frac{y_1-y_0}{h_0}\right)-h_0\left[U-\left.\frac{\partial F}{\partial \dot{e}}\right|_d^+\right]\left(\frac{e_1-e_0}{h_0}\right)\\ =&E(y_1,e_1,y_2,e_2,h_1)-h_1\left[-\left.\frac{\partial F}{\partial \dot{y}}\right|_d^-\right]\left(\frac{y_2-y_1}{h_1}\right)-h_1\left[U-\left.\frac{\partial F}{\partial \dot{e}}\right|_d^-\right]\left(\frac{e_2-e_1}{h_1}\right). \end{aligned} \tag{60}$$

We write the energy condition as follows. Define

$$
\begin{aligned}
&f(y_0,e_0,y_1,e_1,y_2,e_2,h_0,h_1)\\
=&E(y_1,e_1,y_2,e_2,h_1)-E(y_0,e_0,y_1,e_1,h_0)\\
&-h_1\left[-\frac{\partial F}{\partial \dot{y}}\bigg|_d^-\right]\left(\frac{y_2-y_1}{h_1}\right)-h_1\left[U-\frac{\partial F}{\partial \dot{e}}\bigg|_d^-\right]\left(\frac{e_2-e_1}{h_1}\right)\\
&-h_0\left[-\frac{\partial F}{\partial \dot{y}}\bigg|_d^+\right]\left(\frac{y_1-y_0}{h_0}\right)-h_0\left[U-\frac{\partial F}{\partial \dot{e}}\bigg|_d^+\right]\left(\frac{e_1-e_0}{h_0}\right)\\
=&\frac{1}{2}m\frac{(y_2-y_1)^2}{h_1^2}+\frac{1}{4}\frac{\mu_0SN^2}{\left(l-\dfrac{y_1+y_2}{2}\right)}\frac{(e_2-e_1)^2}{h_1^2}+mg\left(\frac{y_2+y_1}{2}\right)-\frac{1}{2}m\frac{(y_1-y_0)^2}{h_0^2}\\
&-\frac{1}{4}\frac{\mu_0SN^2}{\left(l-\dfrac{y_0+y_1}{2}\right)}\frac{(e_1-e_0)^2}{h_0^2}-mg\left(\frac{y_0+y_1}{h_0}\right)+h_1b\frac{(y_2-y_1)^2}{h_1^2}-h_1\left[U-R\frac{(e_2-e_1)^2}{h_1^2}\right]\\
&+h_0b\frac{(y_1-y_0)^2}{h_0^2}-h_0\left[U-R\frac{(e_1-e_0)^2}{h_0^2}\right],
\end{aligned}
\tag{61}
$$

then the discrete Lagrange–Maxwell equations (58) and (59) will be written as

$$
\begin{aligned}
&g_1(y_0,e_0,y_1,e_1,y_2,h_0,h_1)\\
=&h_0B_0-h_1\left[-m\frac{y_2-y_1}{h_1^2}-\frac{1}{2}mg+\frac{1}{8}\frac{\mu_0SN^2}{\left(l-\dfrac{y_2+y_1}{2}\right)^2}\frac{(e_2-e_1)^2}{h_1^2}\right]-h_1b\frac{(y_2-y_1)}{h_1},
\end{aligned}
\tag{62}
$$

where

$$
B_0=h_0m\frac{(y_1-y_0)}{h_0^2}-\frac{1}{2}mg+\frac{1}{8}\frac{\mu_0SN^2}{\left(l-\dfrac{y_0+y_1}{2}\right)^2}\frac{(e_1-e_0)^2}{h_0^2}-h_0b\frac{(y_1-y_0)}{h_0},
\tag{63}
$$

and

$$
g_2(y_0,e_0,y_1,e_1,y_2,h_0,h_1)=h_0C_0+h_1\left[-\frac{1}{2}\frac{\mu_0SN^2}{\left(l-\dfrac{(y_2+y_1)}{2}\right)}\frac{(e_2-e_1)}{h_1}\right]+U+R\left(\frac{e_2-e_1}{h_1}\right)
\tag{64}
$$

with

$$
C_0=+\frac{1}{2}\frac{\mu_0SN^2}{\left(l-\dfrac{(y_2+y_1)}{2}\right)}\frac{(e_1-e_0)}{h_0}]+U-R\left(\frac{e_1-e_0}{h_0}\right),
\tag{65}
$$

which is again a quantity that has been computed at the previous step.

8.3. Numerical technique

The basic equations we are to solve are the following:

$$
f(y_0,e_0,y_1,e_1,y_2,e_2,h_0,h_1)=0,
\tag{66}
$$

$$
g_1(y_0,e_0,y_1,e_1,y_2,e_2,h_0,h_1)=0,
\tag{67}
$$

$$
g_2(y_0,e_0,y_1,e_1,y_2,e_2,h_0,h_1)=0.
\tag{68}
$$

To solve the variables y_2, e_2, and h_1 as a function of y_0, e_0, y_1, e_1, and h_0, the technique we use is to minimize the quantity

$$
\begin{aligned}
B=&[f(y_0,e_0,y_1,e_1,y_2,e_2,h_0,h_1)]^2\\
&+[g_1(y_0,e_0,y_1,e_1,y_2,e_2,h_0,h_1)]^2\\
&+[g_2(y_0,e_0,y_1,e_1,y_2,e_2,h_0,h_1)]^2
\end{aligned}
\tag{69}
$$

over the variables h_1, y_2, and e_2, with the other variables given, and subject to the constraint $h_1>0$. As mentioned in the general theory, this technique is implemented by using descent methods, following Byrd *et al*[17] and Zhu *et al.*[18]

9. Conclusions

The main contribution of this paper is extending the study of symplectic-energy-first integrators to the

discrete mechanico-electrical coupling systems. The results obtained here present a significant approach to seeking for the structural features of mechanico-electrical coupling systems. Therefore, the further investigation should be carried out to discrete nonholonomic mechanico-electrical coupling systems.

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