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The second-order dynamic phase transition and Lee–Yang zeros in Eggers urn model*

Liu Xiao-Xian(刘小贤)† and Tong Pei-Qing(童培庆)‡

Department of Physics, Nanjing Normal University, Nanjing 210097, *China*

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A second-order dynamic phase transition in a non-equilibrium Eggers urn model for the separation of sand is studied. The order parameter, the susceptibility and the stationary probability distribution have been calculated. By applying the Lee–Yang zeros method of equilibrium phase transitions, we study the distributions of the effective partition function zeros and obtain the same result for the model. Thus, the Lee–Yang theory can be applied to a more general non-equilibrium system.

Keywords: urn model, non-equilibrium system, Lee–Yang theory, partition function
PACC: 0250, 0570F, 0570J, 0540

1. Introduction

Recently, there has been a great deal of interest in studying physical properties of granular systems.[1−3] The main motivation of this research is to obtain a basic understanding of these technologically important materials whose behaviour, indeed, very often appears elusive and mysterious. One of the most interesting phenomena is the spatial separation of shaken sand.[4] Theoretical arguments to understand how the symmetry breaking arises in this non-equilibrium system were given by Eggers.[5] Later, the non-equilibrium Eggers urn model was simplified by Lipowski and Droz.[6] Although the Lipowski–Droz urn model can explain the essence of the phenomena and show a spontaneous symmetry breaking in contrast to other generalizations of the Ehrenfest model,[7] it is too simple in terms of the practical experiment and physical questions. The Eggers urn model is obviously more suitable. However, as far as we know, a detailed study of the non-equilibrium Eggers urn model is still lacking.

In 1952, Lee and Yang[8,9] proposed the well-known method to analyse the equilibrium phase transition by studying the partition function zeros (PFZs) in a complex fugacity plane in the thermodynamical limit. Afterwards, the method of PFZs has been extensively applied to different kinds of systems.[10−14] In particular, there have been recent investigations in the applicability of the Lee–Yang theory to nonequilibrium systems.[15−18] In non-equilibrium systems,[19−22] one does not have a partition function, instead a steady-state normalization factor is regarded as an effective partition function (EPF) in a complex plane of the transition rates.

In this work, we focus on a second-order dynamical phase transition in the non-equilibrium Eggers urn model and how the phase transition may be placed in the context of the Lee–Yang equilibrium theory. Indicators of the dynamic transition such as the order parameter and susceptibility are exhibited. The stationary probability distribution in each phase has also been calculated. By applying the Lee–Yang theory we can predict the same result for the model. Thus, it is the further evidence for the application of the Lee–Yang ideas of equilibrium phase transitions to a more general non-equilibrium system.

The paper is organized as follows. In next section we briefly review the model and its master equation. In Section 3 we analyse some of indicators of the dynamic transition. In Section 4 the EPF is introduced as a polynomial of an effective size-dependent fugacity and its zeros are numerically studied in complex fugacity z and control parameter A planes. The analytical

*Project supported by the National Natural Science Foundation of China (Grant No 10175035), and the Foundation for Outstanding Young Teacher of Ministry of Education of China.
†E-mail: liuxiaoxian@njnu.edu.cn
‡Corresponding author. E-mail: pqtong@njnu.edu.cn

result for the distributions of zeros close to the critical point is given. Section 5 is devoted to discussion and conclusions.

2. The model and its master equation

In the Eggers urn model, N particles are distributed between two urns L and R, specifically, the first urn contains M particles and the second one $N-M$. The dynamics is defined as follows. At each time step, one particle is chosen to be random in one of the urns. Then, with a probability that depends on the number of particles present in the chosen urn, i.e. with a state-dependent transition rate, this particle moves to the other urn. Correspondingly, the flux $F(n)$ of particles leaving a given urn at a certain time depends on the fraction n of the total number of particles in the given urn at that moment. In this model the flux $F(n)$ is defined in the following way:

$$F(n) = Bn^2 \exp(-An^2), \tag{1}$$

where the constant B may be eliminated by an appropriate redefinition of the time scale, so in the Eggers urn model A is the single control parameter.

The master equation for the probability distribution $p(M,t)$, where there are M particles in a given urn at time t, is

$$\begin{aligned} p(N,t+1) =& F\left(\frac{1}{N}\right) p(N-1,t) \\ &+ [1-F(1)]\, p(N,t), \\ p(M,t+1) =& F\left(\frac{N-M-1}{N}\right) p(M-1,t) \\ &+ F\left(\frac{M+1}{N}\right) p(M+1,t) \\ &+ \left[1 - F\left(\frac{M}{N}\right) - F\left(\frac{N-M}{N}\right)\right] p(M,t), \end{aligned}$$

and

$$\begin{aligned} p(0,t+1) =& F\left(\frac{1}{N}\right) p(1,t) \\ &+ [1-F(1)]\, p(0,t), \end{aligned} \tag{2}$$

where $M = 1, 2, \ldots, N-1$. Its stationary solution is found to be

$$\begin{aligned} p(M) =& \frac{F\left(\dfrac{N-M+1}{N}\right)}{F\left(\dfrac{M}{N}\right)} p(M-1) \\ =& p(0) \prod_{i=1}^{M} \frac{F\left(\frac{N-i+1}{N}\right)}{F\left(\dfrac{i}{N}\right)}, \end{aligned} \tag{3}$$

and the normalized factor is

$$p(0) = \left[1 + \sum_{M=1}^{N} \prod_{i=1}^{M} \frac{F\left(\dfrac{N-i+1}{N}\right)}{F\left(\dfrac{i}{N}\right)}\right]^{-1}. \tag{4}$$

This model describes the transition from a symmetric distribution of the particles, associated with a single peak of the probability distribution at $M = N/2$ (for N even), to a symmetry breaking state described by a bimodal distribution with peaks at $M = N(1/2 \pm \varepsilon)$ in the two urns. The order parameter $\varepsilon = (N_L - N_R)/(2N)$ measures the difference in occupancy between the two urns.

For the Eggers urn model, one can find a simple, but exact equation that determines the time average ε in a steady state. Indeed, in the steady state the flux of particles changing their positions from L to R equals the flux from R to L. The detailed balance condition is given as

$$F(1/2+\varepsilon) = F(1/2-\varepsilon). \tag{5}$$

Of course, Eq.(5) admits symmetric solution $\varepsilon = 0$ for an arbitrary A. The function $f(\varepsilon) = F(1/2+\varepsilon) - F(1/2-\varepsilon)$ expresses the difference flux from one urn to another. By analysing zero points of $f(\varepsilon)$ in Fig.1, we can see that for $A < A_c = 4$ the equation has only one solution $\varepsilon = 0$; while for $A > A_c = 4$ the equation has three solutions $\varepsilon = 0$ and $\varepsilon = \pm\varepsilon_0$, but the solution $\varepsilon = 0$ is unstable. Namely, in the thermodynamic limit for $A < A_c = 4$ the stationary state is the symmetric one, while for $A > A_c = 4$ the equipartition of particles is broken, i.e. a second-order nonequilibrium

phase transition takes place at $A = A_c = 4$.

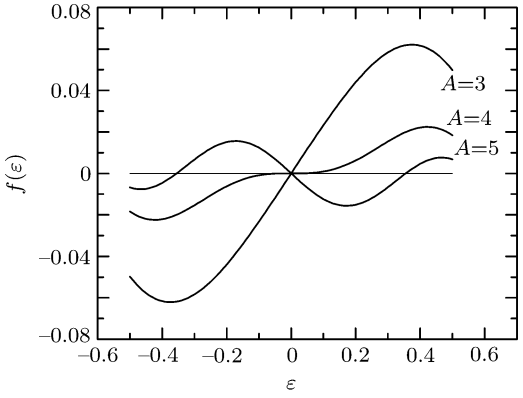


Fig.1. The curves of the function $f(\varepsilon) - \varepsilon$ with different A.

3. Indicators of the dynamics transition

The transition is best characterized by the asymmetry order parameter ε. Figure 2 shows the absolute value of ε as a function of A, which exhibits a behaviour typical of a second order phase transition. This is not surprising in that the present non-equilibrium transition is associated with a spontaneous breaking of the bilateral symmetry. We can easily check that in the vicinity of the critical point $|\varepsilon| \sim (A - A_c)^\beta$, $\beta = 0.475$. The second-order dynamic phase transition takes place at $A = A_c = 4$.

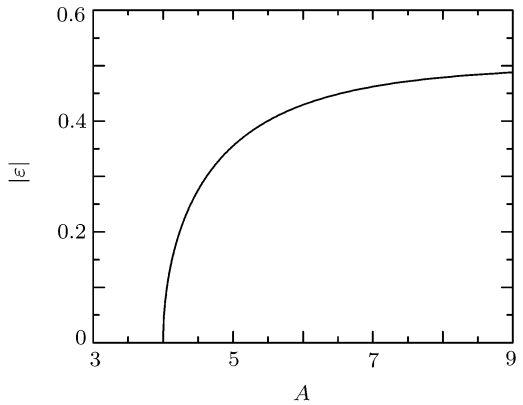


Fig.2. Absolute value of the order parameter ε as a function of A.

Further, we study the fluctuations of the order parameter. Using the variance of the order parameter we define the susceptibility κ as[6]

$$\kappa = N\langle(\varepsilon - \langle\varepsilon\rangle)^2\rangle = \frac{1}{N}\left\{\sum_{i=0}^{N} i^2 p(i,\infty) - \left[\sum_{i=0}^{N} ip(i,\infty)\right]^2\right\}, \quad (6)$$

where $p(\ldots,\infty)$ denotes the steady-state probabilities obtained from Eq.(2). Susceptibility is known to be divergent at the continuous phase transition. Our calculations confirm such a behaviour (Fig.3). For $A < 4$ the inverse susceptibility $\kappa^{-1} - (A_c - A)^\gamma$ decays linearly, which implies $\gamma = 0.958$.

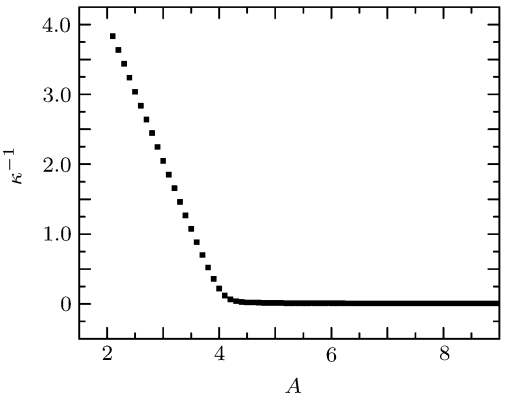


Fig.3. Inverse of the susceptibility κ^{-1} as a function of A for $N = 600$.

Finally, we calculate the stationary probability distribution in each phase. With $M/N = \frac{1}{2} + \varepsilon$, and $(i/N) = \frac{1}{2} + x$, and by scaling the probability distribution by N, the stationary probability distribution for large N now becomes[23]

$$p_s(\varepsilon) \approx \frac{e^{NG(\varepsilon)}}{\int_{-1/2}^{1/2} dx e^{NG(x)}}, \quad (7)$$

where

$$G(\varepsilon) = \int_{-1/2}^{\varepsilon} dx \left[\ln F\left(\frac{1}{2} - x\right) - \ln F\left(\frac{1}{2} + x\right)\right]. \quad (8)$$

From Eqs.(7), for a given N the denominator is a constant and the stationary probability distribution $p_s(\varepsilon)$ can be determined by $G(\varepsilon)$. Therefore, $G(\varepsilon)$ can be regarded as the stationary probability distribution in the limit $N \to \infty$. Note that in Fig.4, for $A < 4$ only the symmetric solution is stable, while for $A > 4$ the two asymmetric solutions with positive and negative particle excesses are stable and have the same maximum. Therefore the stationary probability distribution has only the central peak for $A < 4$ and double

peaks for $A > 4$. This implies that the transition occurs at $A_c = 4$.

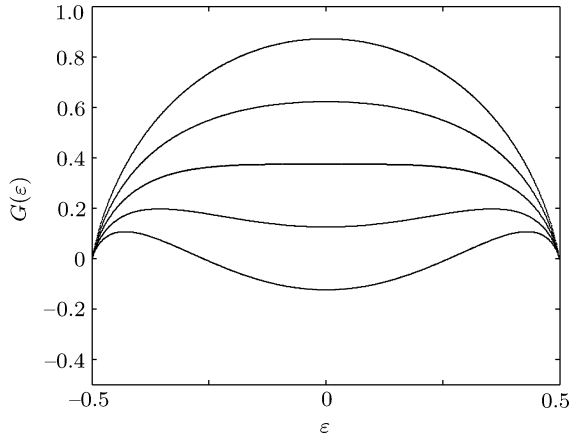


Fig.4. Curves of $G(\varepsilon)$ as a function with the values of A being 2, 3, 4, 5, and 6 (from the top).

4. Application of the Lee–Yang theory

Now we examine the Lee–Yang theory. From Eq.(3) we define the partition function of the system Z_N as the normalization factor given by[18]

$$Z_N = 1 + \sum_{M=1}^{N} \prod_{i=1}^{M} \frac{F\left(\dfrac{N-i+1}{N}\right)}{F\left(\dfrac{i}{N}\right)}. \qquad (9)$$

We substitute $F(n)$ in Eq.(9) and rewrite the EPF as

$$Z_N = \sum_{M=0}^{N} \left(\frac{N!}{M!(N-M)!}\right)^2 z^{M(N-M)}. \qquad (10)$$

Here $z = \exp(-A(N+1)/N^2)$ is the effective fugacity. So EPF is expressed as a polynomial of z, which is related to the control parameter A of the model. Indeed, in this model the EPF cannot be expressed as a polynomial of a size-independent control parameter like the former models.[15−17] It is noticeable that z is a size-dependent quantity.

4.1. The zeros of EPF in a complex z plane

In the following, we analyse the zeros of the EPF numerically. First, we consider the zeros of Z_N given by Eq.(10) in a complex z plane. We obtain numerical estimates of the zeros of Z_N with the values of finite system size N being 11, 21 and 31. The results are shown in Fig.5.

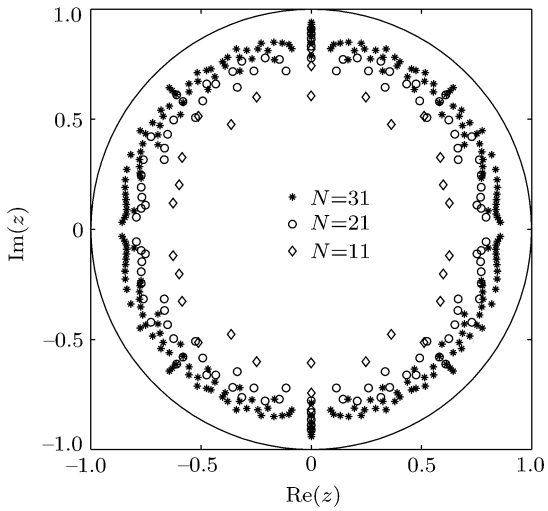


Fig.5. Zeros of the EPF(10) in a complex z plane for three values of N. The solid line represents the unit circle.

Note that the order and the coefficient of the polynomial of Eq.(10) increase rapidly, and the therefore we cannot perform too accurate calculations of the roots beyond $N = 31$. But we can predict that these roots approach a unit circle with N increasing. To investigate it further, we associate with EPF(10) the non-equilibrium complex free energy density $f_N = (1/N)\ln(Z_N)$. For large values of N and $|z| > 1$, the EPF is dominated by the central term, hence, $f_N(|z| > 1) \approx \ln 4 + (N/4)\ln z$. Thus, we define $f^{(1)} = (N/4)\ln z$ as a free energy in the $|z| > 1$ region for large values of N. On the other hand, for $|z| < 1$ the dominant contributions are coming only from the $M = 0$ and $M = N$ terms, and thus $f_N(|z| < 1) \approx (\ln 2)/N$ which in the thermodynamic limit defines $f^{(2)} = 0$ for $|z| < 1$. The above analysis indicates that the model for which Eq.(10) is the EPF with z as a control parameter undergoes an abrupt transition at $|z| = 1$. In the thermodynamic limit N, f_N exhibits an infinite jump at the transition. In order to obtain the location of the zeros of in this limit, we should equate real parts of the non-equilibrium complex free energies on both sides of the transition as follows below:

$$\mathrm{Re} f^{(1)}(z) = f^{(2)}(z). \qquad (11)$$

By calculating, only $|z|$=1 satisfies the above equation. Hence, we can conclude that the zeros should be located on the unit circle in the thermodynamic limit.

4.2. The zeros of EPF in a complex A plane

In order to infer some information about the phase transition in the Eggers urn model, it is useful to change a variable from a z plane to an $A = -N^2/(N+1)\ln z$ plane. From the above results we have known that the model with z as a parameter has a transition with a jump of the effective free energy density, but it is quite different from the original urn model with A as a control parameter (which has a continuous phase transition). We plot the numerically obtained zeros in a complex A plane in Fig.6. With N increasing, the zeros approach the critical point $A_c = 4$ with a slope that is close to $\pi/4$, and with a vanishing density of zeros. These seem to confirm the second-order nature of the phase transition.

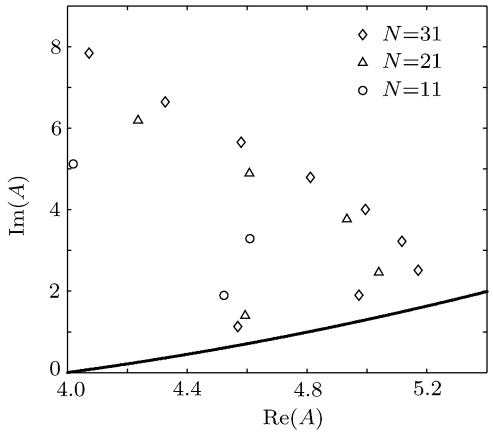


Fig.6. Zeros of Eq.(10) in a complex A plane, nearby the critical value $A_c = 4$, for three values of N. The solid line represents the analytical perturbative solution (17) of the line of zeros in the thermodynamic limit.

4.3. Analytical results

In the following, we show how to obtain these results analytically, and the final results are presented as a solid line in Fig.6. These not only confirm our numerical results but also give us some insight into the nature of the continuous transition in the model. As mentioned already, the macroscopic stationarity condition is rewritten as:

$$\left(\frac{1}{2}+\varepsilon\right)^2 \exp\left(-A\left(\frac{1}{2}+\varepsilon\right)^2\right) = \left(\frac{1}{2}-\varepsilon\right)^2 \exp\left(-A\left(\frac{1}{2}-\varepsilon\right)^2\right). \quad (12)$$

Below the critical point ($A < A_c = 4$) the leading term of Z_N for $N \to \infty$ is given by the central peak $M = N/2$, i.e.

$$Z_N \approx \binom{N}{\frac{1}{2}N}^2 \exp(-A(N+1)/4). \quad (13)$$

On the other hand, for $A > A_c = 4$ there are two leading contributions to Z_N coming, respectively, from $M = N(1/2-\varepsilon)$ and $M = N(1/2+\varepsilon)$. Therefore,

$$Z_N \approx 2\binom{N}{N(1/2-\varepsilon)}^2 \exp(-A(N+1)(1/4-\varepsilon^2)). \quad (14)$$

Correspondingly, the expressions of effective free energy density $f = \lim_{N\to\infty}(1/N)\ln(Z_N)$ are give as

$$f^{(1)} = -A/4 + \ln 4,$$

and

$$f^{(2)} = -A(1/4-\varepsilon^2) - 2(1/2-\varepsilon)\ln(1/2-\varepsilon) - 2(1/2+\varepsilon)\ln(1/2+\varepsilon). \quad (15)$$

We substitute the above expressions into Eqs.(11) and (12) and consider the behaviour of the effective free energy density as a function of the complex parameters A and ε. However, Eq.(11) is now too complicated to allow for a complete analysis of the zeros line in the whole A plane. We follow the I.Bena's perturbative way and are mainly interested in the behaviour of this line in the vicinity of the critical point A_c=4. Therefore, A and ε are expanded around a small real parameter $\alpha = \mathrm{Re}A - 4$ as follows:

$$A = 4 + \alpha + \mathrm{i}\alpha(a_0 + \alpha a_1 + \alpha^2 a_2 + \ldots),$$

$$\varepsilon = \alpha^{1/2}(c_0 + \alpha c_1 + \alpha^2 c_2 + \ldots) + \mathrm{i}\alpha^{1/2}(d_0 + \alpha d_1 + \alpha^2 d_2 + \ldots). \quad (16)$$

After a long calculation, we obtain

$$A = (4+\alpha) + \mathrm{i}\alpha[1 + 0.3\alpha + \vartheta(\alpha^2)]. \quad (17)$$

Here i is the imaginary unit. The result is represented by a solid line in Fig.6. Note that, indeed, the slope of this curve at the critical point is $\pi/4$, the sign of a second-order phase transition. Until now, we have shown that the zeros of the EPF indeed provide the information about the location and the type of the phase transition.

5. Discussion and conclusions

We consider the Eggers urn model in detail. By combining the knowledge of the continuous transition, certain features have enabled us to locate the second-order dynamic phase transition. The order parameter and its fluctuations are discussed firstly. Additional evidence of such a transition in our model comes from the analysis of the stationary probability distribution. Furthermore, we exhibit the non-equilibrium phase transition numerically and analytically by using the Lee–Yang equilibrium theory. Although it is not a straightforward task to apply the concepts of the theory, it is a nontrivial fact that it still works when the microscopic transition rates of the model are state dependent. Indeed, we have made certain of the type and the location of transition qualitatively and quantitatively. Thus, the Lee–Yang theory can be confidently applied to a more general non-equilibrium system.

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