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Multi-symplectic method for generalized fifth-order KdV equation*

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This paper considers the multi-symplectic formulations of the generalized fifth-order KdV equation in Hamiltonian space. Recurring to the midpoint rule, it presents an implicit multi-symplectic scheme with discrete multi-symplectic conservation law to solve the partial differential equations which are derived from the generalized fifth-order KdV equation numerically. The results of the numerical experiments show that this multi-symplectic algorithm is good in accuracy and its long-time numerical behaviour is also perfect.

Keywords: generalized fifth-order KdV equation, multi-symplectic, travelling wave solution, conservation law
PACC: 0200, 0340G, 0365G

1. Introduction

The generalized fifth-order KdV equation, which is called generalized Kawahara equation, a typical model equation for plasma waves, capillary-gravity water waves, as well as other dispersive phenomena when the dispersion in the cubic KdV type equation is weak, can be written in the general form

$$2\partial_t u + \alpha\partial_{xxx}u + \beta\partial_{xxxxx}u = \partial_x f(u, \partial_x u, \partial_{xx}u), \quad \beta \neq 0, \tag{1}$$

where $u = u(x,t)$ is a scalar function, α and β are real parameters and $f(u, \partial_x u, \partial_{xx}u)$ belongs to smooth function. Equation (1) was first presented by Kawahara in 1972.

Recently, an upsurge of interest in the analytic solution as well as the numerical method for the Kawahara equation and the modified Kawahara equation, two typical fifth-order KdV equations, has arisen. Bridges and Derks considered the linear stability problem for solitary wave states of the Kawahara equation and its generalizations.[2] Wazwaz used the sine–cosine method, the tanh method, the extended tanh method, and ansatze of hyperbolic functions for analytic treatment for the modified Kawahara equation.[3,4] Cui and Tao studied solvability of the Cauchy problem of the Kawahara equation in detail.[5,6] Elgarayhi obtained new exact travelling wave solutions of the Kawahara equation with the aid of the symbolic computation system.[7] Zhang presented some doubly periodic solutions of the modified Kawahara equation.[8] Haragus *et al* studied the stability of spatially periodic solutions to the Kawahara equation.[9] Zhang and Chen obtained the novel multi-solitary wave solutions to the fifth-order KdV equation by using Hirota's method.[10] Recently, Polat *et al* obtained an analytic and numerical solution of a modified Kawahara equation and analyse the convergence of the method.[11–14]

The previous works mentioned above mostly focused on the approach to obtain the analytic solutions of two typical fifth-order KdV equations (the Kawahara equation and the modified Kawahara equation) and the stability of the analytic solutions, except for the significative works reported in the Refs.[11–14], the presented numerical methods paid much attention to the convergence rate and the accuracy of the algo-

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rithm. In this paper, focusing on the accuracy as well as the long-time numerical behaviour of the numerical solution, we discuss the multi-symplectic method of the generalized fifth-order KdV equation in detail, which is useful for the numerical method of the high-order nonlinear equations in mathematical physics.

2. Multi-symplectic structure with the multi-symplectic conservation law of the generalized fifth-order KdV equation

The concept of a multi-symplectic integrator, proven to be a very robust framework for the accurate efficient and long-time integration of some partial differential equations (PDEs) systems, has been widely investigated during the last decade:[15−22] Bridges *et al* presented the concept of the multi-symplectic integrator and applied it to solve nonlinear wave equation[15−18] as well as nonlinear Schrödinger equation.[16] Subsequently, the multi-symplectic schemes and the conservation laws of several PDEs, such as KdV equation,[20,22] membrane free vibration equation,[19] nonlinear Schrödinger equation[21,23,24] and so on, were discussed to obtain the solutions of them numerically. In this section, we construct the multi-symplectic structure of the generalized fifth-order KdV equation with multi-symplectic conservation law.

In many cases, the model equation (1) has a Hamiltonian structure, thus it is reasonable to assume that the function f is a variational derivative, in this case Eq.(1) is a Hamiltonian system[2]

$$\partial_t u = J\partial_u H \quad \text{with} \quad J = -\frac{1}{2}\partial_x, \tag{2}$$

and

$$H(u) = \int_R \left[\frac{1}{2}\beta(\partial_{xx}u)^2 - \frac{1}{2}\alpha(\partial_x u)^2 + h(u, \partial_x u, \partial_{xx}u)\right] \mathrm{d}x, \tag{3}$$

where the variational derivative of the functional associated with $h(u, \partial_x u, \partial_{xx}u)$ yields f, that is, $f = \frac{1}{2}\int_R h(u, \partial_x u, \partial_{xx}u)\mathrm{d}x$. A straightforward calculation shows that the function h has the form

$$h(p, q, s) = F(p, q) + sE(p, q), \tag{4}$$

thus the function f has the form

$$\begin{aligned} f(p, q, s) &= \frac{1}{2}\int_R h(u, \partial_x u, \partial_{xx}u)\mathrm{d}x \\ &= F_p(p, q) - qF_{pq}(p, q) - sF_{qq}(p, q) \\ &\quad +2sE_p(p, q) + qsE_{qp}(p, q) \\ &\quad +q^2E_{pp}(p, q). \end{aligned} \tag{5}$$

To reformulate the general form (1) with the variational condition (5) on the function f, as a Hamiltonian system on a multi-symplectic structure, define the potential function $q_1(x, t)$ by $\partial_x q_1 = u$. Then with the canonical momenta

$$\begin{aligned} \partial_x q_1 &= u, \\ p_1 &= \partial_t q_1 - \partial_x p_2 - \partial_u F \\ &\quad -\frac{1}{\beta}(E + p_3)\partial_u E, \\ \partial_x u &= q_2, \\ p_2 &= -\alpha q_2 - \partial_x p_3 - \partial_{q_2} F \\ &\quad -\frac{1}{\beta}(E + p_3)\partial_{q_2} E, \\ p_3 &= \beta\partial_x q_2 - E, \end{aligned} \tag{6}$$

Eq.(1) reduces to the form

$$\partial_t u + \partial_x p_1 = 0. \tag{7}$$

Combining Eqs.(6) and (7), we can write the general form (1) in the multi-symplectic form

$$\boldsymbol{M}\partial_t \boldsymbol{z} + \boldsymbol{K}\partial_x \boldsymbol{z} = \nabla_z S(\boldsymbol{z}), \quad \boldsymbol{z} \in \boldsymbol{R}^6, \tag{8}$$

where $\boldsymbol{z} = (q_1, u, q_2, p_1, p_2, p_3)^{\mathrm{T}}$ is the state variable, the skew-symmetric matrices are

$$\boldsymbol{M} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \boldsymbol{K} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

and the Hamiltonian function is

$$\begin{aligned} S(\boldsymbol{z}) &= \frac{1}{2}\alpha q_2^2 + \frac{1}{2\beta}p_3^2 + p_1 u + p_2 q_2 + F(u, q_2) \\ &\quad +\frac{1}{2\beta}(2p_3 + E(u, q_2))E(u, q_2). \end{aligned} \tag{9}$$

The multi-symplectic reformulation (8) is interesting for several reasons,[17] one perhaps is the existence of the multi-symplectic conservation law,

$$\partial_t \omega + \partial_x \kappa = 0, \tag{10}$$

where $\omega = \frac{1}{2}\mathrm{d}\boldsymbol{z}\Lambda\boldsymbol{M}\mathrm{d}\boldsymbol{z}$, $\kappa = \frac{1}{2}\mathrm{d}\boldsymbol{z}\Lambda\boldsymbol{K}\mathrm{d}\boldsymbol{z}$. For the Eq.(1), the specific expression of the multi-symplectic conservation law is

$$\begin{aligned}&\partial_t(\mathrm{d}u\Lambda\mathrm{d}q_1)+\partial_x(\mathrm{d}p_1\Lambda\mathrm{d}q_1\\&+\mathrm{d}p_2\Lambda\mathrm{d}u+\mathrm{d}p_3\Lambda\mathrm{d}q_2)=0.\end{aligned}\tag{11}$$

3. A multi-symplectic box scheme of the generalized fifth-order KdV equation

It is well known that the midpoint rule is the simplest implicit discretization leading to a symplectic scheme for Hamiltonian ordinary differential equations, and it is the lowest order scheme in the Gauss–Legendre class of discretization. Thus, in this paper, it is well-grounded to use the multi-symplectic box scheme which is derived from the midpoint rule to illustrate the merit of the multi-symplectic method comparing with other numerical methods.

Firstly, introduce an uniform grid $\{(x_i,t_j)\}$ in $\boldsymbol{R}^2$ with time step length Δt in the t-direction and spatial step length Δx in the x-direction. The mesh approximate value of $\boldsymbol{z}(x,t)$ at the point (x_i,t_j) is denoted by $\boldsymbol{z}_i^j$.

We apply the implicit midpoint rule in the t-direction and in the x-direction individually for the semi-discretizations, and then concatenate the two semi-discretizations to get the following discrete version of the multi-symplectic form (8):

$$\begin{aligned}&\boldsymbol{M}\left[(z_{i+1/2}^{j+1}-z_{i+1/2}^{j})/\Delta t\right]\\&+\boldsymbol{K}\left[(z_{i+1}^{j+1/2}-z_{i}^{j+1/2})/\Delta x\right]\\&=\nabla_z S(z_{i+1/2}^{j+1/2}),\end{aligned}\tag{12}$$

where $\boldsymbol{z}_i^{j+1/2}=(\boldsymbol{z}_i^{j+1}+\boldsymbol{z}_i^{j})/2$, $\boldsymbol{z}_{i+1/2}^{j}=(\boldsymbol{z}_{i+1}^{j}+\boldsymbol{z}_i^{j})/2$ and $\boldsymbol{z}_{i+1/2}^{j+1/2}=(\boldsymbol{z}_{i+1}^{j+1}+\boldsymbol{z}_{i+1}^{j}+\boldsymbol{z}_i^{j+1}+\boldsymbol{z}_i^{j})/4$.

We call the scheme (12) the multi-symplectic centred box scheme, since it satisfies the discrete multi-symplectic conservation law:[15]

$$\begin{aligned}&\left(\omega_{i+1/2}^{j+1}-\omega_{i+1/2}^{j}\right)/\Delta t\\&+\left(\kappa_{i+1}^{j+1/2}-\kappa_{i}^{j+1/2}\right)/\Delta x=0,\end{aligned}\tag{13}$$

where $\omega_i^j=\left\langle\boldsymbol{M}U_i^j,V_i^j\right\rangle$ and $\kappa_i^j=\left\langle\boldsymbol{K}U_i^j,V_i^j\right\rangle$, and $\left\{U_i^j\right\}$ and $\left\{V_i^j\right\}$ are any two solutions of the discrete variational equation associated with Eq.(12).

Applying the centred box scheme (12) to the multi-symplectic form of Eq.(1), and eliminating the intermediate variables q_1, q_2, p_1, p_2 and p_3, we can obtain an implicit multi-symplectic scheme which is equivalent to the multi-symplectic centred box scheme:

$$\begin{aligned}&\frac{1}{32\Delta t}\Big(\delta_t u_{i+1}^{j+5}+\delta_t u_i^{j+5}+2\delta_t u_{i+1}^{j+4}+2\delta_t u_i^{j+4}+3\delta_t u_{i+1}^{j+3}+3\delta_t u_i^{j+3}+3\delta_t u_{i+1}^{j+2}+3\delta_t u_i^{j+2}\\&+2\delta_t u_{i+1}^{j+1}+2\delta_t u_i^{j+1}+\delta_t u_{i+1}^{j}+\delta_t u_i^{j}\Big)+\frac{\alpha}{8\Delta x^3}\left(\delta_x^3 u_{i+2}^{j}+2\delta_x^3 u_{i+1}^{j}+\delta_x^3 u_i^{j}\right)+\frac{\beta}{2\Delta x^5}\left(\delta_x^5 u_{i+1}^{j}+\delta_x^5 u_i^{j}\right)\\&=\frac{1}{\Delta x}\left[\left(f_{i+\frac12}^{j+\frac12}\right)^2-\left(f_{i+\frac12}^{j+\frac32}\right)^2+\left(f_{i+\frac32}^{j+\frac12}\right)^2-\left(f_{i+\frac32}^{j+\frac32}\right)^2+\left(f_{i+\frac52}^{j+\frac32}\right)^2-\left(f_{i+\frac32}^{j+\frac52}\right)^2\right.\\&\left.+\left(f_{i+\frac52}^{j+\frac52}\right)^2-\left(f_{i+\frac52}^{j+\frac72}\right)^2+\left(f_{i+\frac72}^{j+\frac52}\right)^2-\left(f_{i+\frac72}^{j+\frac72}\right)^2\right],\end{aligned}\tag{14}$$

where $f_{i+1/2}^{j+1/2}=\left(f_{i+1}^{j+1}+f_{i+1}^{j}+f_i^{j+1}+f_i^{j}\right)/4$, $\delta_t u_i^j=u_i^{j+1}-u_i^{j-1}$, $\delta_x^3 u_i^j=u_{i+1}^j-3u_i^j+3u_{i-1}^j-u_{i-2}^j$, $\delta_x^5 u_i^j=u_{i+2}^j-5u_{i+1}^j+10u_i^j-10u_{i-1}^j+5u_{i-2}^j-u_{i-3}^j$ etc.

The discrete multi-symplectic conservation law is

$$\begin{aligned}&\frac{1}{\Delta t}\left[\mathrm{d}u_{i+\frac12}^{j+1}\Lambda(\mathrm{d}q_1)_{i+\frac12}^{j+1}-\mathrm{d}u_{i+\frac12}^{j}\Lambda(\mathrm{d}q_1)_{i+\frac12}^{j}\right]+\frac{1}{\Delta x}\left[(\mathrm{d}p_1)_{i+1}^{j+\frac12}\Lambda(\mathrm{d}q_1)_{i+1}^{j+\frac12}+(\mathrm{d}p_2)_{i+1}^{j+\frac12}\Lambda\mathrm{d}u_{i+1}^{j+\frac12}\right.\\&\left.+(\mathrm{d}p_3)_{i+1}^{j+\frac12}\Lambda(\mathrm{d}q_2)_{i+1}^{j+\frac12}-(\mathrm{d}p_1)_{i}^{j+\frac12}\Lambda(\mathrm{d}q_1)_{i}^{j+\frac12}-(\mathrm{d}p_2)_{i}^{j+\frac12}\Lambda\mathrm{d}u_{i}^{j+\frac12}-(\mathrm{d}p_3)_{i}^{j+\frac12}\Lambda(\mathrm{d}q_2)_{i}^{j+\frac12}\right]=0.\end{aligned}\tag{15}$$

4. Numerical experiments

In this section, we test the implicit multi-symplectic scheme on the travelling wave solutions and solitons of Eq.(1) over long time intervals.

4.1. Numerical experiments on the travelling wave solutions of the Kawahara equation

In applications, Eq.(1) most often occurs with $f(u, \partial_x u, \partial_{xx} u) = \gamma u^2$, where γ is a nonzero constant, which implies $h(u, \partial_x u, \partial_{xx} u) = 2\gamma u \partial_x u$. For the variational derivative is not identical to zero, then we can obtain $F(u, \partial_x u, \partial_{xx} u) = 2\gamma u \partial_x u$ and $E(u, \partial_x u, \partial_{xx} u) = 0$. Thus, the specific form of the Hamiltonian function in the multi-symplectic form (8) is

$$S(\boldsymbol{z}) = \frac{1}{2}\alpha q_2^2 + \frac{1}{2\beta} p_3^2 + p_1 u + p_2 q_2 + 2\gamma u \partial_x u. \quad (16)$$

Substituting f into Eq.(1), we can obtain the common form of the generalized form (1)

$$2\partial_t u + \alpha \partial_{xxx} u + \beta \partial_{xxxxx} u = 2\gamma u \partial_x u, \quad \beta \neq 0. \quad (17)$$

In the experiments, we test the implicit multi-symplectic scheme on the travelling wave solutions of Eq.(17) with $\alpha = -\beta = 2$, $\gamma = -1$, which implies that the generalized fifth-order KdV equation (1) reduces to the Kawahara equation:

$$\partial_t u + u \partial_x u + \partial_{xxx} u - \partial_{xxxxx} u = 0. \quad (18)$$

The Kawahara equation arises in the special but interesting parameter regime when the Bond number, a dimensionless parameter proportional to the surface tension, is nearly equal to 1/3.[13] And it is a generic approximation equation describing the evolution of long waves in situations in which the coefficient in front of the third derivative term in the KdV equation becomes very small. One situation in which this occurs is for waves on the surface of an inviscid incompressible fluid undergoing irrotational motion, which is subjected to surface tension on the free surface.

According to Ref.[13], the Kawahara equation has the travelling wave solution

$$u_1(x,t) = -\frac{72}{169} + \frac{105}{169}\mathrm{sech}^4\left[\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right], \quad (19)$$

subjected to the initial condition

$$u_1(x,0) = -\frac{72}{169} + \frac{105}{169}\mathrm{sech}^4\left(\frac{1}{2\sqrt{13}}x\right), \quad (20)$$

and the travelling wave solution

$$u_2(x,t) = -\frac{72}{169} + 420\mathrm{sech}^2\left[\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right] \Big/ 169\left\{1 + \mathrm{sech}^2\left[\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right]\right\}, \quad (21)$$

subjected to the initial condition

$$u_2(x,0) = -\frac{72}{169} + 420\mathrm{sech}^2\left(\frac{1}{2\sqrt{13}}x\right) \Big/ 169\left[1 + \mathrm{sech}^2\left(\frac{1}{2\sqrt{13}}x\right)\right]. \quad (22)$$

We simulate the travelling wave solutions (19) and (21) with the corresponding initial conditions using the implicit multi-symplectic scheme (14) in $x \in [-20, 20]$ from $t = 0$ to $t = 50$ respectively. Throughout the numerical experiments, we let the time step length $\Delta t = 0.25$ and spatial step length $\Delta x = 0.1$. Figure 1 shows the development of the waveform of the travelling wave solution (19). For the travelling wave solution (21), the wave amplitude is about 0.800325 and the wave shape is very similar to Fig.1. Tables 1 and 2 report the supremum norm of error between the exact solutions and the numerical solutions derived

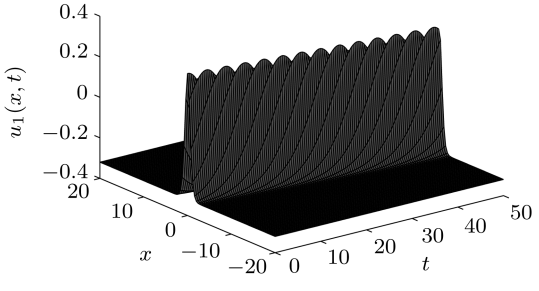


Fig.1. The waveform of $u_1(x,t)$ given by scheme (14).

from the scheme (14) as $\|u\|_s = \sup\left|u_i^j - u(x_i, t_j)\right|$ for different nodes at $t = 0$, 10, 20, 30, 40 and 50, the first column of which reports the supremum norm of error between the exact solutions and the numerical solutions for $x_i = -3.0$ and $t_j = 0$, and so on for other columns.

Table 1. The supremum norm of error between the numerical solution and exact solution (19).

$x_i \backslash t_j$	0	10	20	30	40	50
−3.0	0	5.528835×10^{-18}	0	0	1.190244×10^{-17}	1.132239×10^{-18}
−2.4	1.496677×10^{-18}	0	2.888070×10^{-19}	3.897995×10^{-18}	0	0
−1.8	4.347519×10^{-19}	2.036904×10^{-18}	0	1.381265×10^{-18}	0	9.441997×10^{-19}
−1.2	1.363958×10^{-17}	0	7.282829×10^{-18}	0	0	0
−0.6	0	0	2.112160×10^{-17}	3.155426×10^{-19}	4.620480×10^{-18}	6.446789×10^{-18}
0	0	1.018411×10^{-19}	1.022610×10^{-17}	1.957384×10^{-17}	4.930879×10^{-18}	1.132239×10^{-18}
0.6	7.933022×10^{-18}	0	0	5.045423×10^{-18}	0	6.312796×10^{-17}
1.2	0	1.580402×10^{-18}	0	0	0	0
1.8	0	7.866191×10^{-19}	4.649733×10^{-18}	1.864529×10^{-18}	3.210046×10^{-19}	2.325211×10^{-18}
2.4	4.693837×10^{-17}	6.816568×10^{-17}	9.678735×10^{-18}	0	1.099840×10^{-18}	0
3.0	9.035669×10^{-18}	0	1.357297×10^{-18}	0	0	1.132239×10^{-18}

Table 2. The supremum norm of error between the numerical solution and exact solution (21).

$x_i \backslash t_j$	0	10	20	30	40	50
−3.0	7.257905×10^{-17}	7.143245×10^{-16}	4.119083×10^{-17}	0	6.435952×10^{-17}	3.309573×10^{-17}
−2.4	0	1.623562×10^{-16}	1.290249×10^{-16}	5.287430×10^{-17}	0	5.778573×10^{-17}
−1.8	2.183185×10^{-16}	6.917757×10^{-15}	0	2.193206×10^{-16}	1.009115×10^{-15}	4.031403×10^{-17}
−1.2	1.363958×10^{-16}	0	1.190838×10^{-16}	0	1.951067×10^{-16}	0
−0.6	1.139313×10^{-16}	1.254001×10^{-16}	1.202457×10^{-16}	2.170674×10^{-16}	4.822078×10^{-17}	5.689002×10^{-17}
0	1.066768×10^{-15}	1.593729×10^{-16}	1.978955×10^{-16}	5.918782×10^{-16}	4.319184×10^{-17}	2.556457×10^{-16}
0.6	5.928146×10^{-17}	1.440964×10^{-16}	1.567172×10^{-16}	1.010633×10^{-15}	3.178594×10^{-16}	3.774689×10^{-16}
1.2	0	5.711476×10^{-16}	1.604085×10^{-16}	6.144630×10^{-17}	1.095003×10^{-15}	2.958871×10^{-16}
1.8	8.323494×10^{-17}	3.998855×10^{-16}	2.573042×10^{-16}	5.077407×10^{-16}	1.873990×10^{-17}	1.475134×10^{-15}
2.4	2.944108×10^{-16}	6.899973×10^{-16}	1.056473×10^{-15}	1.692429×10^{-16}	4.281832×10^{-17}	2.340040×10^{-17}
3.0	1.336181×10^{-16}	0	1.415141×10^{-16}	5.912826×10^{-17}	0	1.184748×10^{-16}

From the results above, we find that the waveforms keep their amplitudes and velocities invariable throughout the processes of the simulations, which implies that the implicit multi-symplectic scheme (14) can preserve the local properties of the travelling wave solution perfectly. Comparing with the results of Doğn Kaya,[13,14] we find that the convergence rate of the algorithm presented above is rapid and the minute supremum norm of error between the numerical solutions and exact solutions is limited in the intervals $[0,\ 6.917757\times10^{-15}]$ in our experiments, which suggests that the implicit multi-symplectic scheme (14) has high degree of accuracy. Furthermore, the accuracy does not reduce throughout the processes of the simulations, which implies that the implicit multi-symplectic scheme (14) has excellent long-time numerical behaviours.

4.2. Numerical experiments on the solitons of the modified Kawahara equation

In this section, we simulate the soliton solutions of the modified Kawahara equation

$$2\partial_t u + \alpha\partial_{xxx}u + \beta\partial_{xxxxx}u = \gamma u^2\partial_x u, \quad \beta \neq 0, \quad (23)$$

where α, β, γ are nonzero constants. This equation is also called the singularly perturbed KdV equation.

From Eq.(23), we can get $f(u, \partial_x u, \partial_{xx}u) = \dfrac{\gamma}{3}u^3$, which implies $h(u, \partial_x u, \partial_{xx}u) = \gamma u^2\partial_x u$, for the variational derivative is not identical to zero, then we can get $F(u, \partial_x u, \partial_{xx}u) = \gamma u^2\partial_x u$ and $E(u, \partial_x u, \partial_{xx}u) = 0$. Thus, the specific form of the Hamiltonian function

in the multi-symplectic form (8) is

$$S(\boldsymbol{z}) = \frac{1}{2}\alpha q_2^2 + \frac{1}{2\beta}p_3^2 + p_1 u + p_2 q_2 + \gamma u^2 \partial_x u. \quad (24)$$

According to Ref.[4], if $\alpha\beta < 0$, the modified Kawahara equation (23) has the soliton solutions

$$u_3(x,t) = -\frac{3\alpha}{2\sqrt{5\beta\gamma}} \times \mathrm{sech}^2\left[\frac{1}{2}\sqrt{-\frac{\alpha}{10\beta}}\left(x - \frac{\alpha^2}{25}t\right)\right] \quad (25)$$

and

$$u_4(x,t) = \frac{3\alpha}{2\sqrt{5\beta\gamma}} \times \mathrm{csch}^2\left[\frac{1}{2}\sqrt{-\frac{\alpha}{10\beta}}\left(x - \frac{\alpha^2}{25}t\right)\right]. \quad (26)$$

In the experiments, we let $\alpha = 2, \beta = \gamma = -2$. We simulate the soliton solution (25) subjected to the initial condition

$$u_3(x,0) = -\frac{3}{2\sqrt{5}}\mathrm{sech}^2\left(\frac{1}{2\sqrt{10}}x\right), \quad (27)$$

and the soliton solution (26) subjected to the initial condition

$$u_4(x,0) = \frac{3}{2\sqrt{5}}\mathrm{csch}^2\left(\frac{1}{2\sqrt{10}}x\right), \quad (28)$$

by using the implicit multi-symplectic scheme (14) in $x \in [-20, 20]$ from $t = 0$ to $t = 50$ respectively. Throughout the numerical experiment, we let the time step length $\Delta t = 0.25$ and spatial step length $\Delta x = 0.1$. Figures 2 and 3 show the development of the waveform of the soliton solutions (25) and (26) respectively.

It is also found that the waveforms keep their amplitudes and velocities invariable throughout the processes of the simulations, which implies that the implicit multi-symplectic scheme (14) can preserve the local properties of the soliton solutions well and has excellent long-time numerical behaviours once again.

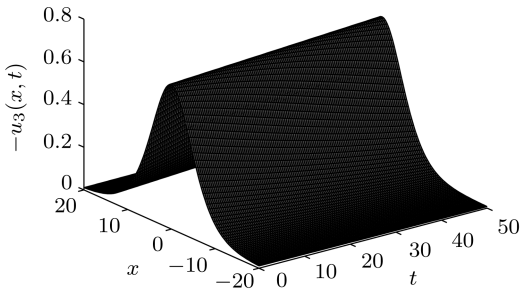


Fig.2. The waveform of $-u_3(x,t)$.

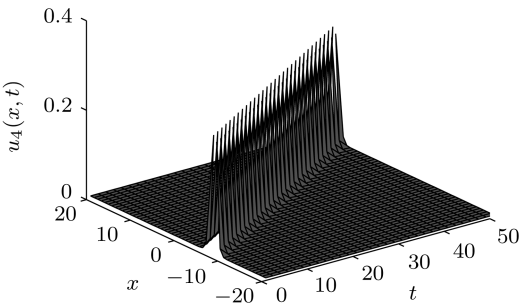


Fig.3. The waveform of $u_4(x,t)$.

5. Conclusions

This paper studies the multi-symplectic formulations according to the multi-symplectic theory of Bridges of the generalized fifth-order KdV equation in Hamiltonian space. Then an implicit multi-symplectic scheme with the discrete multi-symplectic conservation law is constructed to solve the PDEs which are derived from the generalized fifth-order KdV equation. The numerical experiments on the travelling wave solutions of the Kawahara equation and on the solitons of the modified Kawahara equation are also reported finally, the obtained results show that the multi-symplectic method is an efficient algorithm with high accuracy and excellent long-time numerical behaviours.

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