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M^2 factor of four-petal Gaussian beam*

Zhou Guo-Quan(周国泉)† and Fan Yan(樊 艳)

School of Sciences, Zhejiang Forestry University, Lin'an 311300, *China*

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Based on the second-order moments, this paper derives an analytical expression of the M^2 factor of four-petal Gaussian beam. The results show that the M^2 factor is only determined by the beam order n. The corresponding numerical calculations are also given. As the beam order increases, the augment of M^2 factor is disciplinary. As the expression of M^2 factor is expressed in series form and becomes more complicated, a new concise formula of M^2 factor is also presented by using curve fitting of numerical calculations. When $3 \leq n \leq 200$, the maximum error rate of fitting formula will not exceed 2.6% and the average error rate is 0.28%. This research is helpful to the applications of four-petal Gaussian beam.

Keywords: beam propagation, four-petal Gaussian beam, second-order moment
PACC: 4225B, 4210Q

1. Introduction

The characterization of laser beam is a topic of optics and lasers. Within the framework of the paraxial approximation, a comprehensive investigation of the beam quality was performed.[1−7] The beam propagation factor, namely M^2 factor, was proposed to characterize the beam quality of laser beam. Moreover, M^2 factor plays an important role in the characterization of beam quality of laser beams.[8,9] Recently, a new kind of laser beams called the four-petal Gaussian beam (FPGB) has been introduced to describe the laser beams with four petals at the initial source plane. FPGB can be potentially applied in micro-optics, optical communications and beam splitting techniques.[10] In the near field, FPGB reserves the four-petal distribution. While in the far field, FPGB evolves into a number of mirror symmetric petals.[11] The space among the four-petals is only determined by the beam order. To our best knowledge, the research into M^2 factor of FPGB has not been reported elsewhere. In this paper, therefore, an analytical expression for M^2 factor of FPGB will be presented on the basis of the second-order moments.

Organization of the paper is as follows. Based on second-order moments, the beam propagation factor of FPGB is derived in Section 2. A new concise formula of the beam propagation factor is given by using curve fitting of numerical calculations in Section 3. Finally, the main conclusions are outlined in Section 4.

2. Beam propagation factor of FPGB

A FPGB propagates toward the half free space $z \geq 0$, and the z-axis is taken to be the propagation axis. The electric field of a FPGB at the source plane $z = 0$ takes the form as[11,12]

$$E_n(x, y, 0) = \left(\frac{xy}{w_0^2}\right)^{2n} \exp\left(-\frac{x^2 + y^2}{w_0^2}\right), \quad n = 1, 2, 3, \ldots, \tag{1}$$

where n is the beam order. If $n = 0$, Eq.(1) reduces to the well-known Gaussian beam with beam waist size w_0. Here, w_0 is larger than the incident light wavelength λ, which means that FPGB considered in this paper is confined to the paraxial regime. FPGB can be generated by using a Gaussian beam in passage through a screen with transmittance function

$$T(x, y) = \begin{cases} \left(\dfrac{xy}{w_0^2}\right)^{2n}, & |x|, |y| \leq w_0 d, \\ 0, & |x|, |y| > w_0 d, \end{cases} \tag{2}$$

where d is chosen to ensure that the screen completely covers the four-petals at the source plane. The light

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†E-mail: zhouguoquan178@sohu.com

intensity distributions of FPGBs with $n = 1$, 2 and 3 are depicted at the source plane $z = 0$, which is shown in Fig.1. w_0 is set to be 10λ. The space among the four-petals is only determined by the beam order, and it augments as the beam order increases.

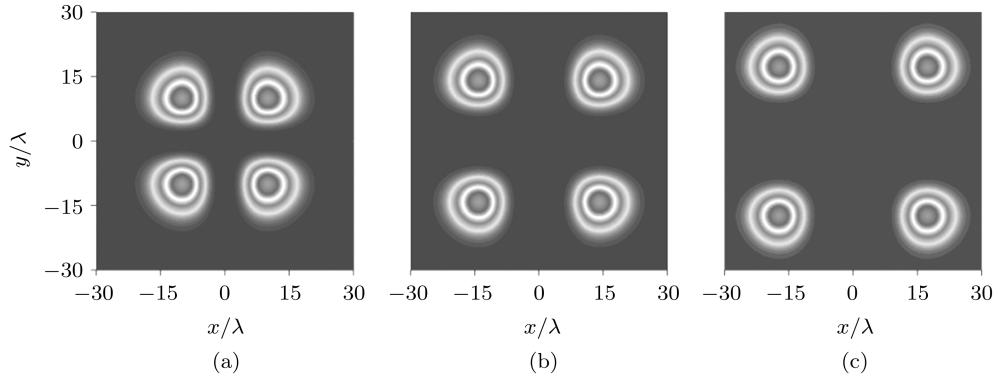


Fig.1. The light intensity distribution of FPGB with $w_0 = 10\lambda$ at the source plane. (a) $n = 1$. (b) $n = 2$. (c) $n = 3$.

The M_j^2 factor is defined as[1]

$$M_j^2 = 4\pi\sigma_{0j}\sigma_{\infty j}, \tag{3}$$

where $j = x$ or y. σ_{0j} and $\sigma_{\infty j}$ are the second-order moments associated with the intensity distribution at the waist plane and the far field plane, respectively. The first-order moment of FPGB is given by

$$\begin{aligned}\bar{x} &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\,|E_n(x,y,0)|^2\mathrm{d}x\mathrm{d}y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|E_n(x,y,0)|^2\mathrm{d}x\mathrm{d}y}\\ &= \frac{\int_{-\infty}^{\infty} x^{4n+1}\exp\left(-\dfrac{2x^2}{w_0^2}\right)\mathrm{d}x}{\int_{-\infty}^{\infty} x^{4n}\exp\left(-\dfrac{2x^2}{w_0^2}\right)\mathrm{d}x}\\ &= 0.\end{aligned} \tag{4}$$

According to the second-order moment definition of σ_{0x}, one can have

$$\begin{aligned}\sigma_{0x}^2 &= \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\bar{x})^2\,|E_n(x,y,0)|^2\mathrm{d}x\mathrm{d}y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|E_n(x,y,0)|^2\mathrm{d}x\mathrm{d}y}\\ &= \frac{\int_{0}^{\infty} x^{4n+2}\exp\left(-\dfrac{2x^2}{w_0^2}\right)\mathrm{d}x}{\int_{0}^{\infty} x^{4n}\exp\left(-\dfrac{2x^2}{w_0^2}\right)\mathrm{d}x}\\ &= \frac{(4n+1)w_0^2}{4},\end{aligned} \tag{5}$$

where the following integral formula is used[13]

$$\int_0^{\infty} x^{2n}\exp(-ax^2)\mathrm{d}x = \frac{(2n-1)!!}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}}. \tag{6}$$

In Eq.(6), a is an arbitrary real parameter. Similarly, σ_{0y} turns out to

$$\sigma_{0y} = \frac{\sqrt{4n+1}w_0}{2}. \tag{7}$$

The optical field in the spatial-frequency domain $E_n(f_x, f_y)$ is given by[14]

$$\begin{aligned}&E_n(f_x,f_y)\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E_n(x,y,0)\exp\left[-\mathrm{i}2\pi(f_x x + f_y y)\right]\mathrm{d}x\mathrm{d}y,\end{aligned} \tag{8}$$

where f_x and f_y are the transverse spatial frequency. Substituting Eq.(1) into Eq.(8) and using the integral formula

$$\begin{aligned}&\int_{-\infty}^{\infty}(x+\mathrm{i}b)^{2n}\exp(-x^2)\mathrm{d}x\\ &= \frac{n!(2n-1)!!}{2^n}\sqrt{\pi}\sum_{m=0}^{n}\frac{(-1)^m(2b)^{2m}}{(2m)!(n-m)!},\\ &\quad b \text{ is a real parameter,}\end{aligned} \tag{9}$$

we can obtain the optical field in the spatial-frequency domain

$$\begin{aligned}E_n(f_x,f_y) &= \frac{\pi w_0^2[n!(2n-1)!!]^2}{2^{2n}}\\ &\quad\times\exp[-\pi^2 w_0^2(f_x^2+f_y^2)]\\ &\quad\times\sum_{m=0}^{n}\frac{(-1)^m(2\pi w_0 f_x)^{2m}}{(2m)!(n-m)!}\\ &\quad\times\sum_{m'=0}^{n}\frac{(-1)^{m'}(2\pi w_0 f_y)^{2m'}}{(2m')!(n-m')!}.\end{aligned} \tag{10}$$

Figure 2 shows the light intensity distributions of FPGBs with $n = 1$, 2 and 3 in the spatial-frequency

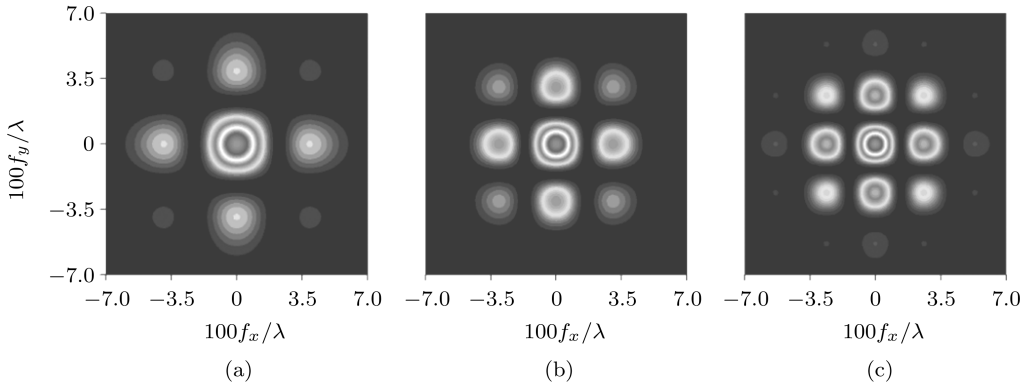


Fig.2. The light intensity distribution of FPGB in the spatial-frequency domain. (a) $n = 1$. (b) $n = 2$. (c) $n = 3$.

domain. As shown in Fig.2, FPGB evolves into a number of mirror symmetric petals. The central petal diminishes as the beam order increases. However, the intensities of the side petals increase as the beam order augments.

Therefore, the first-order moment in the spatial-frequency domain is defined by[1]

$$\bar{f}_x = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_x \left|E_n(f_x, f_y)\right|^2 \mathrm{d}f_x \mathrm{d}f_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|E_n(f_x, f_y)\right|^2 \mathrm{d}f_x \mathrm{d}f_y} = 0. \quad (11)$$

The second-order moment in the spatial-frequency domain is given by

$$\sigma_{\infty x}^2 = \frac{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (f_x - \bar{f}_x)^2 \left|E_n(f_x, f_y)\right|^2 \mathrm{d}f_x \mathrm{d}f_y}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left|E_n(f_x, f_y)\right|^2 \mathrm{d}f_x \mathrm{d}f_y} = \frac{1}{4\pi^2 w_0^2} \frac{\sum_{m=0}^{n}\sum_{m'=0}^{n} \frac{(-1)^{m+m'}[2(m+m')+1]!!}{(2m)!(n-m)!(2m')!(n-m')!}}{\sum_{m=0}^{n}\sum_{m'=0}^{n} \frac{(-1)^{m+m'}[2(m+m')-1]!!}{(2m)!(n-m)!(2m')!(n-m')!}}. \quad (12)$$

Similarly, $\sigma_{\infty y}$ is given by

$$\sigma_{\infty y}^2 = \frac{1}{4\pi^2 w_0^2} \frac{\sum_{m=0}^{n}\sum_{m'=0}^{n} \frac{(-1)^{m+m'}[2(m+m')+1]!!}{(2m)!(n-m)!(2m')!(n-m')!}}{\sum_{m=0}^{n}\sum_{m'=0}^{n} \frac{(-1)^{m+m'}[2(m+m')-1]!!}{(2m)!(n-m)!(2m')!(n-m')!}}. \quad (13)$$

As a result, the M_x^2 and M_y^2 factors of FPGB are found to be

$$M_x^2 = M_y^2 = \left((4n+1) \frac{\sum_{m=0}^{n}\sum_{m'=0}^{n} \frac{(-1)^{m+m'}[2(m+m')+1]!!}{(2m)!(n-m)!(2m')!(n-m')!}}{\sum_{m=0}^{n}\sum_{m'=0}^{n} \frac{(-1)^{m+m'}[2(m+m')-1]!!}{(2m)!(n-m)!(2m')!(n-m')!}} \right)^{1/2}. \quad (14)$$

The M^2 factor of FPGB can be readily obtained by virtue of the above results[15]

$$M^2 = (M_x^2 M_y^2)^{1/2} = \left((4n+1) \frac{\sum_{m=0}^{n}\sum_{m'=0}^{n} \frac{(-1)^{m+m'}[2(m+m')+1]!!}{(2m)!(n-m)!(2m')!(n-m')!}}{\sum_{m=0}^{n}\sum_{m'=0}^{n} \frac{(-1)^{m+m'}[2(m+m')-1]!!}{(2m)!(n-m)!(2m')!(n-m')!}} \right)^{1/2}. \quad (15)$$

If n=0, M^2=1, which corresponds to the fundamental Gaussian beam. The M^2 factor of FPGB is only determined by the beam order.

3. Numerical calculation and curve fitting

For the sake of intuition, the M^2 factor of FPGB is plotted in Fig.3(a). When the beam order increases, the M^2 factor also augments slowly. The reason is that the space among the four-petals increases as the beam order increases. Though the formula of M^2 factor is analytical, it is expressed in series form, which is unintuitive and difficult to calculate. Therefore, a new concise formula is given by using curve fitting of Fig.3(a), which turns out to be

$$M_f^2 = 3.23 + 50.85[1 - \exp(-n/229.81)] + 7.14[1 - \exp(-n/14.48)], \quad R^2 = 0.99994, \tag{16}$$

where R is the coefficient of correlation. Table 1 shows the M^2 factor calculated by Eqs.(15) and (16). The error rate is also presented in Table 1. The error rate is calculated by the following equation

$$E = \frac{|M^2 - M_f^2|}{M^2} \times 100\%. \tag{17}$$

When n = 1 and 2, the error rates are 14.98% and 4.56% respectively. n = 3, 4 and 5 are special cases. When $5 \le n \le 200$, the error rate decreases as n increases. When $3 \le n \le 200$, the maximum error rate will not exceed 2.60% and the average error rate is 0.28%. Therefore, Eq.(16) is valid within the range of $3 \le n \le 200$ and is convenient to practical calculations.

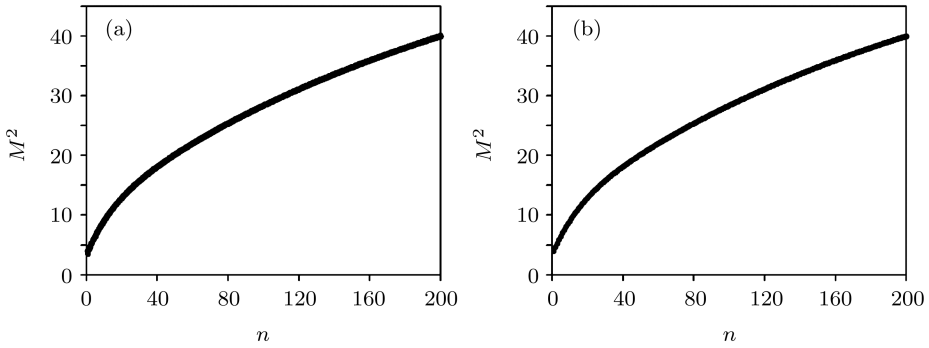


Fig.3. (a) The M^2 factor of FPGB versus the beam order n given by Eq.(15). (b) Curve fitting of Fig.2 (a) to give the Eq.(16).

Table 1. The M^2 factor calculated by Eqs.(15) and (16) and the error rate.

n	M^2	M_f^2	The error rate /%
1	3.42	3.93	14.98
2	4.39	4.59	4.56
3	5.21	5.23	0.22
4	5.93	5.83	1.64
5	6.57	6.41	2.39
6	7.15	6.96	2.60
7	7.69	7.49	2.53
8	8.19	8.00	2.32
9	8.66	8.49	2.04
10	9.11	8.95	1.74
11	9.54	9.40	1.44
12	9.95	9.84	1.15
13	10.35	10.26	0.87
14	10.72	10.66	0.63
15	11.09	11.05	0.41

4. Conclusions

On the basis of the second-order moments, an analytical expression of the M^2 factor of four-petal Gaussian beam has been derived. The corresponding numerical calculations are also given. The M^2 factor of FPGB is only determined by the beam order, and it increases slowly with the increase of the beam order. As the expression of M^2 factor is expressed in series form and unintuitive, a more concise formula of M^2 factor is also presented by using curve fitting of numerical calculations, which is very convenient to computation. When $3 \le n \le 200$, the maximum error rate of fitting formula is smaller than 2.6% and the average error rate is 0.28%. Therefore, the fitting formula is valid within the range of $3 \le n \le 200$. This research is helpful to the practical applications of four-petal Gaussian beam. Here, the four-petal Gaus-

sian beam is limited to the paraxial region. When we extend to the non-paraxial regime, the M^2 factor of FPGB can be calculated by the method suggested by Refs.[16, 17].

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