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A practical approach to robust impulsive lag synchronization between different chaotic systems*

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In this paper, a practical impulsive lag synchronization scheme for different chaotic systems with parametric uncertainties is proposed. By virtue of the new definition of synchronization and the theory of impulsive differential equations, some new and less conservative sufficient conditions are established to guarantee that the error dynamics can converge to a predetermined level. The idea and approach developed in this paper can provide a more practical framework for the synchronization between identical and different chaotic systems in parameter perturbation circumstances. Simulation results finally demonstrate the effectiveness of the method.

Keywords: chaotic systems, impulsive control, lag synchronization, parametric uncertainty
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1. Introduction

In the past decade, synchronization of the chaotic systems has attracted considerable attention due to its great potential applications in secure communication, chemical reactions and biological systems.[1] The first idea of synchronization between two chaotic systems with different initial conditions was introduced by Pecora and Carroll.[2] Since then, many different methods have been applied theoretically and experimentally to synchronize the chaotic systems, such as linear and nonlinear feedback control,[3−5] backstepping control,[6−8] variable structure control,[9−12] adaptive control,[12−15] impulsive control,[16−20] active control,[21,22] etc. Among these methods, the impulsive control provides an efficient method to deal with the dynamical systems which cannot be controlled by the continuous control. Additionally, in synchronization process, the response system receives the information from the drive system only at the discrete time instants, which drastically reduces the amount of synchronization information transmitted from the drive system to the response system and makes this method more efficient in a great number of real-life applications.

As is well known, for many applications, the parametric uncertainties of systems are inevitable, and they can destroy the synchronization and even break it. In the past few years, the analysis of synchronization for chaotic systems with parametric uncertainties has gained much researchers' attention.[4,11,16] Most of the researchers mentioned above just concern the synchronization between two identical chaotic systems with known parameters or identical unknown parameters. However, in reality there is hardly the case where the structures of the drive and the response systems can be assumed to be identical. Moreover, the parametric uncertainties of the drive and response systems are always different and time-varying. Therefore, it is significant to investigate the synchronization between two different chaotic systems in the presence of different time-varying parametric uncertainties.

Strictly speaking, in a practical environment with signal propagation delays, it is not reasonable to require the response system to synchronize with the drive system at exactly the same time. For this reason, Ref.[23] re-defined the chaotic synchronization in such a way that the state of the drive system at time $t-\eta$ asymptotically synchronizes with the response system at time t, namely

$$\lim_{t\to\infty} \|x(t-\eta)-\tilde{x}(t)\| = \lim_{t\to\infty} \|e(t)\| = 0,$$

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where x and $\tilde{x}$ are the state vectors of the drive system and the response system, respectively, where $e(t) = x(t-\eta) - \tilde{x}(t)$ is the synchronization error, and η is a finite channel time-delay, which is often unknown. Due to the existence of different structures and different time-varying parametric uncertainties, it is hard to realize the complete synchronization defined in Ref.[23]. Here, this paper aims at finding a reasonable estimation of the synchronization error bound. To formulate a more practical synchronization scheme, we re-define the chaotic synchronization in such a way that the synchronization error magnitude, i.e. $\|e\|$, converges to a value below a predetermined bound $\xi > 0$ whenever $\|e\|$ is larger than ξ.

Motivated by the above discussion, we address a practical issue by using the impulsive control method to synchronize two different chaotic systems with channel time-delay and different time-varying parametric uncertainties, derive some new and less conservative conditions for the robust impulsive synchronization criteria, and reduce the synchronization error bound which can be reduced arbitrarily as long as some specific conditions hold. In particular, some simpler and more convenient conditions are derived by taking the same impulsive distances and constant control gains. Finally, some numerical simulations for the Lorenz system and the Chen system are given to demonstrate the effectiveness and feasibility of the proposed method.

The findings of this paper can be summarized as follows: 1) the practical synchronization scheme for different chaotic systems with different time-varying parametric uncertainties via impulsive control is proposed, which has not been fully investigated in the literature to the best of authors' knowledge; 2) a more practical definition of chaotic synchronization is given, and the boundary of stable region of the synchronization error system can be reduced arbitrarily as long as some specific conditions hold; 3) sufficient conditions for the synchronization between different chaotic systems can also be extended to identical ones with or without parametric uncertainties.

The rest of the present paper is organized as follows. In Section 2, the robust impulsive lag synchronization scheme for different chaotic systems is described. In Section 3, some sufficient conditions are given to ascertain the robust stability of the error system such that the synchronization is achieved. Also, some corollaries and remarks are given to show the advantages of the proposed method. In Section 4, an illustrative example is provided to demonstrate the theoretical results. Finally, conclusions are drawn from the present study are given in Section 5.

Notations: Throughout the paper, R and R^n denote the real number and n-dimensional Euclidean space, respectively. $R^{n\times m}$ is a set of all $n \times m$ real matrices. I denotes the identity matrix with appropriate dimensions. $\|\cdot\|$ refers to the Euclidean vector norm or the induced matrix 2-norm.

2. System description and synchronization problem

Consider the drive system as follows:

$$\dot{x}(t) = (A + \Delta A(t))x(t) + f(x(t)), \tag{1}$$

where $x(t) \in R^n$ is the state vector, $A \in R^{n\times n}$ is a known constant matrix, and continuous nonlinear function $f(x(t)) \in R^n$ satisfies the Lipschitz condition with a Lipschitz constant $\sigma > 0$, i.e.

$$\|f(x(t)) - f(\tilde{x}(t))\| \le \sigma\|x(t) - \tilde{x}(t)\|, \tag{2}$$

where $x(t) \in R^n$ and $\tilde{x}(t) \in R^n$.

Owing to modelling errors and parametric perturbations, the system parameter may contain an uncertainty, which is often time-varying and unknown. Now, we assume that the uncertainty satisfies the following assumptions:

Assumption 1 The parametric uncertainty $\Delta A(t)$ is in the form

$$\Delta A(t) = EF(t)H, \tag{3}$$

where E and H are known real constant matrices with appropriate dimensions, and the uncertain matrix $F(t) \in R^{n\times n}$ satisfies

$$\|F(t)\| \le 1. \tag{4}$$

Assumption 2 The parametric uncertainty $\Delta A(t)$ does not destroy the chaotic behaviour of the chaotic system (1).

In fact, if the parameters of chaotic system vary in a finite span, the system is still chaotic.[24] Therefore, in many cases, Assumption 2 is reasonable. For example, in Refs.[19, 25], under the parametric uncertainty ΔA_1 or ΔA, the chaotic signal is still bounded.

Considering channel time-delay η, we replace t with $t - \eta$ in Eq.(11), and give the drive system at time $t - \eta$ as follows:

$$\dot{x}(t-\eta) = (A + \Delta A(t-\eta))x(t-\eta) + f(x(t-\eta)). \tag{5}$$

We consider the response system as follows:

$$\dot{\tilde{x}}(t) = (\tilde{A} + \Delta\tilde{A}(t))\tilde{x}(t) + g(\tilde{x}(t)), \tag{6}$$

where $\tilde{x}(t) \in R^n$ is the state vector, $\tilde{A} \in R^{n\times n}$ is the known constant matrix, and $g(x(t)) \in R^n$ is a continuous nonlinear function. The parametric uncertainty $\Delta\tilde{A}(t)$ satisfies the same assumptions as those of $\Delta A(t)$, i.e. $\Delta\tilde{A}(t) = \tilde{E}\tilde{F}(t)\tilde{H}$ and $\|\tilde{F}(t)\| \le 1$.

Assumption 3 For the nonlinear functions $f(\cdot)$ and $g(\cdot)$, there exists a positive constant μ such that the following condition holds

$$\|f(\tilde{x}(t)) - g(\tilde{x}(t))\| \le \mu\|\tilde{x}(t)\|, \tag{7}$$

where $\tilde{x}(t) \in R^n$.

Remark 1 Most of typical chaotic systems with parametric uncertainties can be described by Eq.(1) or Eq.(6), and the continuous nonlinear functions can satisfy conditions (2) and (7), such as the Lorenz system, Chen system, Rössler system, Lü system, unified system, Liu system, several variants of Chua circuits, etc.

Suppose that a discrete instant set $\{t_k\}$ satisfies $0 < t_1 < t_2 < \ldots < t_k < t_{k+1} < \ldots$, $\lim\limits_{k\to\infty} t_k = \infty$, $k = 1, 2, \ldots$, $0 \le t_0 < t_1$. At discrete time t_k, the state variables of the drive system are converted into those of the response system as the control input such that the state variables of the response system are suddenly changed at these instants. Therefore, the response system can be written in the following form:

$$\begin{aligned}
\dot{\tilde{x}}(t) =& (\tilde{A} + \Delta\tilde{A}(t))\tilde{x}(t) + g(\tilde{x}(t)),\ t \ne t_k,\\
\Delta\tilde{x}(t_k) =& \tilde{x}(t_k^+) - \tilde{x}(t_k^-)\\
=& \tilde{x}(t_k^+) - \tilde{x}(t_k)\\
=& -B_k e(t_k),\ t = t_k,\ k = 1, 2, \ldots,\\
\tilde{x}(t_0^+) =& \tilde{x}_0,
\end{aligned} \tag{8}$$

where $e(t) = x(t-\eta) - \tilde{x}(t) = [x_1(t-\eta) - \tilde{x}_1(t), x_2(t-\eta) - \tilde{x}_2(t), \ldots, x_n(t-\eta) - \tilde{x}_n(t)]^{\mathrm{T}}$ is the synchronization error, $\lim\limits_{t\to t_k^+} \tilde{x}(t) = \tilde{x}(t_k^+)$, and $\lim\limits_{t\to t_k^-} \tilde{x}(t) = \tilde{x}(t_k^-)$. $\tilde{x}(t_k) = \tilde{x}(t_k^-)$ implies that $\tilde{x}(t)$ is left-continuous at $t = t_k$, $B_k \in R^{n\times n}$ are the impulsive control gains. $\Delta x(t_k) = x(t_k^+) - x(t_k^-) = 0$. Then from systems (5) and (8), the following error system is obtained:

$$\begin{aligned}
\dot{e}(t) =& (A + \Delta A(t-\eta))e(t) + h(t)\\
&+ (A - \tilde{A} + \Delta A(t-\eta) - \Delta\tilde{A}(t))\tilde{x}(t)\\
&+ f(\tilde{x}(t)) - g(\tilde{x}(t)),\ t \ne t_k,\ k = 1, 2, \ldots,\\
\Delta e|_{t=t_k} =& e(t_k^+) - e(t_k) = B_k e(t_k),\ t = t_k,\\
e(t_0^+) =& e_0,
\end{aligned} \tag{9}$$

where $h(t) \doteq f(x(t-\eta)) - f(\tilde{x}(t))$.

Definition 1 The synchronization between systems (5) and (8) is said to be achieved if, for arbitrary initial conditions x_0 and $\tilde{x}_0$, the trivial solution of the error system (9) converges to a predetermined neighbour of the origin for any admissible parametric uncertainty that satisfies assumptions 1 and 2.

Here the objective is to set the control gains B_k and the impulsive distances $\delta_k = t_k - t_{k-1}$ ($k = 1, 2, \ldots$) such that the error magnitude $\|e\|$ converges to a value below some given constant $\xi > 0$, which implies the impulsively controlled response system (8) is synchronized with drive system (5) for arbitrary initial conditions.

For simplicity, in the following sections, $x(t-\eta)$, $\tilde{x}(t)$, $h(t)$, $\Delta A(t-\eta)$, $\Delta\tilde{A}(t)$ are denoted by x, $\tilde{x}$, h, ΔA, $\Delta\tilde{A}$, respectively.

Remark 2 Owing to the boundedness of the chaotic signals, there exists a constant $\varsigma > 0$ such that $\|\tilde{x}\| \le \varsigma$.

3. Main results

The following theorem will give the sufficient conditions for robust stability of error system (9), which implies that impulsively controlled response system (8) is synchronized with drive system (5).

Theorem 1 Let β_k and λ_A be the largest eigenvalues of $(I + B_k)^{\mathrm{T}}(I + B_k)$ and $A + A^{\mathrm{T}}$, respectively. Response system (8) is synchronized with drive system (5), if one of the following conditions holds:

(i) when $\theta \ge 0$, and there exists a constant $\rho > 1$ such that

$$\ln(\rho\beta_{2k-1}\beta_{2k}) + \theta(t_{2k+1} - t_{2k-1}) \le 0,\ k = 1, 2, \ldots, \tag{10}$$

and

$$\sup_k\{\beta_k \exp[\theta(t_{k+1} - t_k)]\} = \gamma < \infty; \tag{11}$$

(ii) when $\theta < 0$, and there exists a constant ρ ($0 \le \rho < -\theta$) such that

$$\ln(\beta_k) - \rho(t_k - t_{k-1}) \le 0,\ k = 1, 2, \ldots, \tag{12}$$

where $\xi > 0$ is the bound of the error magnitude $\|e\|$ and can be chosen to be small enough, $\theta = \lambda_A + 2\sigma + 2\|E\|\|H\| + \frac{2\varsigma}{\xi}\big(\|A - \tilde{A}\| + \|E\|\|H\| + \|\tilde{E}\|\|\tilde{H}\| + \mu\big)$.

Proof Choose the following Lyapunov function:

$$V(e) = e^{\mathrm{T}}e. \tag{13}$$

For $t \in (t_{k-1}, t_k]$, $k = 1, 2, \ldots$, the time derivative of $V(e)$ along with the solution of Eq.(9) is

$$\begin{aligned}
&\dot{V}(e(t))\\
=&\dot{e}^{\mathrm{T}}e + e^{\mathrm{T}}\dot{e}\\
=&[(A+\Delta A)e + h + (A - \tilde{A} + \Delta A - \Delta\tilde{A})\tilde{x}\\
&+ f(\tilde{x}) - g(\tilde{x})]^{\mathrm{T}}e + e^{\mathrm{T}}[(A+\Delta A)e + h\\
&+ (A - \tilde{A} + \Delta A - \Delta\tilde{A})\tilde{x} + f(\tilde{x}) - g(\tilde{x})]\\
=&e^{\mathrm{T}}(A + A^{\mathrm{T}})e + 2e^{\mathrm{T}}h + e^{\mathrm{T}}(\Delta A + \Delta A^{\mathrm{T}})e\\
&+ 2e^{\mathrm{T}}(A - \tilde{A} + \Delta A - \Delta\tilde{A})\tilde{x} + 2e^{\mathrm{T}}(f(\tilde{x}) - g(\tilde{x}))\\
\leq&(\lambda_A + 2\sigma + 2\|E\|\|H\|)e^{\mathrm{T}}e\\
&+ 2\big(\|A - \tilde{A}\| + \|E\|\|H\| + \|\tilde{E}\|\|\tilde{H}\| + \mu\big)\varsigma\|e\|.
\end{aligned} \tag{14}$$

To make the error magnitude converge to a value below the constant ξ, here we concern only the case of $\|e\| > \xi$, which implies that $\|e\| < \|e\|^2/\xi$, and therefore inequality (14) further becomes

$$\begin{aligned}
\dot{V}(e(t)) \leq&(\lambda_A + 2\sigma + 2\|E\|\|H\|)e^{\mathrm{T}}e + \frac{2\varsigma}{\xi}\big(\|A - \tilde{A}\|\\
&+ \|E\|\|H\| + \|\tilde{E}\|\|\tilde{H}\| + \mu\big)\|e\|^2\\
\leq&\Big[\lambda_A + 2\sigma + 2\|E\|\|H\| + \frac{2\varsigma}{\xi}\big(\|A - \tilde{A}\|\\
&+ \|E\|\|H\| + \|\tilde{E}\|\|\tilde{H}\| + \mu\big)\Big]V(e(t))\\
=&\theta V(e(t)),
\end{aligned} \tag{15}$$

Inequality (15) implies that

$$\begin{aligned}
V(e(t)) \leq&V(e(t_{k-1}^+))\exp[\theta(t - t_{k-1})],\\
&t \in (t_{k-1}, t_k],\ k = 1, 2, \ldots.
\end{aligned} \tag{16}$$

On the other hand, from the second equation of Eq.(9), we obtain

$$\begin{aligned}
V(e(t_k^+)) =&[(I + B_k)e(t_k)]^{\mathrm{T}}(I + B_k)e(t_k)\\
=&e^{\mathrm{T}}(t_k)(I + B_k)^{\mathrm{T}}(I + B_k)e(t_k)\\
\leq&\beta_k V(e(t_k)).
\end{aligned} \tag{17}$$

Thus, let $k = 1$ in the inequality (16), we have

$$V(e(t)) \leq V(e(t_0^+))\exp[\theta(t - t_0)], \quad t \in (t_0, t_1],$$

which leads to

$$V(e(t_1)) \leq V(e(t_0^+))\exp[\theta(t_1 - t_0)],$$

and from inequality (17), we have

$$V(e(t_1^+)) \leq \beta_1 V(e(t_1)) \leq V(e(t_0^+))\beta_1\exp[\theta(t_1 - t_0)].$$

Similarly, for $t \in (t_1, t_2]$, we have

$$\begin{aligned}
V(e(t)) \leq& V(e(t_1^+))\exp[\theta(t - t_1)]\\
\leq& V(e(t_0^+))\beta_1\exp[\theta(t - t_0)].
\end{aligned}$$

In general, for $t \in (t_k, t_{k+1}]$, we have

$$V(e(t)) \leq V(e(t_0^+))\beta_1\beta_2, \ldots, \beta_k\exp[\theta(t - t_0)]. \tag{18}$$

(i) If $\theta \geq 0$, there exists a constant $\rho > 1$, and from inequality (10), expression (11) and inequality (18), when

a) $t \in (t_{2k-1}, t_{2k}]$, we have

$$\begin{aligned}
V(e(t)) \leq&V(e(t_0^+))\prod_{i=1}^{2k-1}\beta_i\exp[\theta(t - t_0)]\\
\leq&V(e_0)\prod_{i=1}^{2k-1}\beta_i\exp[\theta(t_{2k} - t_0)]\\
=&V(e_0)\beta_1\beta_2\exp[\theta(t_3 - t_1)]\ldots\\
&\times\beta_{2k-3}\beta_{2k-2}\exp[\theta(t_{2k-1} - t_{2k-3})]\\
&\times\beta_{2k-1}\exp[\theta(t_{2k} - t_{2k-1})]\exp[\theta(t_1 - t_0)]\\
\leq&\gamma\frac{V(e_0)}{\rho^{k-1}}\exp[\theta(t_1 - t_0)],
\end{aligned} \tag{19}$$

b) $t \in (t_{2k}, t_{2k+1}]$, we have

$$\begin{aligned}
V(e(t)) \leq&V(e(t_0^+))\prod_{i=1}^{2k}\beta_i\exp[\theta(t - t_0)]\\
\leq&V(e_0)\prod_{i=1}^{2k}\beta_i\exp[\theta(t_{2k+1} - t_0)]\\
=&V(e_0)\beta_1\beta_2\exp[\theta(t_3 - t_1)]\ldots\\
&\times\beta_{2k-1}\beta_{2k}\exp[\theta(t_{2k+1} - t_{2k-1})]\\
&\times\exp[\theta(t_1 - t_0)]\\
\leq&\frac{V(e_0)}{\rho^k}\exp[\theta(t_1 - t_0)].
\end{aligned} \tag{20}$$

(ii) If $\theta < 0$, and there exists a constant ρ $(0 \leq \rho < -\theta)$, such that $\ln(\beta_k) - \rho(t_k - t_{k-1}) \leq 0$, i.e. $\beta_k \leq \exp(\rho(t_k - t_{k-1}))$, then

$$\begin{aligned}
V(e(t)) \leq& V(e(t_0^+))\beta_1\beta_2\ldots\beta_k\exp[\theta(t - t_0)]\\
\leq& V(e_0)\exp[\rho(t_k - t_0)]\exp[\theta(t - t_0)]\\
\leq& V(e_0)\exp[(\theta + \rho)(t - t_0)].
\end{aligned} \tag{21}$$

It follows from inequalities (19), (20) and (21) that the error magnitude $\|e\|$ will converge to a value below the constant ξ whenever $\|e\|$ is larger than ξ. Up till here the proof is completed.

Remark 3 If $A = \tilde{A}$ and $f(\cdot) = g(\cdot)$, then the structures of systems (5) and (6) are identical. Therefore, Theorem 1 is also applicable to the impulsive synchronization of two identical chaotic systems with or without parametric uncertainties.

Remark 4 From inequality (10) in Theorem 1, it can be seen that we need only choose the odd switch sequence $\{t_{2k-1}\}$ instead of the whole switch sequence $\{t_k\}$ as in Theorem 1 in Ref.[17].

Remark 5 Inequality (10) can be generalized to the following condition.

There exist a finite integer $n_0 > 0$ and a constant $\rho > 1$ such that

$$\ln(\rho\beta_{n_0(k-1)+1}\cdots\beta_{n_0k}) + \theta(t_{n_0k+1} - t_{n_0(k-1)+1}) \le 0, \qquad k = 1, 2, \ldots. \tag{22}$$

The choice of n_0 in inequality (22) depends on the practical system to be considered. Especially, $n_0 = 1$ corresponds to the case of the whole switch sequence $\{t_k\}$, and $n_0 = 2$ corresponds to the case of the odd switch sequence $\{t_{2k-1}\}$.

Remark 6 In Eq.(8), the value of channel time-delay η is not required to be known for executing the impulsive control, for we can obtain the time-delay signal $x(t-\eta)$ at the discrete time t_k easily and use it in the response system blindly when it is received.

Based on the matrix theory, we can obtain $\lambda_A \le \left\|A + A^{\mathrm{T}}\right\|$, then we can have the following corollary.

Corollary 1 Let β_k be the largest eigenvalue of $(I + B_k)^{\mathrm{T}}(I + B_k)$. If there exists a constant $\rho > 1$ such that

$$\ln(\rho\beta_{2k-1}\beta_{2k}) + \bar{\theta}(t_{2k+1} - t_{2k-1}) \le 0, \ k = 1, 2, \ldots, \tag{23}$$

and

$$\sup_k\{\beta_k \exp[\bar{\theta}(t_{k+1} - t_k)]\} = \gamma < \infty, \tag{24}$$

then response system (8) is synchronized with drive system (5), where $\xi > 0$ is the bound of the error magnitude $\|e\|$ and can be chosen to be small enough, $\bar{\theta} = \left\|A + A^{\mathrm{T}}\right\| + 2\sigma + 2\|E\|\|H\| + \frac{2\varsigma}{\xi}\big(\|A - \tilde{A}\| + \|E\|\|H\| + \|\tilde{E}\|\|\tilde{H}\| + \mu\big)$.

Remark 7 Generally speaking, the computation of $\left\|A + A^{\mathrm{T}}\right\|$ will be easier than that of λ_A. In addition, $\bar{\theta}$ is always positive and so we need not discuss the negative case like Theorem 1. Therefore, in practice, the result in Corollary 1 will be simpler and more convenient than that in Theorem 1, though Corollary 1 is a little more conservative than Theorem 1.

In practice, for convenience, the gains B_k are always selected as a constant matrix and the impulsive distances Δ_k are set to be a positive constant, then we have the following corollary.

Corollary 2 Assume that $t_{2k+1} - t_{2k-1} = \Delta > 0$ and $B_k = B$ $(k = 1, 2, \ldots)$. If there exists a constant $\rho > 1$ such that

$$\ln(\rho\beta^2) + \Delta\bar{\theta} \le 0 \tag{25}$$

and

$$\beta\exp(\Delta\bar{\theta}) = \gamma < \infty, \tag{26}$$

then response system (8) is synchronized with drive system (5), where $\xi > 0$ is the bound of the error magnitude $\|e\|$ and can be chosen to be small enough, β is the largest eigenvalue of $(I + B)^{\mathrm{T}}(I + B)$, $\bar{\theta}$ is defined in Corollary 1.

Remark 8 It is worth pointing out that the topic of synchronization between different chaotic systems with different parametric uncertainties has attracted a great deal of attention, and the scheme considered here is more general in practical environments. In this paper, to compensate for the different structures of the drive and the response systems, stronger impulsive strengths and higher impulsive frequencies are necessary. On the other hand, to save the energy of impulsive control and improve its performance, one can introduce an additional feedback controller to compensate for the different structures, and this will lead the controller to have more complexities. Hence, to achieve the synchronization, we can choose one of the two schemes mentioned above by considering the implementation simplicity and the performance improvement, which depends on the practical situation.

4. Simulation results

In the simulation, the impulsive distances Δ_k are set to be a positive constant Δ, and the gains B_k are selected as a constant matrix, i.e. $B_k = B = \mathrm{diag}\{d, d, d\}$, and then $\beta = (d+1)^2$.

In order to observe the synchronization behaviour in different chaotic systems, we assume that Lorenz system[26] drives the Chen system.[27] Therefore, the drive and the response systems are respectively as follows:

$$\begin{aligned} \dot{x}_1 &= \alpha_1(x_2 - x_1), \\ \dot{x}_2 &= \alpha_2 x_1 - x_1x_3 - x_2, \\ \dot{x}_3 &= x_1x_2 - \alpha_3 x_3 \end{aligned} \tag{27}$$

and

$$\begin{aligned} \dot{\tilde{x}}_1 &= \theta_1(\tilde{x}_2 - \tilde{x}_1), \\ \dot{\tilde{x}}_2 &= (\theta_2 - \theta_1)\tilde{x}_1 - \tilde{x}_1\tilde{x}_3 + \theta_2\tilde{x}_2, \\ \dot{\tilde{x}}_3 &= \tilde{x}_1\tilde{x}_2 - \theta_3\tilde{x}_3. \end{aligned} \tag{28}$$

From Systems (9), (27) and (28), the following compact form of the error system is obtained:

$$\begin{aligned}\dot{e} =&(A+\Delta A)e + h + (A-\tilde{A}+\Delta A-\Delta\tilde{A})\tilde{x}\\ &+ f(\tilde{x}) - g(\tilde{x}), \quad t \neq t_k, \quad k = 1, 2, \ldots,\\ \Delta e|_{t=t_k} =&e(t_k^+) - e(t_k) = B_k e(t_k), \quad t = t_k,\\ e(t_0^+) =&e_0.\end{aligned}$$

Let $\alpha_1 = 10$, $\alpha_2 = 28$, $\alpha_3 = 8/3$, $\theta_1 = 35$, $\theta_2 = 28$ and $\theta_3 = 3$. Systems (27) and (28) are chaotic for the above parameters. Then we have

$$A = \begin{pmatrix} -\alpha_1 & \alpha_1 & 0 \\ \alpha_2 & -1 & 0 \\ 0 & 0 & -\alpha_3 \end{pmatrix} = \begin{pmatrix} -10 & 10 & 0 \\ 28 & -1 & 0 \\ 0 & 0 & -8/3 \end{pmatrix},$$

$$A + A^{\mathrm{T}} = \begin{pmatrix} -20 & 38 & 0 \\ 38 & -2 & 0 \\ 0 & 0 & -16/3 \end{pmatrix},$$

$$\tilde{A} = \begin{pmatrix} -\theta_1 & \theta_1 & 0 \\ \theta_2 - \theta_1 & \theta_2 & 0 \\ 0 & 0 & -\theta_3 \end{pmatrix} = \begin{pmatrix} -35 & 35 & 0 \\ -7 & 28 & 0 \\ 0 & 0 & -3 \end{pmatrix},$$

$$A - \tilde{A} = \begin{pmatrix} 25 & -25 & 0 \\ 35 & -29 & 0 \\ 0 & 0 & 1/3 \end{pmatrix}.$$

and

$$\Delta A = -\Delta\tilde{A} = 0.04 \begin{pmatrix} \sin t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & \sin t \end{pmatrix},$$

where

$$E = H = \tilde{E} = \tilde{H} = 0.2I,$$

and

$$F(t) = -\tilde{F}(t) = \begin{pmatrix} \sin t & 0 & 0 \\ 0 & \cos t & 0 \\ 0 & 0 & \sin t \end{pmatrix}.$$

From $f(\tilde{x}) = g(\tilde{x}) = (0,\ -\tilde{x}_1\tilde{x}_3,\ \tilde{x}_1\tilde{x}_2)^{\mathrm{T}}$, we have $\mu = 0$.

In this simulation, we choose $\sigma = 27.5$, and $\varsigma = 60$. By computation, we can obtain $\left\|A + A^{\mathrm{T}}\right\| = 50.0512$, $\|A - \tilde{A}\| = 57.5257$, and $\|E\|\,\|H\| = \|\tilde{E}\|\|\tilde{H}\| = 0.04$. Figure 1 shows a stable region for different values of ρ, and the whole region under the curve of $\rho = 1$ is a stable region. When $\rho \to \infty$, the stable region approaches a vertical line, $d = -1$. Similarly, Fig.2 shows a stable region for different values of ξ.

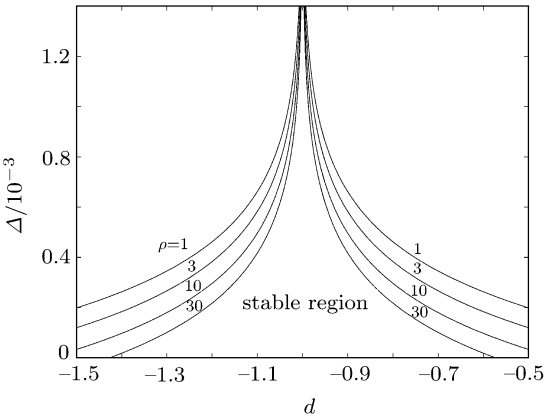


Fig.1. The boundaries of stable region with different values of ρ.

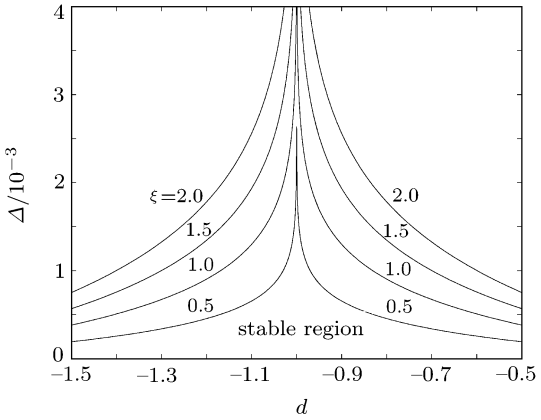


Fig.2. The boundaries of stable region with different values of ξ.

We choose $\rho = 1.1$, $\xi = 0.5$, $d = -0.98$, and the "unknown" channel time-delay $\eta = 3$s, while the initial conditions of the drive and the response systems are taken as $x_0 = (4,\ 1,\ 5)^{\mathrm{T}}$ and $\tilde{x}_0 = (3.6,\ 0.8,\ 5.4)^{\mathrm{T}}$, respectively. Hence, from Corollary 2, the estimate of bound of stable region is given by

$$0 < \Delta \leq -\frac{\ln\rho + \ln\beta^2}{13930} = 0.0011.$$

Using these parameters, conditions in Corollary 2 are satisfied for $\Delta \leq 0.0011$. Robust impulsive lag synchronization between the Lorenz system and the Chen system with impulsive distance $\Delta = 0.001$ is shown in Fig.3. As can be seen, the synchronization has been achieved practically and $\|e\|$ is smaller than ξ ($\xi = 0.5$).

Remark 9 As shown in Fig.3, under consecutive impulsive signals δ_k, after the synchronization error $e(t)$ has entered into the region $\Omega \doteq \{e \in R^n\colon \|e\| \leq \xi\}$, it is hardly possible for $e(t)$ to leave from Ω again. Therefore, in practice, we can use the designed impulsive signals δ_k uninterruptedly though the impulsive controller designed in this paper is unsupported for $\|e\| \leq \xi$.

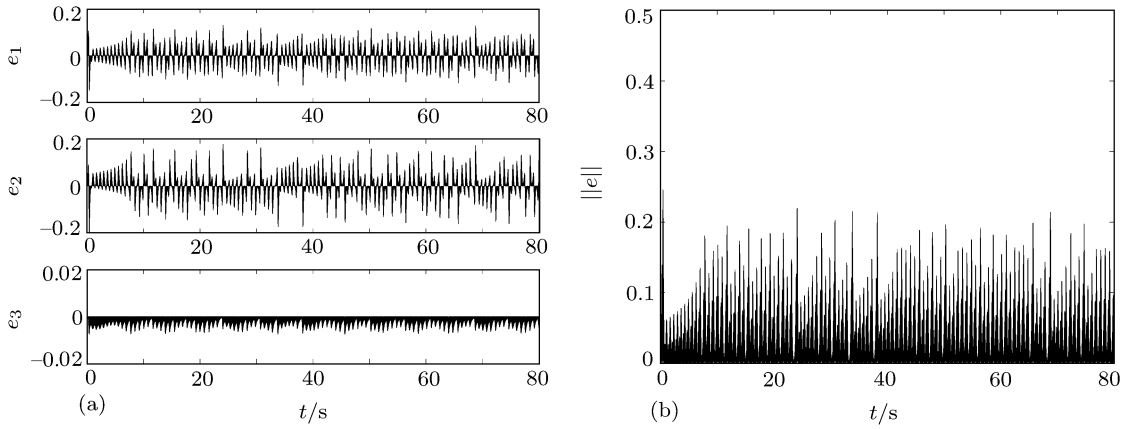


Fig.3. Simulation results for synchronization between the Lorenz system and the Chen system with $\Delta = 0.001$, where panel (a) synchronization errors and panel (b) for synchronization error magnitude $\|e\|$.

5. Conclusions

In this paper, we have investigated the problem about synchronizing different chaotic systems via impulsive control method. Robust synchronization scheme in the presence of parametric uncertainties and channel time-delay is discussed. Based on the theory of impulsive differential equations, some new and less conservative sufficient conditions are established. Finally, numerical simulations are given to demonstrate the effectiveness of the method. It should be pointed out that the method is also applicable to most of typical chaotic systems that can be described by Eqs.(1) and (2), such as Rössler system, Lü system, unified system, Liu system, several variants of Chua's circuits, etc. Owing to the similarity and analogy among the results obtained, the details are not further discussed here.

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