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Adaptive generalized functional synchronization of chaotic systems with unknown parameters*

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A universal adaptive generalized functional synchronization approach to any two different or identical chaotic systems with unknown parameters is proposed, based on a unified mathematical expression of a large class of chaotic system. Self-adaptive parameter law and control law are given in the form of a theorem. The synchronization between the three-dimensional Rössler chaotic system and the four-dimensional Chen's hyper-chaotic system is studied as an example for illustration. The computer simulation results demonstrate the feasibility of the method proposed.

Keywords: chaotic system, adaptive synchronization, generalized synchronization, functional synchronization

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1. Introduction

Synchronizing and controlling chaotic dynamical systems have recently attracted a great deal of attention since the early work on the synchronizing of chaos by Pecora and Carroll[1] and on the controlling of chaos by Ott *et al*[2] was published in 1990. Up to now, several synchronization schemes have been proposed due to their many potential applications in secure communication, biological systems, chemical reactions, information processing, social science, and many other fields.[3]

A significant result is the discovery of different synchronization phenomena,[4−27] such as complete synchronization,[4,5] anti-synchronization,[6,7] projective synchronization,[8,9] lag synchronization,[10,11] anticipated synchronization,[12,13] phase synchronization,[14,15] generalized synchronization,[16,17] and Bluetooth Glossary (Q-S) synchronization.[18,19] In the realizing of the synchronization, an adaptive controller is usually used while the chaotic systems to be synchronized are uncertain. To the best of authors' knowledge, the adaptive synchronization is used mainly for the case of complete synchronization with the identical systems, and adaptive algorithms are similar in form but different from each other in essence due to the chaotic systems involved.[28−32] For the above-mentioned other synchronization phenomena, the synchronizations are applied mainly between two identical systems with deterministic structures and parameters.[6−19] More recently, two different studies have come out. One is the study by Zhang *et al*,[33] where a unified mathematical expression of a large class of chaotic system was used and an adaptive synchronization was studied between two chaotic systems with unknown parameters but with the same dimensions, and the other is the study by Zhang *et al*[34] and Li,[35] where two different dimensional chaotic dynamical systems were generalized synchronized, but the chaotic systems studied there had the parameters that were exactly known. We are enlightened from the studies aforementioned.[33−35] In this paper, we combine the adaptive synchronization method with the generalized synchronization methods and propose a concept of adaptive generalized functional synchronization that can synchronize any two chaotic systems with the same or different dimensions and with the parameters that are unknown. The approach proposed here can be regarded as an extension of the existing complete synchronization, anti-synchronization, projective synchronization, lag synchronization anticipated synchronization, phase synchronization, and generalized synchronization, as long as we choose an appropriate function that will be introduced thereinafter.

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2. Problem statement

Considering a class of chaotic systems described as follows:

Driving system

$$\dot{x} = f(x) + F(x)\alpha, \tag{1}$$

and

Response system

$$\dot{y} = h(y) + H(y)\beta + u, \tag{2}$$

where $x \in R^m$ and $y \in R^n$ (n can be equivalent to m or not) are the state vectors of the driving system and response system respectively; f and h are $m \times 1$ and $n \times 1$ continuous functional matrices respectively; F and H are $m \times p$ and $n \times q$ continuous functional matrices respectively; $\alpha \in R^p$ and $\beta \in R^q$ are the unknown parameter vectors of the systems; $u \in R^n$ is the control input vector. In the following, we assume that (i) the element $F_{ij}(x)$ of $F(x)$ satisfies $F_{ij}(x) \in L_\infty$, (ii) the element $H_{ij}(y)$ of $H(y)$ satisfies $H_{ij}(y) \in L_\infty$, (iii) the parameter vectors of the systems, i.e. $\alpha \in R^p$ and $\beta \in R^q$, are unknown but time invariant, or slow time variant so that they can be considered as time invariant.

Definition Systems (1) and (2) are said to be adaptive generalized functional synchronization, if there exist a manifold $G(x)$: $R^m \to R^n$, a corresponding control input vector u and an adaptive law for unknown parameters such that the trajectory of the response system (2) with initial conditions y_0 and unknown parameters β can asymptotically approach the trajectory of the manifold $G(x)$ with respect to driving system (1) with initial condition x_0 and unknown parameter α, and finally the synchronization is implemented in the sense that

$$\lim_{t\to\infty} \|e(t)\| = \lim_{t\to\infty} \|y(t, y_0) - G\left(x(t, x_0)\right)\| = 0,$$

where $||\bullet||$ is the Euclidean norm.

3. Design of controller for adaptive generalized functional synchronization

Now write the manifold $G(x)$, i.e. the function with respect to system (1) that y will follow, in an n-dimensional vector,

$$G\left(x(t)\right) = \left[\begin{array}{cccc} G_1 & G_2 & \cdots & G_n \end{array}\right]^{\mathrm{T}}, \tag{3}$$

denote J_G as the Jacobian matrix of G with respect to vector x,

$$J_G = \frac{\partial G}{\partial x^{\mathrm{T}}} = \left[\begin{array}{cccc} \dfrac{\partial G_1}{\partial x_1} & \dfrac{\partial G_1}{\partial x_2} & \cdots & \dfrac{\partial G_1}{\partial x_m} \\ \dfrac{\partial G_2}{\partial x_1} & \dfrac{\partial G_2}{\partial x_2} & \cdots & \dfrac{\partial G_2}{\partial x_m} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial G_n}{\partial x_1} & \dfrac{\partial G_n}{\partial x_2} & \cdots & \dfrac{\partial G_n}{\partial x_m} \end{array}\right]_{n\times m}, \tag{4}$$

and refer to $\tilde{\alpha}$ and $\tilde{\beta}$ as the estimation of α and β respectively, thereby the errors between $\tilde{\alpha}$ and α and between $\tilde{\beta}$ and β being given respectively as

$$\Delta\alpha = \tilde{\alpha} - \alpha, \quad \text{and} \quad \Delta\beta = \tilde{\beta} - \beta,$$

then the derivative of error dynamics with respect to time will be

$$\begin{aligned} \dot{e} &= \dot{y} - \left(\frac{\partial G}{\partial t} + \frac{\partial G}{\partial x^{\mathrm{T}}}\dot{x}\right) \\ &= h(y) + H(y)\beta + u - J_G f(x) - J_G F(x)\alpha. \end{aligned} \tag{5}$$

If we choose the following candidate Lyapunov function V:

$$V(e, \Delta\alpha, \Delta\beta) = \frac{1}{2}\left(e^{\mathrm{T}}e + \Delta\alpha^{\mathrm{T}}\Delta\alpha + \Delta\beta^{\mathrm{T}}\Delta\beta\right),$$

the time derivative of V along the error dynamics (5) will be

$$\begin{aligned} &\dot{V}(e, \Delta\alpha, \Delta\beta) \\ &= \dot{e}^{\mathrm{T}}e + \Delta\alpha^{\mathrm{T}}\Delta\dot{\alpha} + \Delta\beta^{\mathrm{T}}\Delta\dot{\beta} \\ &= (h + H\beta + u - J_G f - J_G F\alpha)^{\mathrm{T}}e + \Delta\alpha^{\mathrm{T}}\Delta\dot{\alpha} + \Delta\beta^{\mathrm{T}}\Delta\dot{\beta} \\ &= [h + H\beta - J_G f - J_G F\alpha + (J_G f + J_G F\tilde{\alpha} - h - H\tilde{\beta} - Ke) - (J_G f + J_G F\tilde{\alpha} - h - H\tilde{\beta} - Ke) + u]^{\mathrm{T}}e \\ &\quad + \Delta\alpha^{\mathrm{T}}\Delta\dot{\alpha} + \Delta\beta^{\mathrm{T}}\Delta\dot{\beta} \\ &= -e^{\mathrm{T}}Ke + \Delta\alpha^{\mathrm{T}}(J_G F)^{\mathrm{T}}e - \Delta\beta^{\mathrm{T}}H^{\mathrm{T}}e + [-(J_G f + J_G F\tilde{\alpha} - h - H\tilde{\beta} - Ke) + u]^{\mathrm{T}}e + \Delta\alpha^{\mathrm{T}}\Delta\dot{\alpha} + \Delta\beta^{\mathrm{T}}\Delta\dot{\beta} \end{aligned}$$

$$= -e^{\mathrm{T}}Ke + [-(J_G f + J_G F\tilde{\alpha} - h - H\tilde{\beta} - Ke) + u]^{\mathrm{T}}e + \Delta\alpha^{\mathrm{T}}(J_G F)^{\mathrm{T}}e + \Delta\alpha^{\mathrm{T}}\dot{\tilde{\alpha}} - \Delta\beta^{\mathrm{T}}H^{\mathrm{T}}e + \Delta\beta^{\mathrm{T}}\dot{\tilde{\beta}}.$$

Now, it is clear that if we let

$$u = J_G f + J_G F\tilde{\alpha} - h - H\tilde{\beta} - Ke,$$

and

$$\dot{\tilde{\alpha}} = -(J_G F)^{\mathrm{T}}e$$
$$\dot{\tilde{\beta}} = H^{\mathrm{T}}e,$$

where $K = \mathrm{diag}\{k_1, k_2, \cdots, k_n\}$ $k_i > 0$, $i = 1, 2, \cdots, n, \mathrm{diag}(\cdot)$ is a diagonal matrix, then $\dot{V}(e, \Delta\alpha, \Delta\beta) = -e^{\mathrm{T}}Ke < 0$, if $e \neq 0$. Thus adaptive generalized functional synchronization is realized and the error dynamical system is globally and asymptotically stable.

Theorem 1 For any differentiable vector function $G(x)$, if controller u is selected as

$$u = J_G f + J_G F\tilde{\alpha} - h - H\tilde{\beta} - Ke, \tag{6}$$

and adaptive law of parameters are taken as

$$\dot{\tilde{\alpha}} = -(J_G F)^{\mathrm{T}}e,$$
$$\dot{\tilde{\beta}} = H^{\mathrm{T}}e, \tag{7}$$

then the response system (2) can synchronize the drive system (1) with respect to vector function $G(x)$ globally and asymptotically.
where $K = \mathrm{diag}\{k_1, k_2, \cdots, k_n\}$ $k_i > 0$, $i = 1, 2, \cdots, n$, $\mathrm{diag}(\cdot)$ is a diagonal matrix.

Remark 1 The control method can realize many kinds of chaotic synchronization according to the form of vector function $G(x(t))$. For instance, under the condition where the dimension of drive system n is equivalent to the dimension of response system m, we obtain that if $G(x) = x$, it is complete synchronization; if $G(x) = -x$, it is anti-synchronization; if $G(x) = ax(|a| \neq 1)$, it is projective synchronization; if $G(x) = G[x(t-\tau)]$ with $\tau > 0$, it is lag synchronization; if $G(x) = G[x(t+\tau)]$ with $\tau > 0$, it is anticipated synchronization; And if $G(x)$ is a more general mathematical function of x even the dimension of drive system n is different from the dimension of response system m, it is a more generalized synchronization.

4. Numerical simulation results

We observe the synchronization behaviour in the Rössler chaotic system used as a driving system, and the Chen's hyper-chaotic system used as a response system.

Driving system, the Rössler chaotic system,[36] is written as

$$\dot{x}_1(t) = -x_2(t) - x_3(t),$$
$$\dot{x}_2(t) = x_1(t) + \alpha_1 x_2(t),$$
$$\dot{x}_3(t) = \alpha_2 x_1(t) - \alpha_3 x_3(t) + x_1(t)x_3(t).$$

If we rewrite the system in the form of Eq.(1), we can obtain $f(x) = \begin{bmatrix} -x_2(t) - x_3(t) \\ x_1(t) \\ x_1(t)x_3(t) \end{bmatrix}$, $F(x) = \begin{bmatrix} 0 & 0 & 0 \\ x_2(t) & 0 & 0 \\ 0 & x_1(t) & -x_3(t) \end{bmatrix}$, and $\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^{\mathrm{T}}$.

Response system, the Chen's hyper-chaotic system,[37] is expressed as

$$\dot{y}_1(t) = \beta_1 (y_2(t) - y_1(t)) + y_4(t) + u_1(t),$$
$$\dot{y}_2(t) = \beta_2 y_1(t) - y_1(t)y_3(t) + \beta_3 y_2(t) + u_2(t),$$
$$\dot{y}_3(t) = y_1(t)y_2(t) - \beta_4 y_3(t) + u_3(t),$$
$$\dot{y}_4(t) = y_2(t)y_3(t) + \beta_5 y_4(t) + u_4(t).$$

If we rewrite the system in the form of Eq.(2), we can achieve $h(y) = \begin{bmatrix} y_4(t) \\ -y_1(t)y_3(t) \\ y_1(t)y_2(t) \\ y_2(t)y_3(t) \end{bmatrix}$, $H(y) = \begin{bmatrix} y_2(t) - y_1(t) & 0 & 0 & 0 & 0 \\ 0 & y_1(t) & y_2(t) & 0 & 0 \\ 0 & 0 & 0 & -y_3(t) & 0 \\ 0 & 0 & 0 & 0 & y_4(t) \end{bmatrix}$, and $\beta = \begin{bmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \end{bmatrix}^{\mathrm{T}}$. As an illustrative example, we assume the manifold $G(x)$ to be as follows:

$$G(x) = \begin{bmatrix} x_1(t-1) \\ 0.5x_2(t-1) \\ 2x_3(t-1) \\ \sin x_1(t-1) + \cos(x_2(t-1) + x_3(t-1)) \end{bmatrix},$$

that is to say, the synchronization error $e(t) = [e_1(t)e_2(t)e_3(t)e_4(t)]^{\mathrm{T}}$ can be written as

$$
\begin{aligned}
e_1(t) &= y_1(t) - x_1(t-1),\\
e_2(t) &= y_2(t) - 0.5x_2(t-1),\\
e_3(t) &= y_3(t) - 2x_3(t-1),
\end{aligned}
$$

and

$$
\begin{aligned}
e_4(t) = {} & y_4(t) - \sin x_1(t-1) - \cos(x_2(t-1)\\
& + x_3(t-1)).
\end{aligned}
$$

According to Eq.(5), we have

$$
J_G = \begin{bmatrix}
1 & 0 & 0\\
0 & 0.5 & 0\\
0 & 0 & 2\\
\cos x_1(t-1) & -\sin\left(x_2(t-1)+x_3(t-1)\right) & -\sin\left(x_2(t-1)+x_3(t-1)\right)
\end{bmatrix}.
$$

For simplicity, in the following procedure, x_1, x_2 and x_3 represent $x_1(t)$, $x_2(t)$ and $x_3(t)$ respectively; y_1, y_2, y_3 and y_4 represent $y_1(t)$, $y_2(t)$, $y_3(t)$ and $y_4(t)$ respectively; e_1, e_2, e_3 and e_4 represent $e_1(t)$, $e_2(t)$, $e_3(t)$ and $e_4(t)$ respectively and f, F, h and H represent $f(x)$, $F(x)$, $h(y)$ and $H(y)$ respectively.

According to Eq.(6), we have a control law as follows:

$$
u = J_G f + J_G F\tilde{\alpha} - h - H\tilde{\beta} - Ke.
$$

Each element of u can be written as

$$
\begin{aligned}
u_1 &= -x_2(t-1) - x_3(t-1) - y_4 - (y_2 - y_1)\tilde{\beta}_1 - k_1 e_1,\\
u_2 &= 0.5x_1(t-1) + 0.5x_2(t-1)\tilde{\alpha}_1 + y_1 y_3 - y_1\tilde{\beta}_2 - y_2\tilde{\beta}_3 - k_2 e_2,\\
u_3 &= 2x_1(t-1)x_3(t-1) + 2x_1(t-1)\tilde{\alpha}_2 - 2x_3(t-1)\tilde{\alpha}_3 - y_1 y_2 + y_3\tilde{\beta}_4 - k_3 e_3,
\end{aligned}
$$

and

$$
\begin{aligned}
u_4 = {} & -\cos x_1(t-1)\cdot\left(x_2(t-1)+x_3(t-1)\right) - \sin\left(x_2(t-1)+x_3(t-1)\right)\cdot[x_1(t-1) + x_1(t-1)x_3(t-1)\\
& + x_2(t-d)\tilde{\alpha}_1 + x_1(t-d)\tilde{\alpha}_2 - x_3(t-d)\tilde{\alpha}_3] - y_2 y_3 - y_4\tilde{\beta}_5 - k_4 e_4,
\end{aligned}
$$

and according to Eq.(7), we have an adaptive regulating law of unknown parameters as follows:

$$
\dot{\tilde{\alpha}} = -(J_G F)^{\mathrm{T}} e = \begin{bmatrix}
-0.5x_2(t-1)e_2 + x_2(t-1)e_4\sin\left(x_2(t-1)+x_3(t-1)\right)\\
-2x_1(t-1)e_3 + x_1(t-1)e_4\sin\left(x_2(t-1)+x_3(t-1)\right)\\
2x_3(t-1)e_3 - x_3(t-1)e_4\sin\left(x_2(t-1)+x_3(t-1)\right)
\end{bmatrix},
$$

and

$$
\dot{\tilde{\beta}} = H^{\mathrm{T}} e = \begin{bmatrix}
(y_2 - y_1)e_1\\
y_1 e_2\\
y_2 e_2\\
-y_3 e_3\\
y_4 e_4
\end{bmatrix}.
$$

In the simulation, the real system parameters are $\alpha_1 = 0.34$, $\alpha_2 = 0.4$, and $\alpha_3 = 4.5$ for the Rössler chaotic system; $\beta_1 = 35$, $\beta_2 = 7$, $\beta_3 = 12$, $\beta_4 = 3$, and $\beta_5 = 0.4$ for the Chen's hyper-chaotic system; the initial values of estimation of real parameters are $\tilde{\alpha}_1 = 0.5$, $\tilde{\alpha}_2 = 0.6$, $\tilde{\alpha}_3 = 4$, $\tilde{\beta}_1 = 30$, $\tilde{\beta}_2 = 5$, $\tilde{\beta}_3 = 10$, $\tilde{\beta}_4 = 5$, and $\tilde{\beta}_5 = 0.6$ and the coefficients k_i in Theorem 1 are chosen as $k_i = 10$, $i = 1, 2, 3, 4$. Synchronization errors e_1, e_2, e_3 and e_4 are shown in Fig.1; adaptive parameters $\tilde{\alpha}$ and $\tilde{\beta}$ are shown in Figs.2 and 3 respectively.

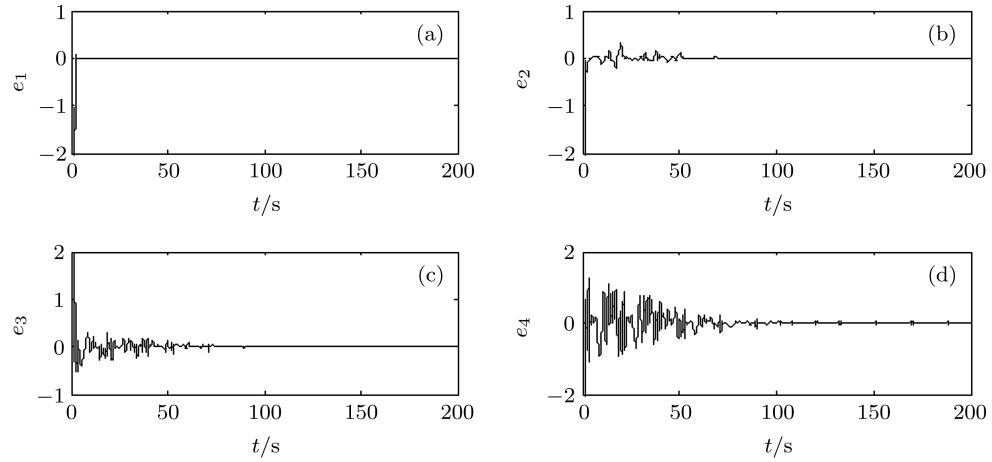


Fig.1. Synchronization errors: (a) error e_1; (b) error e_2; (c) error e_3; (d) error e_4.

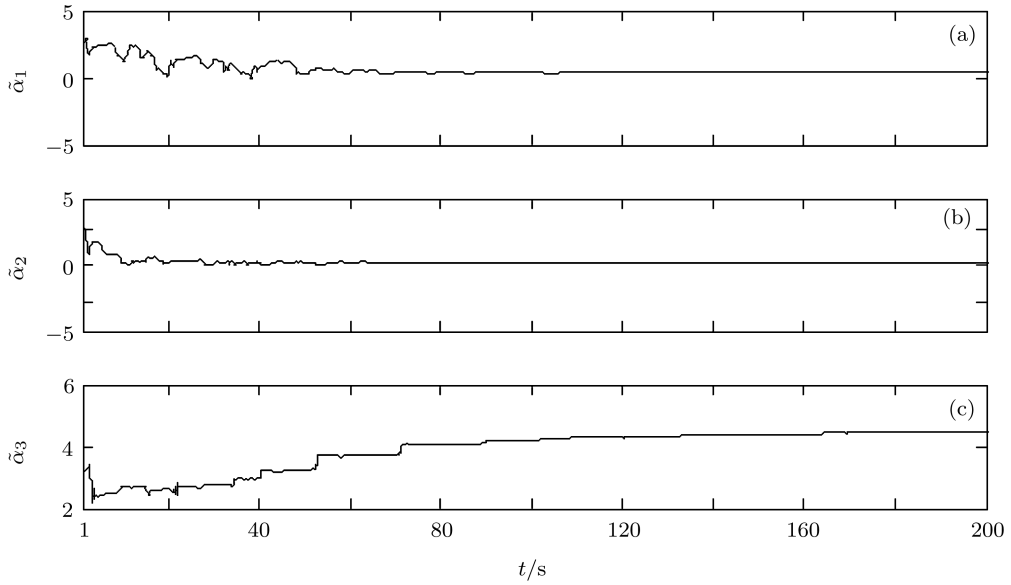
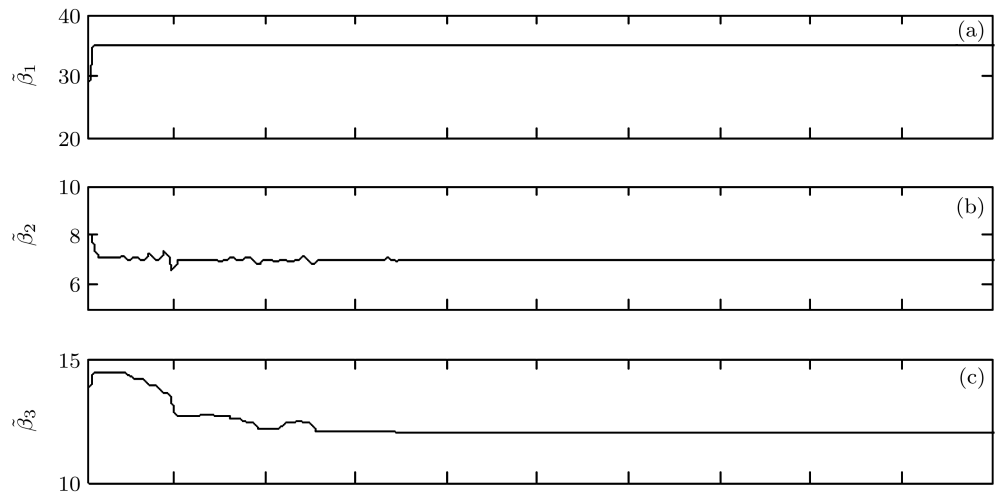


Fig.2. Adaptive parameters: (a) $\tilde{\alpha}_1$; (b) $\tilde{\alpha}_2$; (c) $\tilde{\alpha}_3$.



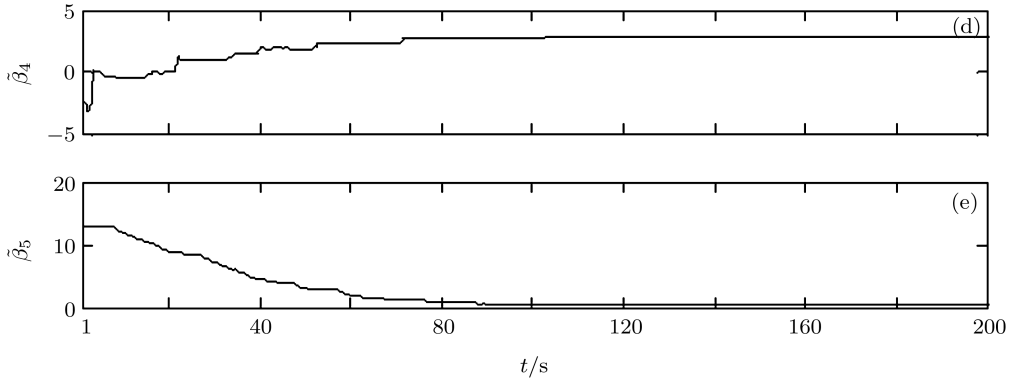


Fig.3. Adaptive parameters: (a) $\tilde{\beta}_1$; (b) $\tilde{\beta}_2$; (c) $\tilde{\beta}_3$; (d) $\tilde{\beta}_4$; (e) $\tilde{\beta}_5$.

5. Conclusions

The concept of and the approach to adaptive generalized functional synchronization are proposed in this paper. The synchronization controller and the adaptive regulating law of unknown parameters are derived by using the Lyapunov stability theory. The existent several synchronization phenomena are its special examples. An experimental example of synchronization between the three-dimensional Rössler chaotic system and the four-dimensional Chen's hyper-chaotic system with unknown parameters is studied. And it should be pointed out that the proposed adaptive generalized functional synchronization approach is very easy to extend to the generalized Q-S synchronization in the adaptive case.

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