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Inverse scattering method and soliton solution family for the Einstein–Maxwell theory with multiple Abelian gauge fields*

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A Hauser–Ernst-type extended hyperbolic complex linear system given in our previous paper [Gao Y J 2004 *Chin. Phys.* **13** 602] is slightly modified and used to develop a new inverse scattering method for the stationary axisymmetric Einstein–Maxwell theory with multiple Abelian gauge fields. The reduction procedures in this inverse scattering method are found to be fairly simple, which makes the inverse scattering method be fine and effective in practical application. As an example, a concrete family of soliton solutions for the considered theory is obtained.

Keywords: Einstein–Maxwell theory with multiple Abelian gauge fields, inverse scattering method, soliton solution family

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1. Introduction

Owing to their great importance in theoretical and mathematical physics, the stationary axisymmetric (SAS) Einstein–Maxwell (EM) equations with multiple (say p) Abelian gauge fields (SAS EM-p equations, for short) attracted much attention in the past years, e.g. in Refs.[1–19]. However, the complications arising from the existence of Maxwell fields make the study of the field equations more difficult and many methods for studying the reduced vacuum gravity theory (i.e. EM-0 theory) are no longer applicable. Thus, deeper researches and further extended studying methods are needed.

Undoubtedly, exact solutions are particularly valuable for understanding the related theories. In the case of vacuum gravity, Belinsky and Zakharov[20] developed an inverse scattering method (ISM) for solving the two-dimensional (2-dim) vacuum gravity field equations. For the reduced Einstein–Maxwell theory (i.e. EM-1 theory), some ISMs were also proposed.[9,11,14] However, more general application of these schemes to the reduced EM-p for $p > 1$ still remains a problem. The difficulties have to do with the complicated mathematical structures, such as the fundamental field matrices in the reduced EM-p theories in general have dimensions greater than two and, at the same time, must satisfy some non-trivial Riemann symmetric space (or coset) conditions,[9,10,18] etc. (As will be seen in the following, the fundamental field matrix in the reduced EM-p theory belongs to the symmetric space $SU(p+1,1)/(SU(p+1)\times U(1))$). Therefore, it would be worthwhile to develop some more effective ISM and obtain more general new exact (soliton) solutions of the EM-p theories.

In the present paper, we develop a new ISM for the SAS EM-p theory. This ISM is different from the above-mentioned Belinsky–Zakharov-like ones. In Refs.[18, 19] we have developed and used a so-called extended hyperbolic complex (EHC) function method and found that all of the SAS EM-p theories for $p = 0, 1, 2, \ldots$ can be studied in a unified form (as special cases, $p = 0$ corresponds to the vacuum gravity; $p = 1$, the EM theory). Moreover, for each p, a pair of Hauser–Ernst (HE)-type EHC linear systems were established for constructing infinitesimal[18] and finite[19] symmetry transformations of the EM-p theory. In this paper we show that, with some minor modifications, one of these linear systems is also suitable for the development of ISM in the SAS EM-p theories. These demonstrate that the applicable range of the HE-type linear system is larger than previously expected. As is well known, for an ISM in solving

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nonlinear equations, the so-called reduction problem is very important. However, the reduction procedures vary with the equations and the associated linear systems, and in general are very difficult; so far there is no generally applicable method. In this paper we unexpectedly find that, based on the modified HE-type EHC linear system, the reduction procedure for the ISM given here is fairly simple.

In Section 2, some related concepts and notations of the so-called EHC function and the EHC field equations for the SAS EM-p theories[18] are briefly recalled. In Section 3, a modified HE-type EHC linear system for the theories studied is established. In Section 4, an EHC dressing transformation is introduced and the associated theorem is proved. In Section 5, an ISM for the EM-p theory is presented. As an example, a family of soliton solutions for the SAS EM-p theory is explicitly constructed in Section 6. Finally, Section 7 gives a summary.

2. EHC function and EM-p field equations

For later use, here we briefly recall some related concepts and notations of the EHC function and EHC matrix field equations of the EM-p theories.[18]

2.1. EHC function[18]

Let ε be the extended hyperbolic imaginary unit, i.e. $\varepsilon^2 = 1$, $\varepsilon \neq \pm 1$. When a and b are ordinary complex numbers, we call $c(\varepsilon) = a + \varepsilon b$ an EHC number and denote $\mathrm{Re_E}(c(\varepsilon)) := a$, $\mathrm{Im_E}(c(\varepsilon)) := b$. All EHC numbers with usual addition and multiplication constitute a commutative ring. Corresponding to ε and the ordinary imaginary unit i ($\mathrm{i}^2 = -1$) in this ring, we have two complex conjugations:

$$c(\varepsilon)^\star := a - \varepsilon b, \quad \overline{c(\varepsilon)} := \bar{a} + \varepsilon \bar{b}, \tag{1}$$

where $\overline{a}$ and $\overline{b}$ are ordinary complex conjugations of a and b. If a and b are ordinary complex functions of some ordinary complex variables $z_1, \ldots, z_n$, then $c(z_1, \ldots, z_n; \varepsilon) = a(z_1, \ldots, z_n) + \varepsilon b(z_1, \ldots, z_n)$ is called an EHC function. We say that $c(z_1, \ldots, z_n; \varepsilon)$ is continuous, analytical, etc. if $a(z_1, \ldots, z_n)$ and $b(z_1, \ldots, z_n)$ both, as ordinary complex functions, have the same properties. We also need EHC (function) matrices, and for an EHC matrix $\boldsymbol{W}$, we define

$$\boldsymbol{W}^\dagger := \overline{\boldsymbol{W}}^\top, \quad \boldsymbol{W}^\ddagger := (\boldsymbol{W}^\dagger)^\star, \tag{2}$$

here '$\top$' denotes the transposition.

2.2. EHC matrix form of the SAS EM-p field equations

When there exist p (nonnegative integer) Maxwell fields A_μ^a ($a = 1, 2, \ldots, p$) in space-time, the essentially complete set of SAS EM-p field equations can be written as[18]

$$\mathrm{d}(\rho^{-1}\boldsymbol{M}\gamma^*\mathrm{d}\boldsymbol{M}) = 0, \tag{3}$$

where the $(p+2) \times (p+2)$ ordinary complex matrix function $\boldsymbol{M} = \boldsymbol{M}(x, y)$ is defined as

$$\boldsymbol{M} = \frac{\rho}{f}\begin{pmatrix} 1 & \sqrt{2}\bar{\Phi}^1 & \sqrt{2}\bar{\Phi}^2 & \ldots & \sqrt{2}\bar{\Phi}^p & \mathrm{i}(f-\mathcal{E}) \\ \sqrt{2}\Phi^1 & 2\Phi^1\bar{\Phi}^1 - f & 2\Phi^1\bar{\Phi}^2 & \ldots & 2\Phi^1\bar{\Phi}^p & -\mathrm{i}\sqrt{2}\Phi^1\bar{\mathcal{E}} \\ \sqrt{2}\Phi^2 & 2\Phi^2\bar{\Phi}^1 & 2\Phi^2\bar{\Phi}^2 - f & \ldots & 2\Phi^2\bar{\Phi}^p & -\mathrm{i}\sqrt{2}\Phi^2\bar{\mathcal{E}} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \sqrt{2}\Phi^p & 2\Phi^p\bar{\Phi}^1 & 2\Phi^p\bar{\Phi}^2 & \ldots & 2\Phi^p\bar{\Phi}^p - f & -\mathrm{i}\sqrt{2}\Phi^p\bar{\mathcal{E}} \\ \mathrm{i}(\bar{\mathcal{E}} - f) & \mathrm{i}\sqrt{2}\mathcal{E}\bar{\Phi}^1 & \mathrm{i}\sqrt{2}\mathcal{E}\bar{\Phi}^2 & \ldots & \sqrt{2}\mathcal{E}\bar{\Phi}^p & \mathcal{E}\bar{\mathcal{E}} \end{pmatrix} \tag{4}$$

and satisfies the conditions

$$\boldsymbol{M}^\dagger = \boldsymbol{M}, \tag{5a}$$

$$\boldsymbol{M}\gamma\boldsymbol{M} = \rho^2\boldsymbol{\gamma}, \tag{5b}$$

$$\boldsymbol{\gamma} := \begin{pmatrix} 0 & 0 & 1 \\ 0 & \mathrm{i}I_p & 0 \\ -1 & 0 & 0 \end{pmatrix}. \tag{5c}$$

In Eqs.(3)–(5), $\mathcal{E}$ is an ordinary complex Ernst potential and Φ^a $(a = 1, 2, \ldots, p)$ are ordinary complex (multiple) Ernst electromagnetic potentials; I_p and I are, respectively, the p-dim (dimensional) and $(p+2)$-dim unit matrices; the real function $\rho = \rho(x, y) > 0$ satisfies $\mathrm{d}{}^*\mathrm{d}\rho = 0$ and "*'' is the dual operation in 2-dim Euclidian space $\{x, y\}$. From Eqs.(5a,b,c), we can see that M essentially belongs to the symmetric space $SU(p+1,1)/(SU(p+1)\times U(1))$.

3. Modified HE-type EHC linear system

Equation (3) implies that we can introduce a $(p+2)\times(p+2)$ matrix twist potential $Q(x, y)$ by $\mathrm{d}Q = -\rho^{-1}\boldsymbol{M}\gamma\,{}^*\mathrm{d}\boldsymbol{M}$, then from Eq.(5) and the harmony of $\rho(x, y)$ we can obtain $Q - Q^\dagger = -2z\gamma$ with the real field $z = z(x, y)$ introduced by ${}^*\mathrm{d}\boldsymbol{\rho} = \mathrm{d}\boldsymbol{z}$. Thus, if we define an EHC matrix $\boldsymbol{H}$-potential as

$$\boldsymbol{H} := \boldsymbol{M} + \varepsilon\boldsymbol{Q}, \tag{6}$$

and denote $\boldsymbol{\Omega} := \varepsilon\boldsymbol{\gamma}$, then the equations about Q and M can be written together as

$$2(z+\rho^*)\mathrm{d}\boldsymbol{H} = (\boldsymbol{H} + \boldsymbol{H}^\ddagger)\boldsymbol{\Omega}\mathrm{d}\boldsymbol{H}. \tag{7}$$

Now we introduce an ordinary complex parameter $\boldsymbol{w}$ and define

$$\boldsymbol{A}(\boldsymbol{w}) := \boldsymbol{w} - (\boldsymbol{H} + \boldsymbol{H}^\ddagger)\boldsymbol{\Omega}, \tag{8}$$

$$\boldsymbol{\Gamma}(\boldsymbol{w}) := \boldsymbol{\Lambda}(\boldsymbol{w})^{-1}\mathrm{d}\boldsymbol{H}, \tag{9}$$

$$\boldsymbol{\Lambda}(\boldsymbol{w}) := \boldsymbol{w} - 2(z+\rho^*),$$

$$\boldsymbol{\Lambda}(\boldsymbol{w})^{-1} = \boldsymbol{\lambda}(\boldsymbol{w})^{-2}[\boldsymbol{w} - 2(\boldsymbol{z} - \boldsymbol{\rho}^*)], \tag{10}$$

$$\boldsymbol{\lambda}(\boldsymbol{w}) := [(\boldsymbol{w} - 2z)^2 + (2\rho)^2]^{1/2},$$

$$\boldsymbol{\lambda}(\boldsymbol{w})\big|_{\boldsymbol{w}\to\infty} = \boldsymbol{w}, \tag{11}$$

then from Eqs.(7), (8) and (9) we can obtain

$$\mathrm{d}\boldsymbol{\Gamma}(\boldsymbol{w}) = \boldsymbol{\Gamma}(\boldsymbol{w})\boldsymbol{\Omega}\wedge\boldsymbol{\Gamma}(\boldsymbol{w}), \tag{12}$$

where "$\wedge$" denotes the exterior product of differential forms. Equation (12) is just the complete compatibility condition of the following EHC linear differential equation:

$$\mathrm{d}\boldsymbol{F}(w) = \boldsymbol{\Gamma}(w)\boldsymbol{\Omega}\boldsymbol{F}(\boldsymbol{w}), \tag{13}$$

where $\boldsymbol{F}(w) = \boldsymbol{F}(x, y, w; \varepsilon)$ is a $(p+2)\times(p+2)$ matrix EHC function of x, y and w. Moreover, from the definitions and properties of $\boldsymbol{\rho}$ and $\boldsymbol{z}$, without loss of generality, we shall redefine coordinates by setting $\boldsymbol{x} = \rho$, $y = z$ for simplicity in the following.

Equation (13) does not define $\boldsymbol{F}(w)$ uniquely, hence we shall impose some subsidiary conditions that are consistent with the above equations and the requirement that $\boldsymbol{F}(w)$ be holomorphic in a neighbourhood of $\boldsymbol{w} = \infty$. To this end, we note that if $\boldsymbol{F}(w)$ is expanded in power of $\boldsymbol{w}^{-1}$ around $\boldsymbol{w} = \infty$ as

$$\boldsymbol{F}(\boldsymbol{w}) = \boldsymbol{F}^{(0)}(\rho, z; \varepsilon) + w^{-1}\boldsymbol{F}^{(1)}(\rho, z; \varepsilon) + \ldots, \tag{14}$$

then from Eqs.(7), (13) and the relation $2\boldsymbol{\Lambda}^{-1}(w)\mathrm{d}z = -\boldsymbol{\lambda}(\boldsymbol{w})^{-1}\mathrm{d}\boldsymbol{\lambda}(w)$ we have

$$\mathrm{d}\boldsymbol{F}^{(0)} = \mathrm{d}[\boldsymbol{F}^{(1)} - \boldsymbol{H}\boldsymbol{\Omega}\boldsymbol{F}^{(0)}] = 0,$$

$$\mathrm{d}[\boldsymbol{\lambda}(w)\boldsymbol{F}(w)^+\boldsymbol{\Omega}\boldsymbol{F}(w)] = 0,$$

$$\mathrm{d}[\boldsymbol{F}(w)^\times\boldsymbol{\Omega}\boldsymbol{A}(w)\boldsymbol{F}(w)] = 0,$$

where $\boldsymbol{F}(w)^+ := \boldsymbol{F}(\bar{w})^\dagger$, $\boldsymbol{F}(w)^\times := \boldsymbol{F}(\bar{w})^\ddagger$. These equations determine $\boldsymbol{F}(w)$ up to right-multiplication by an arbitrary non-degenerate $(p+2)\times(p+2)$ matrix function of w, so we use this freedom and choose the integral constants consistently such that

$$\boldsymbol{F}^{(0)} = \boldsymbol{I}, \tag{15a}$$

$$\boldsymbol{F}^{(1)} = \boldsymbol{H}\boldsymbol{\Omega}, \tag{15b}$$

$$\boldsymbol{\lambda}(w)\boldsymbol{F}(w)^+\boldsymbol{\Omega}\boldsymbol{F}(w) = w\boldsymbol{\Omega}, \tag{16a}$$

$$\boldsymbol{F}(w)^\times\boldsymbol{\Omega}\boldsymbol{A}(w)\boldsymbol{F}(w) = w\boldsymbol{\Omega}. \tag{16b}$$

Equations (13)–(16) are essentially the HE-type EHC linear system for $\boldsymbol{F}(t)$ in Ref.[18] with t being replaced by w^{-1}. Besides, from definition (8) we see that

$$\boldsymbol{A}(w) \text{ is a linear function of } w. \tag{17}$$

We call Eqs.(13)–(17) a modified HE-type EHC linear system for the EM-p theories.

Now we give some other useful relations. Introducing ordinary complex coordinates $\eta := z + \mathrm{i}\rho$, $\xi := z - \mathrm{i}\rho$, we can rewrite Eq.(7) as

$$[2\eta - (H + H^\ddagger)\boldsymbol{\Omega}]\partial_\eta H = 0, \tag{18a}$$

$$[2\xi - (H + H^\ddagger)\boldsymbol{\Omega}]\partial_\xi H = 0, \tag{18b}$$

From the definition of $\boldsymbol{A}(w)$ in Eq.(8), Eqs.(18) may be written as

$$\boldsymbol{A}(w = 2\eta)\partial_\eta\boldsymbol{H} = 0, \tag{19a}$$

$$\boldsymbol{A}(w = 2\xi)\partial_\xi\boldsymbol{H} = 0. \tag{19b}$$

Moreover, for any $\boldsymbol{A}(w)$ satisfying Eqs.(16a) and 16(b), we have

$$\boldsymbol{A}(w)^+\Omega\boldsymbol{A}(w) = \boldsymbol{\lambda}^2(w)\boldsymbol{\Omega}. \tag{20}$$

4. Dressing transformation

Assuming that we have known a solution $\boldsymbol{M}_0$ (seed solution) of Eqs.(3) and (5), then we can obtain the corresponding $\boldsymbol{H}_0$, $\boldsymbol{A}_0(w)$ and solution $\boldsymbol{F}_0(w)$ of Eqs.(13)–(16). Now we take a dressing transformation

$$\boldsymbol{F}(w) = \boldsymbol{\chi}(w)\boldsymbol{F}_0(w), \tag{21}$$

where $\boldsymbol{\chi}(w) = \boldsymbol{\chi}(\rho, z, w; \varepsilon)$ is a $(p+2)\times(p+2)$ matrix EHC function of ρ, z and w. The requirement that $\boldsymbol{F}(w)$ be also a solution of Eqs.(13)–(16) (for some $\boldsymbol{H}$ and the associated $\boldsymbol{A}(w)$) implies

$$\boldsymbol{\chi}(\infty) = \boldsymbol{I}, \tag{22}$$

$$\mathrm{d}\chi(w) = \boldsymbol{\Lambda}^{-1}(w)[\mathrm{d}\boldsymbol{H}\boldsymbol{\Omega}\boldsymbol{\chi}(w) - \boldsymbol{\chi}(w)\mathrm{d}\boldsymbol{H}_0\boldsymbol{\Omega}], \tag{23}$$

$$\boldsymbol{\chi}^{+}(\boldsymbol{w})\boldsymbol{\Omega}\boldsymbol{\chi}(\boldsymbol{w}) = \boldsymbol{\Omega}, \tag{24a}$$

$$\boldsymbol{\chi}^{\times}(\boldsymbol{w})\boldsymbol{\Omega}\boldsymbol{A}(\boldsymbol{w})\boldsymbol{\chi}(\boldsymbol{w}) = \boldsymbol{\Omega}\boldsymbol{A}_0(\boldsymbol{w}). \tag{24b}$$

Expressing the expansion of $\boldsymbol{\chi}(w)$ in the neighbourhood of $w = \infty$ as

$$\boldsymbol{\chi}(w) = \boldsymbol{I} + w^{-1}\boldsymbol{\chi}^{(1)}(\rho, z; \varepsilon) + \ldots, \tag{25}$$

then we have

Theorem 1 If $\boldsymbol{H}_0$ is an $\boldsymbol{H}$-potential for some known $\boldsymbol{M}_0$ and $\boldsymbol{\chi}(\boldsymbol{w})$ satisfies Eqs.(22)–(24) with $\boldsymbol{A}(\boldsymbol{w})$ fulfilling Eq.(17), then

$$\boldsymbol{H} := \boldsymbol{H}_0 - \boldsymbol{\chi}^{(1)}\boldsymbol{\Omega} \tag{26}$$

is an EHC $\boldsymbol{H}$-potential of the SAS EM-p theory, i.e. $\boldsymbol{M} = \mathrm{Re}_{\mathrm{E}}(\boldsymbol{H})$ is a solution of Eq.(3) under conditions (5a,b). **Proof** Firstly, Eqs.(24b), (25) and (17) give

$$\boldsymbol{A}(w) = \boldsymbol{w} - (\boldsymbol{H} + \boldsymbol{H}^{\ddagger})\boldsymbol{\Omega} \tag{27}$$

with $\boldsymbol{H}$ defined by Eq.(26). Moreover, in terms of coordinates η and ξ, Eq.(23) can be written as

$$\begin{aligned}\partial_\eta \boldsymbol{H}\boldsymbol{\Omega} =\ & \boldsymbol{\chi}(\boldsymbol{w})\partial_\eta \boldsymbol{H}_0\boldsymbol{\Omega}\boldsymbol{\chi}^{-1}(\boldsymbol{w}) \\ & -(\boldsymbol{w} - 2\eta)\partial_\eta\boldsymbol{\chi}(\boldsymbol{w})\chi^{-1}(\boldsymbol{w}),\\ \partial_\xi \boldsymbol{H}\boldsymbol{\Omega} =\ & \boldsymbol{\chi}(\boldsymbol{w})\partial_\xi \boldsymbol{H}_0\boldsymbol{\Omega}\boldsymbol{\chi}^{-1}(\boldsymbol{w}) \\ & -(\boldsymbol{w} - 2\xi)\partial_\xi\boldsymbol{\chi}(\boldsymbol{w})\chi^{-1}(\boldsymbol{w}).\end{aligned}$$

Thus, if $\boldsymbol{\chi}(\boldsymbol{w})$ and $\boldsymbol{\chi}^{-1}(\boldsymbol{w})$ are not singular at $\boldsymbol{w} = 2\boldsymbol{\eta}$ and $\boldsymbol{w} = 2\boldsymbol{\xi}$ (for the soliton transformation in the following these conditions are automatically fulfilled), the above two equations give

$$\partial_\eta \boldsymbol{H}\boldsymbol{\Omega} = \boldsymbol{\chi}(\boldsymbol{w} = 2\boldsymbol{\eta})\partial_\eta \boldsymbol{H}_0\boldsymbol{\Omega}\boldsymbol{\chi}^{-1}(\boldsymbol{w} = 2\boldsymbol{\eta}), \tag{28a}$$

$$\partial_\xi \boldsymbol{H}\boldsymbol{\Omega} = \boldsymbol{\chi}(\boldsymbol{w} = 2\boldsymbol{\xi})\partial_\xi \boldsymbol{H}_0\boldsymbol{\Omega}\boldsymbol{\chi}^{-1}(\boldsymbol{w} = 2\boldsymbol{\xi}). \tag{28b}$$

Since $\boldsymbol{H}_0$, $\boldsymbol{A}_0(\boldsymbol{w})$ satisfy Eqs.(19), from Eqs.(24b), (28) and (19) (for $\boldsymbol{H}_0$, $\boldsymbol{A}_0(w)$) we have

$$\boldsymbol{A}(\boldsymbol{w} = 2\boldsymbol{\eta})\partial_\eta \boldsymbol{H} = 0, \tag{29a}$$

$$\boldsymbol{A}(\boldsymbol{w} = 2\boldsymbol{\xi})\partial_\xi \boldsymbol{H} = 0. \tag{29b}$$

These indicate that $\boldsymbol{H}$ given by Eq.(26) satisfies Eq.(7).

Now, Eqs.(23) and (24a) give

$$\begin{aligned}\mathrm{d}\boldsymbol{H} =\ & -\Lambda(\boldsymbol{w})\mathrm{d}\boldsymbol{\chi}(\boldsymbol{w})\boldsymbol{\Omega}\boldsymbol{\chi}^{+}(\boldsymbol{w}) \\ & +\boldsymbol{\chi}(\boldsymbol{w})\mathrm{d}\boldsymbol{H}_0\boldsymbol{\chi}^{+}(\boldsymbol{w}).\end{aligned} \tag{30}$$

Since for a known $\boldsymbol{M}_0$ we have $H_0 - H_0^{\dagger} = -2z\Omega$ by definitions of H_0 and z, Eq.(30) gives

$$\begin{aligned}\mathrm{d}\boldsymbol{H} - \mathrm{d}\boldsymbol{H}^{\dagger} =\ & -\boldsymbol{\Lambda}(\boldsymbol{w})\mathrm{d}[\chi(w)\boldsymbol{\Omega}\boldsymbol{\chi}^{+}(\boldsymbol{w})] \\ & -2\mathrm{d}z\boldsymbol{\chi}(\boldsymbol{w})\boldsymbol{\Omega}\boldsymbol{\chi}^{+}(\boldsymbol{w}),\end{aligned} \tag{31}$$

where we have noted that $\boldsymbol{H}^{+} = \boldsymbol{H}^{\dagger}$; thus using Eq.(24a) again, we obtain

$$H - \boldsymbol{H}^{\dagger} = -2z\boldsymbol{\Omega} \tag{32}$$

by choosing some suitable integral constant. Equation (32) immediately implies that $\boldsymbol{M} := \mathrm{Re}_E(\boldsymbol{H})$ satisfies Eq.(5a). Moreover, by Eqs.(24), (21) and (16) (for $\boldsymbol{F}_0(\boldsymbol{w})$ and $\boldsymbol{A}_0(\boldsymbol{w})$), it follows that $\boldsymbol{A}(\boldsymbol{w})$ satisfies Eq.(20), then from Eqs.(27) and (32) we see that $\boldsymbol{M} := \mathrm{Re}_{\mathrm{E}}(\boldsymbol{H})$ satisfies condition (5b). Finally, under conditions (5a,b), Eq.(7) implies Eq.(3).

5. Inverse scattering method

Based on the modified HE-type EHC linear system and dressing transformation given above, we can develop a new ISM for the EM-p theories and find that the reduction procedure by this ISM is fairly simple.

To construct pure N-soliton solutions of the SAS EM-p field equations, we take the following ansatz for $\boldsymbol{\chi}(\boldsymbol{w})$ and $\boldsymbol{\chi}^{-1}(\boldsymbol{w})$:

$$\boldsymbol{\chi}(\boldsymbol{w}) = \boldsymbol{I} + \sum_{k=1}^{N}\frac{\boldsymbol{R}_k}{\boldsymbol{w} - \boldsymbol{\mu}_k}, \tag{33a}$$

$$\boldsymbol{\chi}^{-1}(\boldsymbol{w}) = \boldsymbol{I} + \sum_{k=1}^{N}\frac{\boldsymbol{S}_k}{\boldsymbol{w} - \boldsymbol{\nu}_k}, \tag{33b}$$

where $\boldsymbol{R}_k = \boldsymbol{R}_k(\rho, z; \varepsilon)$ and $\boldsymbol{S}_k = \boldsymbol{S}_k(\rho, z; \varepsilon)$ are $(p+2)\times(p+2)$ matrix EHC functions, the poles μ_k and ν_k are ordinary complex, and they are all independent of w. The relations $\boldsymbol{\chi}(\boldsymbol{w})\boldsymbol{\chi}^{-1}(\boldsymbol{w}) = \boldsymbol{\chi}^{-1}(\boldsymbol{w})\boldsymbol{\chi}(\boldsymbol{w}) = \boldsymbol{I}$ imply

$$\begin{aligned} &\boldsymbol{R}_k\boldsymbol{\chi}^{-1}(\boldsymbol{\mu}_k) = \boldsymbol{\chi}^{-1}(\boldsymbol{\mu}_k)\boldsymbol{R}_k = 0, \\ &\boldsymbol{S}_k\boldsymbol{\chi}(\boldsymbol{\nu}_k) = \boldsymbol{\chi}(\boldsymbol{\nu}_k)\boldsymbol{S}_k = 0, \end{aligned} \tag{34}$$

thus $\boldsymbol{R}_k$ and $\boldsymbol{S}_k$ are singular matrices. In the following, we take R_k and S_k to be of rank 1[1)] and write them as

$$\begin{aligned} &\boldsymbol{R}_k = \boldsymbol{m}_k\boldsymbol{n}_k, \qquad \boldsymbol{S}_k = \boldsymbol{p}_k\boldsymbol{q}_k, \\ &k = 1, \quad \text{(no sum in } k), \ldots, N, \end{aligned} \tag{35}$$

where $\boldsymbol{m}_k = \boldsymbol{m}_k(\rho, z; \varepsilon)$ and $\boldsymbol{p}_k = \boldsymbol{p}_k(\rho, z; \varepsilon)$ are $(p+2)$-dim column vectors, $\boldsymbol{n}_k = \boldsymbol{n}_k(\rho, z; \varepsilon)$ and $\boldsymbol{q}_k = \boldsymbol{q}_k(\rho, z; \varepsilon)$ are $(p+2)$-dim row vectors. From Eqs.(33) and (35), Eqs.(34) give

$$\begin{aligned} &\boldsymbol{n}_k + \sum_{l=1}^{N} \frac{\boldsymbol{n}_k\boldsymbol{p}_l}{\boldsymbol{\mu}_k - \boldsymbol{\nu}_l}\boldsymbol{q}_l = 0, \\ &\boldsymbol{p}_k - \sum_{l=1}^{N} \frac{\boldsymbol{n}_l\boldsymbol{p}_k}{\boldsymbol{\mu}_l - \boldsymbol{\nu}_k}\boldsymbol{m}_l = 0, \end{aligned} \tag{36}$$

where $\boldsymbol{n}_k\boldsymbol{p}_l$ is scalar product of row vector $\boldsymbol{n}_k$ and column vector $\boldsymbol{p}_l$. Equations (36) enable us to express all of the vector functions $\boldsymbol{m}_k$'s and $\boldsymbol{q}_k$'s in terms of the set of $2N$ vector functions $\boldsymbol{n}_k$'s and $\boldsymbol{p}_k$'s:

$$\boldsymbol{m}_k = \sum_{l=1}^{N} \boldsymbol{G}^{-1}_{lk}\boldsymbol{p}_l, \qquad \boldsymbol{q}_k = -\sum_{l=1}^{N} \boldsymbol{G}^{-1}_{kl}\boldsymbol{n}_l, \tag{37}$$

where $\boldsymbol{G}^{-1}_{kl}$'s are elements of the inverse of matrix $\boldsymbol{G} = \{\boldsymbol{G}_{lk}\}^N_{l,k=1}$ with $\boldsymbol{G}_{lk} := \dfrac{\boldsymbol{n}_l\boldsymbol{p}_k}{\mu_l - \nu_k}$. Therefore, we only need to determine $\boldsymbol{n}_k$, $\boldsymbol{p}_k$ and μ_k, ν_k in the following.

The functions $\boldsymbol{R}_k$, $\boldsymbol{S}_k$, μ_k and ν_k must also obey Eqs.(23) and (24). Writing Eq.(23) as

$$\begin{aligned} &\boldsymbol{\Lambda}^{-1}(\boldsymbol{w})\mathrm{d}\boldsymbol{H}\boldsymbol{\Omega} \\ &= [\boldsymbol{\chi}(\boldsymbol{w})\boldsymbol{\Gamma}_0(\boldsymbol{w})\boldsymbol{\Omega} + \mathrm{d}\boldsymbol{\chi}(\boldsymbol{w})]\,\boldsymbol{\chi}^{-1}(\boldsymbol{w}) \\ &= \boldsymbol{\chi}(\boldsymbol{w})\left[\boldsymbol{\Gamma}_0(\boldsymbol{w})\boldsymbol{\Omega}\boldsymbol{\chi}^{-1}(\boldsymbol{w}) - \mathrm{d}\boldsymbol{\chi}^{-1}(\boldsymbol{w})\right] \end{aligned} \tag{38}$$

with $\boldsymbol{\Gamma}_0(\boldsymbol{w}) = \boldsymbol{\Lambda}^{-1}(\boldsymbol{w})\mathrm{d}\boldsymbol{H}_0$, then the substitution of Eqs.(33) and (35) into Eq.(38) shows that $\boldsymbol{\mu}_k$, $\boldsymbol{\nu}_k$ should all be ordinary complex constants and $\boldsymbol{n}_k$, $\boldsymbol{p}_k$ obey the equations

$$\begin{aligned} &\mathrm{d}\boldsymbol{n}_k + \boldsymbol{n}_k\boldsymbol{\Gamma}(\boldsymbol{\mu}_k)_0\boldsymbol{\Omega} = 0, \\ &\mathrm{d}\boldsymbol{p}_k - \boldsymbol{\Gamma}(\boldsymbol{\nu}_k)_0\boldsymbol{\Omega}\boldsymbol{p}_k = 0. \end{aligned} \tag{39}$$

Comparing Eqs.(39) with (13) (for seed solution $\boldsymbol{H}_0$), we see that the solutions of Eqs.(39) can be written as

$$\boldsymbol{n}_k = \boldsymbol{n}^{(0)}_k\boldsymbol{F}^{-1}_0(\boldsymbol{\mu}_k), \quad \boldsymbol{p}_k = \boldsymbol{F}_0(\boldsymbol{\nu}_k)\boldsymbol{p}^{(0)}_k, \tag{40}$$

where $\boldsymbol{n}^{(0)}_k$ and $\boldsymbol{p}^{(0)}_k$ are EHC constant $(p+2)$-dim row and column vectors, respectively. Further, when substituting Eqs.(33), (35), (37) and (40) into Eqs.(24a) and (24b), noticing (17) and by some algebraic calculations, we surprisingly find that the conditions (24a) and (24b) can be simultaneously fulfilled provided that we take

$$\boldsymbol{\nu}_k = \bar{\boldsymbol{\mu}}_k, \tag{41a}$$

$$\boldsymbol{n}^{(0)}_k = \boldsymbol{p}^{(0)\dagger}_k\boldsymbol{\gamma} \tag{41b}$$

with $\boldsymbol{p}^{(0)}_k$'s being ordinary complex and $\boldsymbol{\gamma}$ being defined by Eq.(5c).

So far, we have obtained $\boldsymbol{\chi}(w)$ in terms of $\boldsymbol{F}_0(w)$; from Theorem 1 and noticing the relation (16a) for $\boldsymbol{F}_0(w)$ we can obtain the new N-soliton EHC H-potential $\boldsymbol{H}_N$ by formula (26) with

$$\begin{aligned} &\boldsymbol{\chi}^{(1)}_N \\ &= -\sum_{l,k=1}^{N} \frac{\boldsymbol{\lambda}(\boldsymbol{\mu}_k)}{\boldsymbol{\mu}_k}\boldsymbol{G}^{-1}_{lk}\boldsymbol{F}_0(\bar{\boldsymbol{\mu}}_l)\boldsymbol{p}^{(0)}_l\,\boldsymbol{p}^{(0)\dagger}_k[\boldsymbol{F}_0(\bar{\boldsymbol{\mu}}_k)]^\dagger\boldsymbol{\gamma}, \end{aligned} \tag{42}$$

where

$$\begin{aligned} &\boldsymbol{G}_{lk} = \frac{\boldsymbol{\lambda}(\boldsymbol{\mu}_l)}{\boldsymbol{\mu}_l}\,\frac{\boldsymbol{p}^{(0)\dagger}_l[\boldsymbol{F}_0(\bar{\boldsymbol{\mu}}_l)]^\dagger\boldsymbol{\gamma}\boldsymbol{F}_0(\bar{\boldsymbol{\mu}}_k)\boldsymbol{p}^{(0)}_k}{\boldsymbol{\mu}_l - \bar{\boldsymbol{\mu}}_k}, \\ &\mathrm{Im}(\boldsymbol{\mu}_k) \neq 0, \qquad k, l = 1, \ldots, N \end{aligned} \tag{43}$$

and $\boldsymbol{p}^{(0)}_k$'s are arbitrary ordinary complex constant $(p+2)$-dim column vectors.

6. Application: generation of family of soliton solutions

To illustrate the ISM given above, we construct a family of soliton solutions, in which we choose the Minkowski spacetime as a seed solution. The corresponding $\boldsymbol{M}_0(\rho, z)$, $\boldsymbol{H}_0(\rho, z; \varepsilon)$ and $\boldsymbol{F}_0(\rho, z, w; \varepsilon)$ are

[1)] This is not the only possible choice. In general, the rank r of R_k can be taken as $1 \leq r \leq p+1$, and similarly for S_k.

$$\boldsymbol{M}_0 = \rho\tilde{\boldsymbol{I}}, \quad \boldsymbol{H}_0 = \rho\tilde{\boldsymbol{I}} - z\boldsymbol{\Omega}; \quad \tilde{\boldsymbol{I}} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -I_p & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{44}$$

$$\boldsymbol{F}_0(w) = \frac{w^{1/2}}{2\boldsymbol{\lambda}(w)\boldsymbol{T}^{1/2}(w)} \begin{pmatrix} \boldsymbol{T}(w)+\boldsymbol{\lambda}(w) & 0 & -\mathrm{i}(\boldsymbol{T}(w)-\boldsymbol{\lambda}(w)) \\ 0 & 2\boldsymbol{\lambda}(w)\boldsymbol{I}_{\mathrm{p}} & 0 \\ \mathrm{i}(\boldsymbol{T}(w)-\boldsymbol{\lambda}(w)) & 0 & \boldsymbol{T}(w)+\boldsymbol{\lambda}(w) \end{pmatrix}, \tag{45}$$

where $\boldsymbol{T}(w) = \boldsymbol{T}(\rho, z, w; \varepsilon) := (w - 2z) + \mathrm{i}\varepsilon 2\rho$.

Now for simplifying the notations, we write $\boldsymbol{p}_k^{(0)} = \begin{pmatrix} \boldsymbol{a}_k \\ \boldsymbol{b}_k \\ \boldsymbol{c}_k \end{pmatrix}$ ($\boldsymbol{a}_k$, $\boldsymbol{c}_k$ are ordinary complex constants and $\boldsymbol{b}_k$ is a p-dim ordinary complex constant column vector) and introduce

$$\boldsymbol{\lambda}_k = \boldsymbol{\lambda}_k(\rho, z) := \boldsymbol{\lambda}(\bar{\boldsymbol{\mu}}_k),$$
$$\boldsymbol{T}_k = \boldsymbol{T}_k(\rho, z; \varepsilon) := \boldsymbol{T}(\bar{\boldsymbol{\mu}}_k).$$

Then from Eqs.(42), (43) and (26), we can obtain the corresponding N-soliton EHC H-potentials as follows:

$$\boldsymbol{H}_N(\rho, z; \varepsilon) = \boldsymbol{H}_0 - \varepsilon \sum_{l,k=1}^{N} \hat{\boldsymbol{G}}_{lk}^{-1}\, \boldsymbol{u}_l\, \boldsymbol{u}_k^{\dagger},$$
$$N = 1, 2, \ldots, \tag{46}$$

where

$$\hat{\boldsymbol{G}}_{lk} = \frac{1}{\boldsymbol{\mu}_l - \bar{\boldsymbol{\mu}}_k}\, \boldsymbol{u}_l^{\dagger}\, \boldsymbol{\gamma}\, \boldsymbol{u}_k,$$
$$\mathrm{Im}(\boldsymbol{\mu}_k) \neq 0, \qquad k, l = 1, \ldots, N,$$

and

$$\boldsymbol{u}_k = \begin{pmatrix} (a_k - \mathrm{i}c_k)\boldsymbol{T}_k + (a_k + \mathrm{i}c_k)\boldsymbol{\lambda}_k \\ 2\boldsymbol{\lambda}_k b_k \\ (c_k + \mathrm{i}a_k)\boldsymbol{T}_k + (c_k - \mathrm{i}a_k)\boldsymbol{\lambda}_k \end{pmatrix}.$$

Thus, for each p we have obtained a soliton solution family $\{\boldsymbol{M}_N = \mathrm{Re}_{\mathrm{E}}(\boldsymbol{H}_N),\ N = 1, 2, \ldots\}$ for the EM-p theories.

7. Summary

A previously established HE-type EHC linear system[18] is slightly modified and used to develop an ISM for the EM-p theories. We show that the reduction procedure for this ISM is fairly simple, which makes the ISM be fine and effective in practical application. As an example in application, we use this method to obtain a concrete family of soliton solutions for the SAS EM-p theory.

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